Perfect sets of 1-factors in complete bipartite graphs

Tyler Bloom
tyler.bloom@uvm.edu

Maddie Burke
mburke3@uvm.edu

Cole Charbonneau
cole.charbonneau@uvm.edu

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1 Introduction

1.1 Definitions

A *1-factor* of a graph G is a 1-regular spanning subgraph of G. As such, a 1-factor is very similar to a perfect matching; the only difference is that matchings are sets of edges alone, whereas factors contain vertices as well.

A pair of 1-factors is said to be *perfect* provided that their union is a Hamiltonian cycle in G. Similarly, we say a set of n 1-factors is perfect when any pair of 1-factors from the set is perfect. As a shorthand, we will call a set of n disjoint 1-factors an n-set.

A 1-factorization of a graph G is a collection of disjoint 1-factors that covers G. As such, G must be regular for a 1-factorization to exist. When a 1-factorization does exist, it is analogous to an edge coloring of G. We can naturally extend the idea of being "perfect" to a 1-factorization as well: a 1-factorization is perfect provided that every pair of 1-factors within it is perfect.

1.2 Questions

For a given 1-factor F from a 1-factorization of $K_{r,r}$, how many other 1-factors form a perfect pair with F?

What is the largest perfect n-set for a $K_{r,r}$? What bearing does r have on this size? For which values of r does there exist a perfect r-set, or in other words, a perfect 1-factorization?

How may perfect pairs, triples, etc. exist? More specifically, given an n, how many perfect n-sets are there?

2 Perfect pairs

To start making claims about perfect pairs and sets in $K_{r,r}$, it will help to first show that a 1-factorization exists for $K_{r,r}$.

Proposition 2.1. *A 1-factorization exists for every complete bipartite graph $K_{r,r}$.*

Proof. Consider the complete bipartite graph $K_{r,r}$ with bipartitions A and B. For any subset $S \subseteq A$, $|S| \leq |N(S)|$ since the $|S| \leq r$ and $|N(S)|$ will always be r by definition of a complete bipartite graph. Thus, Hall's condition holds and a matching on the side of A holds. We know this is a 1-factor covering all vertices of $K_{r,r}$ since $|A| = |B|$. If we remove this matching, we are left with an $(r-1)$-regular bipartite graph. The above argument holds for this graph, so a perfect matching exists for the $(r-1)$-regular bipartite graph. Repeat the removal of the matching and Hall's argument until no edges remain in the graph and r 1-factors have been found that together create the 1-factorization of $K_{r,r}$. □

Proposition 2.2. *There is at least 1 perfect pair of 1-factors for every complete bipartite graph, $K_{r,r}$.*

Proof. To show that a perfect pair exists in $K_{r,r}$ with bipartitions A and B, we will begin by numbering the vertices of each bipartition

$$\{a_1, a_2, ...a_r\} \in A, \quad \{b_1, b_2, ..., b_r\} \in B.$$

We construct our first 1-factor on the edges $\{a_1b_1, a_2b_2, ..., a_rb_r\}$ and let the next 1-factor in our 1-factorization be $\{a_1b_2, a_2b_3, ..., a_{r-1}b_r, a_rb_1\}$. Moving from B to A, we follow edges in the first matching, then moving from A to B we follow edges in the second. Starting at the vertex b_1 we move through the vertices in A and B in order until we reach a_r which sends us to b_1 by the second 1-factor. Now we have reached the repeated vertex, closing the cycle, and in doing so, we traversed every vertex in the graph $K_{r,r}$. We successfully found a Hamiltonian cycle, and thus a perfect pair. □

3 Hamiltonian cycles

Since each perfect pair represents a Hamiltonian cycle, knowing the number of unique Hamiltonian cycles in $K_{r,r}$ will be helpful. To determine this, first label the vertices in one class with numbers and the vertices in the other class with letters and consider what a Hamiltonian cycles would look like if we were to list its vertices. Without loss of generality, let's start our enumeration of the cycle at the vertex labeled "1". There are r many options to choose from for the second vertex, namely a, b, c, etc. Now, there are r-1 many ways to choose the next vertex, namely 2, 3, 4, etc. For the next vertex, there are also r-1 ways to choose it. This pattern will continue.

By multiplying these together, we get that there are $r!(r-1)!$ many Hamiltonian cycles in $K_{r,r}$; however, we have over-counted slightly. Since a path is the same regardless of the direction that we travel, half of these Hamiltonian cycles should be identical to another cycle in the list except they have been written "backwards". So by dividing by 2, we should get that the total number Hamiltonian cycles in $K_{r,r}$ is $\frac{r!(r-1)!}{2}$.

4 Perfect 1-factorizations

A perfect 1-factorization exists in $K_{r,r}$ for some but not all values of r. This section will highlight some examples of when perfect 1-factorizations do or do not exist for certain r.

Proposition 4.1. *There is a single unique 1-factorization of* $K_{3,3}$*, and it is perfect.*

Proof. Any union of two disjoint 1-factors is a collection of disjoint cycles (a 2-factor). The smallest cycle that could exist in $K_{3,3}$ is a 4-cycle, which would leave two leftover vertices that could not form a cycle. The only other possibility is a 6-cycle, which must be Hamiltonian. Thus, any two disjoint 1-factors of $K_{3,3}$ are a perfect pair, and their structure is isomorphic to the black and blue edges in Figure 1. The only missing edges form a single 1-factor (dashed red) that is disjoint from the others and is therefore perfect with the others.

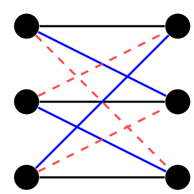

Figure 1: The only 1-factorization of $K_{3,3}$ is perfect.

□

Proposition 4.2. *There is no perfect 1-factorization of* $K_{4,4}$.

Proof. We will assume that a perfect 1-factorization of $K_{4,4}$ exists. Any perfect pair in $K_{4,4}$ is isomorphic to the black and blue edges in Figure 2a, and the red edge shown must exist in some other 1-factor. We will attempt to construct this red 1-factor. Given this first edge, Figure 2b shows the only two options for a red edge coming from a particular vertex. The dashed edge results in a problematic 4-cycle with the black edges, so the other edge is forced into the red 1-factor. From here, Figure 2c shows the only two options for a red edge coming from another particular vertex. Both of these options are problematic; one of them results in a 6-cycle with black, and the other results in a 6-cycle with blue. By this construction, no perfect 1-factorization exists for $K_{4,4}$.

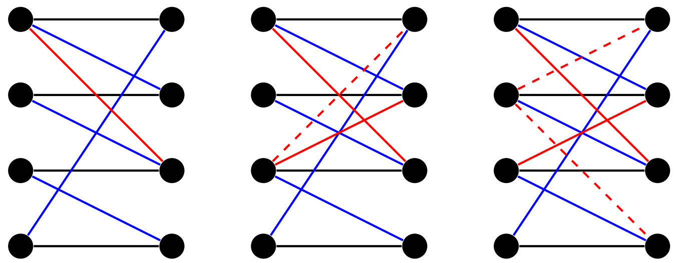

Figure 2: There is no perfect 1-factorization for $K_{4,4}$

□

Proposition 4.3. *If a 1-factorization of $K_{r,r}$ is unique, then it is perfect*

Proof. Assume that a 1-factorization of $K_{r,r}$ is not perfect. Then there exists a pair 1-factors whose union is not a Hamiltonian cycle. Without loss of generality, color the edges of these 1-factors red and blue. Since they are members of the same 1-factorization, they must be disjoint, so their union creates a 2-factor (each vertex has degree 2). Thus, the union of the red and blue edges is a collection of cycles spanning the vertices of $K_{r,r}$. We can create a new 1-factorization of $K_{r,r}$ by selecting a cycle and swapping the colors of the edges. The color classes are still disjoint 1-factors of $K_{r,r}$, so the recoloring of the edges creates a new distinct 1-factorization. Thus an imperfect 1-factorization is not unique. □

5 1-factors and the symmetric group

A 1-factor of $K_{r,r}$ pairs each vertex in one class of the partition with a vertex in the other class. This, essentially, is a permutation of r vertices; we will define a bijection between the set of perfect matchings and the elements of S_r. This bijection will map a 1-factor in which a and b share an edge to an element of S_r of the form $(1\ \ldots\ a\, b \ldots)$.

As an example, consider the matchings pictured below. That matching would be mapped to $(1\ 2\ 3\ 4)$.

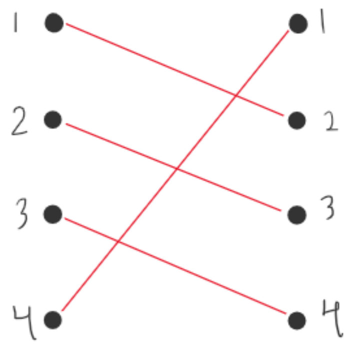

Figure 3: The matching on $K_{4,4}$ that is mapped to $(1\ 2\ 3\ 4)$.

How can we use this equivalence? Let x, y be 1-factors of $K_{r,r}$, and let X, Y be their equivalent members of S_r. From our bijection, x, y forms a perfect pair if and only if XY^{-1} and YX^{-1} are algebraic r-cycles. This is because all subcycles in the product need to have order r, which means that the only subcycle the product can have is itself.

This provides us with another way to calculate the number of perfect pairs. For each r-cycle in S_r, there are $r!$ many ways that r-cycle can be represented as a product of two elements of S_r. There are $(r-1)!$ many r-cycles in S_r, so we see that the number of ways to represent an r-cycle in S_r is a familiar number, $r!(r-1)!$. In much the same way as before, we are over counting here too. For a pair to be perfect, both XY^{-1} and YX^{-1} need to be r-cycles. As such, we need to divide our previous answer in half, which gives us our original answer.

5.1 Stepped elements of S_r

When referring to the symmetric group in this section, we will enumerate the elements starting at 0. This will allow for simpler expressions using the modulo operator.

Definition 5.1. We call an element of the symmetric group, S_r, *stepped* provided that permutation takes x to $(x+k) \bmod r$ for some constant k. The smallest positive value for k that satisfies the above we will call the *step* of f.

With the aid of a python script, we have verified that for prime $r < 100$, $K_{r,r}$ has a perfect 1-factorization composed of factors generated by stepped members of S_r. We conjecture that this is true for all prime r.

Theorem 5.1. *Let F and G be 1-factors of $K_{r,r}$ that are generated by stepped permutations $f, g \in S_r$ with steps a and b, respectively. If $(a-b)$ and r are relatively prime, F and G are a perfect pair.*

Proof. Let A and B be the bipartitions of $K_{r,r}$. The composition of F and G is equivalent to $h = f \circ g^{-1}$. Since f and g have steps a and b, the composition h effectively takes a vertex in a partition with index i to another in the same partition with index $(i + a - b) \bmod r$. Consequently, $h \in S_r$ is a stepped permutation with step $c = (a - b) \bmod r$. A similar argument can be made for $f^{-1} \circ g$, the composition that starts and ends in the other bipartition.

From our observations in the beginning of Section 5, we know that h must be an algebraic r-cycle in order for F and G to be perfect. Note that an r-cycle with step c in S_r can be written as

$$\begin{pmatrix} 0 & c \bmod r & 2c \bmod r & \dots & (r-1)c \bmod r \end{pmatrix}.$$

If h has the above structure and attains every value, then $nc \neq r$ for $n < r$. In other words, $\operatorname{lcm}(c, r) = cr$, or $c = (a - b)$ and r are relatively prime. □

Corollary 5.1.1. *$K_{r,r}$ has a perfect 1-factorization for prime r.*

Proof. Note that for two permutations in S_r with steps a and b, respecitvely, $(a - b)$ will always be smaller than r. As a result, $(a - b)$ and a prime r are always relatively prime. When r is prime, we can construct a perfect 1-factorization comprised of the stepped permutations

$$\begin{pmatrix} 0 & k \bmod r & 2k \bmod r & \dots & (r-1)k \bmod r \end{pmatrix}, \quad 0 \leq k < r.$$

□

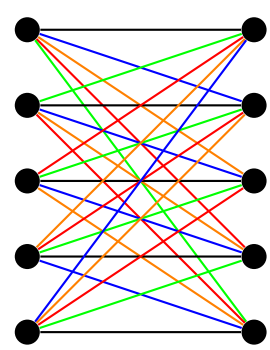

Figure 4: A perfect 1-factorization of $K_{5,5}$ constructed using the method from Theorem 5.1.

6 Remaining open questions

There exists a perfect 1-factorization for $K_{r,r}$ when r is prime, but it remains to be seen how large a perfect n-set can be for composite r. Using our framework for stepped 1-factors of $K_{r,r}$, a lower bound for n given composite r would be the cardinality of the largest set of natural numbers less than r such that for any a, b in the set, $(a - b)$ does not divide r.

References

[1] C. Charbonneau *python script that verifies the existence of perfect 1-factorizations for small prime* r: *https://github.com/colecharb/Graph-Theory-Final-Project*