KNOTS AND KNOT INVARIANTS: AN INTRODUCTION

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Abstract. A mathematical knot is an embedding of a circle into Euclidean three-space. In general, distinguishing knots is nontrivial. Tricolorability and writhe provide elementary insight for invariance of knots. Knot polynomials provide further accuracy in distinction of knots while maintaining a relative ease of computability.

Contents

1. Introduction

The purpose of this paper is to cover the fundamentals in the field of knot theory, providing an adequate jumping-off point for anyone wishing to study the mathematics of knots. We assume the reader has knowledge covering roughly that of a first course in topology, including a basic understanding of homeomorphisms and embeddings. Section 2 is a review of homotopy and isotopy, followed by the definition of ambient isotopy. We begin Section 3 with a precise definition of a knot, followed by a first look at knot diagrams and the Reidemeister moves. Section 4 explains tricolorability and the writhe number. Lastly, Section 5 details the construction of the Jones polynomial by combining the Kauffman bracket polynomial with the writhe number. We conclude with a few calculations of knot polynomials for small diagrams.

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2. Prerequisites

Before discussing knots, we will review homotopy and isotopy.

Definition 2.1. Let X and Y be topological spaces, and let f and g be continuous functions from X to Y. A ***homotopy*** between f and g is a continuous function $H : X \times [0,1] \to Y$ such that

(1) $H(x, 0) = f(x)$
(2) $H(x, 1) = g(x)$.

Definition 2.2. Let X and Y be topological spaces, and let f and g be *embeddings* of X into Y. An ***isotopy*** between f and g is a continuous function $H : X \times [0,1] \to Y$ such that

(1) $H(x, 0) = f(x)$
(2) $H(x, 1) = g(x)$
(3) $H(x, t)$ is an embedding of X into Y for all $t \in [0, 1]$.

It is common for knot equivalence to be stated in terms of *ambient isotopy*. The principle is the same as isotopy; they are both continuous families of homeomorphisms taking one embedding to another. However, an ambient isotopy is, essentially, a deformation of the space in which the embeddings live; embeddings themselves merely come along for the ride.

Definition 2.3. Let X and Y be topological spaces, and let f and g be embeddings of X into Y. An ***ambient isotopy*** between f and g is a continuous family of homeomorphisms $H : Y \times [0,1] \to Y$ with the following properties.

(1) $H(f, 0) = f$
(2) $H(f, 1) = g$
(3) $H(f, t)$ is an embedding for all $t \in [0, 1]$.

3. Mathematical Knots

3.1. **What is a knot?** In the physical world, knots are constructed by taking a cord, wrapping it around itself somehow, and pulling the structure tight. This knot may be untied by loosening the structure and undoing the steps that were taken to tie it in the first place. It might make sense, then, to define a mathematical knot as some embedding of the interval $[0, 1]$ into $\mathbb{R}^3$. The trouble with this definition lies in how we wish to distinguish knots. Mathematical knots are considered equivalent under isotopy; using the above definition, any knot would be equivalent to any other, because they would all be isotopic with the interval. To quote Crowell and Fox, "we must get rid of the ends" [1]. So, to that end, we use the following definition.

Definition 3.1. A ***knot*** is an embedding of the unit circle into $\mathbb{R}^3$.

FIGURE 1. A tight and loose trefoil (overhand) knot tied with paracord, followed by its analogous knot diagram.

The trivial knot, often called the *unknot*, is the mathematical equivalent of an unknotted structure. In general, we call an embedding of one or more circles into $\mathbb{R}^3$ a *link*. We refer to the maximally connected components of a link simply as its components. Thus, a knot is a 1-component link.

Definition 3.2. The ***unknot*** is any knot isotopic to the trivial embedding of the unit circle into $\mathbb{R}^3$, given by $K(x, y) = (x, y, 0)$.

Definition 3.3. A ***link*** is an embedding of one or more circles into $\mathbb{R}^3$.

Remark. In this paper, we will focus purely on so-called *tame knots*. Loosely speaking, they are knots which may be represented in the real world with a physical piece of string. Somewhat more technically, they are isotopic to a finite closed chain of straight line segments in $\mathbb{R}^3$. All tame knots are isotopic to one that is continuously differentiable. More on tame knots can be found in [1, pp. 5-6].

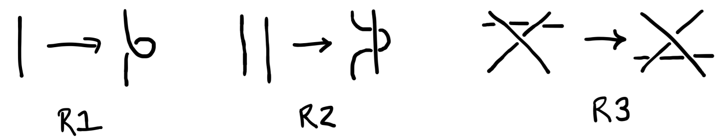
Theorem 3.1. *Two tame knots are isotopic if, and only if, they are ambiently isotopic.*

This theorem is a consequence of the fact that a tame knot is isotopic with a smooth manifold, for which isotopy and ambient isotopy are equivalent.

3.2. **Knot diagrams.** With the exception of the photographs of real-world knots, the figures we have seen thus far are all examples of *knot diagrams*. A knot diagram is simply a projection of a knot onto a plane, with information about how strands cross one another. In fact, all diagrams we have seen have been in *regular position*. A diagram in regular position has the following properties: at no point in the diagram do more than two arcs cross at once, and no two arcs in the diagram are tangent to one another. Regular diagrams avoid any ambiguity in their representation of a knot. From this point onward, the term “diagram” will refer to one in regular position.

Knot diagrams are the standard way to represent knots, though they have more utility than visualization alone. By making small adjustments to a knot diagram, we can obtain different representations of the same knot. These adjustments are completely encapsulated in the Reidemeister moves.

Definition 3.4. The ***Reidemeister moves*** are the following three elementary operations for knot diagrams.



Remark. There are obvious variants of all three moves that we exclude from this diagram, as they are very similar.

It is clear that a diagram obtained by some finite sequence of Reidemeister moves on D is isotopic to D. Conversely, if D and D' are two regular diagrams of isotopic knots, then there *always* exists some finite sequence of Reidemeister moves to get from one to the other.

Theorem 3.2. *Two knots are ambiently isotopic if, and only if, any regular diagram of either knot may be obtained from that of the other by means of a finite sequence of Reidemeister moves.*

A complete proof of this theorem is given by Reidemeister himself [4, pp. 8–10]. The result makes it possible to study knot equivalence through the equivalence of their diagrams alone. The rest of this paper focuses exclusively on invariants computed on knot diagrams.

The last foundational topic we cover in this section is *regular isotopy*. This stricter form of isotopy is another way in which knots may be considered equivalent. Certain invariants only hold only for regular isotopy and not ambient isotopy, such as writhe (Section 4) and the Kauffman bracket polynomial (Section 5).

Definition 3.5. Two knot diagrams are ***regularly isotopic*** if one may be transformed into the other without the use of Reidemeister move 1.

4. Simple Invariants

4.1. **Tricolorability.**

Definition 4.1. A knot K is ***tricolorable*** if, for all regular diagrams of K, each continuous curve in the diagram may be colored with one of three colors such that:

(1) At a crossing, either all three or only one color is present;
(2) All three colors are used in the diagram.

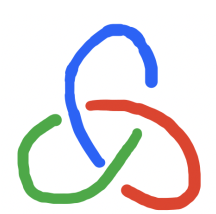
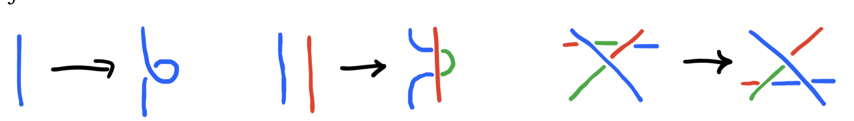


Figure 2. A tricoloring of the trefoil knot.

In Figure 2, we see that the usual diagram of the trefoil knot is tricolorable. The figure-eight knot is not tricolorable, the proof of which is simple and is left to the reader.

Theorem 4.1. *Tricolorability is an invariant of ambient isotopy.*

Proof.



□

Remark. We are omitting proof for the previously mentioned variants of each Reidemeister move.

If any one diagram of a knot is tricolorable, then any other diagram of that knot is as well, so tricolorability places a knot in one of two groups: knots with a tricoloring, and knots without a tricoloring. It is a weak invariant of knots. For example, given any three distinct knots, it is impossible to distinguish them using tricolorability alone. Nevertheless, it is an invariant.

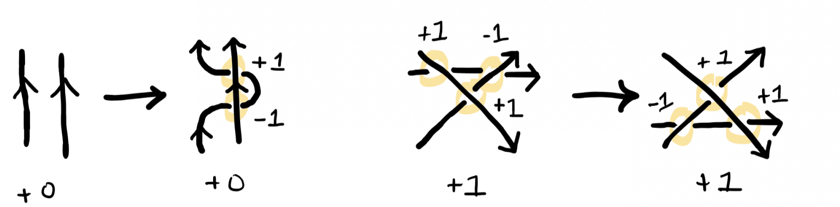
4.2. **Writhe.** The *writhe* of a diagram, also called the *twist number*, is an invariant of regular isotopy that is easily calculated from an oriented knot diagram. To give a diagram orientation, we simply choose a directionality of the knot, notated with arrows, as in Figure 4.

Definition 4.2. The ***writhe*** $w(K)$ of an oriented knot diagram K is the sum of the signs of all crossings in the diagram.

The convention for crossing signs is shown in figure 3.

Theorem 4.2. *Writhe is an invariant of regular isotopy.*

Proof. It is easy to see that the first Reidemeister move does not preserve writhe. The following shows invariance under moves 2 and 3.



□

The writhe numbers of the trefoil and figure-eight knots are 3 and 0, respectively, as seen in Figure 4. We can be sure, then, that the trefoil is not equivalent to the unknot under regular isotopy. On the other hand, the figure-eight knot is indistinguishable from the unknot considering both writhe and tricolorability. Though seemingly not very useful on its own, an important application of writhe will be in our definition of the Jones polynomial.

FIGURE 3. Sign conventions for crossings in an oriented diagram.

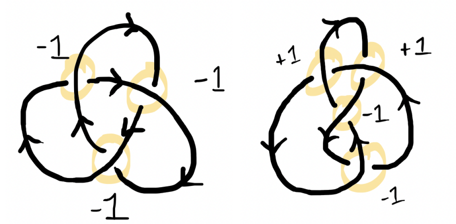


FIGURE 4. Calculating writhes of the trefoil and figure-eight.

5. POLYNOMIAL INVARIANTS

The Jones Polynomial was originally formulated by means of von Neumann algebras on elements of the braid group [2]. Louis Kauffman formulated the bracket polynomial, in part, as a "self-contained introduction to the Jones polynomial" [3, p. 395]. We will take Kauffman's approach, as Jones's original method is beyond the scope of this paper. We conclude with some computations of knot polynomials for small links.

Definition 5.1. Let K be a knot or link diagram. The ***bracket polynomial*** $\langle K \rangle$ is the Laurent polynomial in the variable a defined by:

(i) $\langle \bigcirc \rangle = 1$
(ii) $\langle \bigcirc \cup K \rangle = (-a^2 - a^{-2}) \langle K \rangle$
(iii) $\langle \times \rangle = a \langle \asymp \rangle + a^{-1} \langle)(\rangle$

Here, the symbol "$\bigcirc$" refers to a diagram of the unknot with no crossings. In rule (iii), which we will call the *splitting rule*, the diagrams in the brackets are assumed to differ only in the presented area. By applying these rules for $\langle K \rangle$ recursively, we will always terminate at some final configuration of disjoint unknots. This follows from the observation that an application of the splitting rule essentially wipes out a single crossing from the diagram.

We can symbolically represent the action of *splitting* that takes place in the splitting rule by placing a marker on a crossing that denotes which way it is to be split, as in Figure 5. If we apply a marker to every crossing in a diagram K, we obtain a *state* of K.

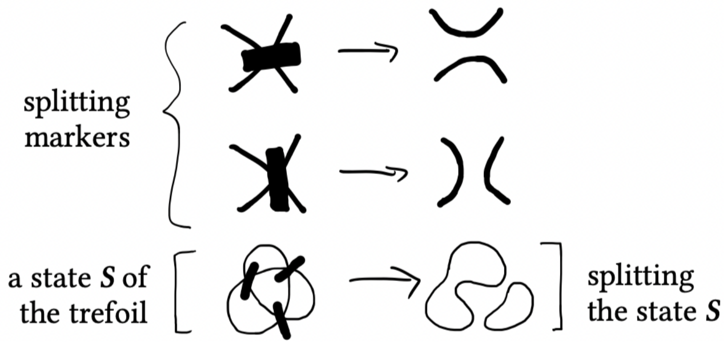


FIGURE 5

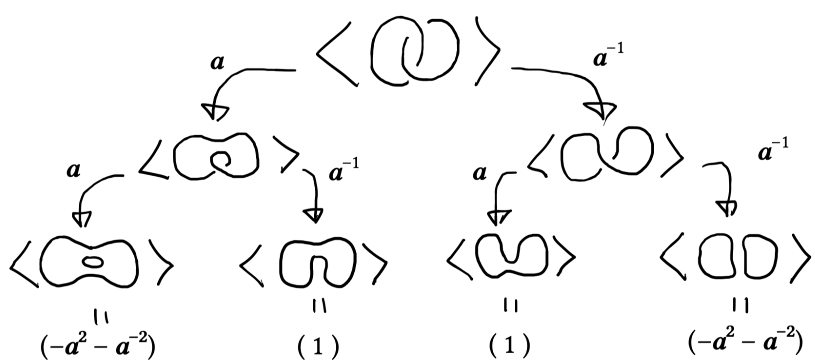
Because the splitting of all crossings in a state results in a disjoint collection of unknots, we see that the set of all states of K are in bijection with all final configurations in the expansion of $\langle K \rangle$. Thus, for a diagram K and a state S, we define

$$\langle K|S \rangle = a^{i-j}$$

where i is the number of markers corresponding with the coefficient a in the splitting rule, and j is the number of markers corresponding with the coefficient a^{-1}. Each state, then, will contribute precisely $\langle K|S \rangle\,(-a^2-a^{-2})^{|S|-1}$ to the polynomial. Therefore, we can express $\langle K \rangle$ as the sum over all states of K as follows;

$$\langle K \rangle = \sum_S \langle K|S \rangle\,(-a^2 - a^{-2})^{|S|-1}$$

where $|S|$ is the number of components in the configuration after splitting the state S completely.



$$\implies \langle K \rangle = a^2(-a^2 - a^{-2}) + 1 + 1 + a^{-2}(-a^2 - a^{-2})$$
$$= \boxed{-a^4 - a^{-4}}$$

FIGURE 6. Calculation of the bracket for the Hopf link.

Proposition 1. *The bracket polynomial is an invariant of regular isotopy.*

Proof of this proposition can be found in Kauffman's article, where much of the theory in this section is from [3, p. 398].

To achieve invariance of ambient isotopy, we must deal with the behavior of the bracket under the first Reidemeister move. Through direct calculation, we have that $\langle\ \rangle = -a^{-3}\langle\ \rangle$ as well as $\langle\ \rangle = -a^{3}\langle\ \rangle$. We know, as well, that writhe is an invariant of regular isotopy and that the preceding diagrams have writhes of +1 and −1, respectively. By augmenting the bracket polynomial to account for this relationship, we obtain

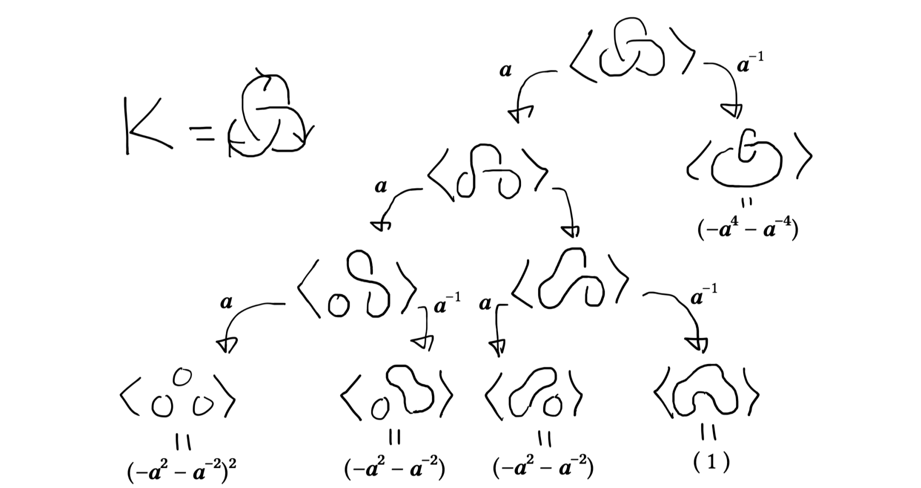
$$f[K] = -a^{-3w(K)}\langle K\rangle,$$

an invariant under all three Reidemeister moves, and thus an invariant of oriented knots and links. When the above polynomial is evaluated at $t^{-1/4}$, it is equivalent to the Jones polynomial [2] [3].

Definition 5.2. Let K be an oriented link diagram. The ***Jones polynomial*** of K, denoted $V_K(t)$ is defined as follows.

$$V_K(t) := f[K]\Big|_{t^{-1/4}}$$

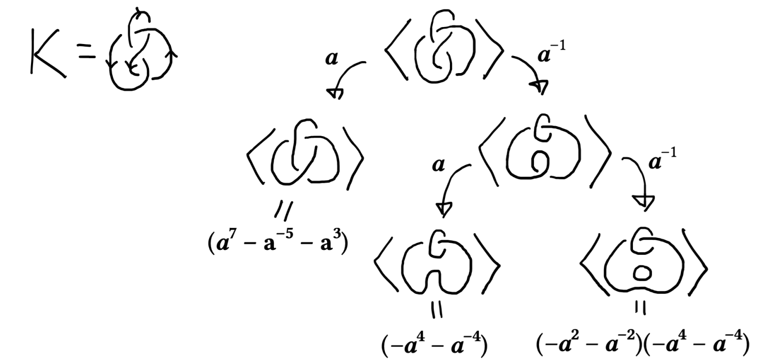
5.1. **Example Calculations.**



$$\implies \langle K\rangle = a^3(-a^2-a^{-2})^2 + 2a(-a^2-a^{-2}) + a^{-1} + a^{-1}(-a^4-a^{-4})$$
$$= \boxed{a^7 - a^{-5} - a^3}$$

$$w(K) = -3, \implies f[K] = -a^{-3(-3)}(a^7 - a^{-5} - a^3) = \boxed{-a^{16} + a^{12} + a^4}$$
$$\implies V_K(t) = \boxed{-t^{-4} + t^{-3} + t^{-1}}$$

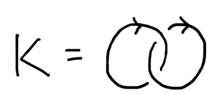
FIGURE 7. Polynomials for the trefoil knot.



$$\implies \langle K \rangle = a(a^7 - a^{-5} - a^3) + (-a^4 - a^{-4}) + a^{-2}(-a^2 - a^{-2})(-a^4 - a^{-4})$$
$$= \boxed{a^8 + a^{-8} - a^4 - a^{-4} + 1}$$

$$w(K) = 0, \implies f[K] = -a^{-3(0)}(a^8 + a^{-8} - a^4 - a^{-4} + 1)$$
$$= \boxed{a^8 + a^{-8} - a^4 - a^{-4} + 1}$$
$$\implies V_K(t) = \boxed{t^2 + t^{-2} - t - t^{-1} + 1}$$

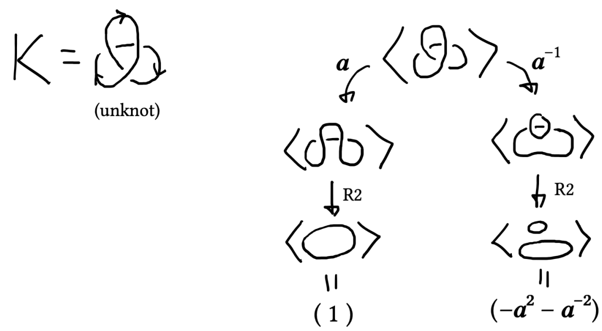
FIGURE 8. Polynomials for the figure-eight knot.



$w(K) = 2$, and we know that $\langle K \rangle = \boxed{-a^4 - a^{-4}}$, so

$$f[K] = -a^{-3(2)}(-a^4 - a^{-4}) = \boxed{a^{-10} + a^{-2}}$$
$$\implies V_K(t) = \boxed{t^{5/2} + t^{1/2}}$$

FIGURE 9. Polynomials for the Hopf link.



$$\implies \langle K \rangle = a + a^{-1}(-a^2 - a^{-2}) = \boxed{-a^{-3}}$$

$$w(K) = -1, \implies f[K] = -a^{-3(-1)}(-a^{-3}) = \boxed{1}$$

$$\implies V_K(t) = \boxed{1}$$

Figure 10. Polynomials for a nontrivial unknot.

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[2] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.

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