COMPUTING THE WIRTINGER PRESENTATION OF A KNOT GROUP FROM A KNOT DIAGRAM

COLE CHARBONNEAU

Contents

1. Introduction

1.1. **What is a knot?** In the physical world, knots are constructed by taking a cord, wrapping it around itself somehow, and pulling the structure tight. This knot may be untied by loosening the structure and undoing the steps that were taken to tie it in the first place. It might make sense, then, to define a mathematical knot as some embedding of the interval $[0,1]$ into $\mathbb{R}^3$. The trouble with this definition lies in how we wish to distinguish knots. Mathematical knots are considered equivalent under isotopy; using the above definition, any knot would be equivalent to any other because they would all be isotopic with the interval. To quote Crowell and Fox, "we must get rid of the ends" [2]. So, to that end, a *knot* is defined as an embedding of the unit circle into $\mathbb{R}^3$.

The trivial knot, often called the *unknot*, is the mathematical representation of an unknotted structure, defined as the trivial embedding of the unit circle into $\mathbb{R}^3$, given by $K(x,y) = (x,y,0)$. In general, we call an embedding of one or more

Date: December 22, 2021.

circles into $\mathbb{R}^3$ a *link*, and we refer to the maximally connected components of a link simply as its components. Thus, a knot is a 1-component link.

Remark. In this paper, we will focus purely on so-called *tame knots*. Loosely speaking, they are mathematical knots which are representations of real-world knots. Somewhat more technically, they are isotopic to a finite closed chain of straight line segments in $\mathbb{R}^3$. All tame knots are isotopic to one that is continuously differentiable. More on tame knots can be found in [2, pp. 5-6].

FIGURE 1. A tight and loose trefoil (overhand) knot tied with paracord, followed by its analogous knot diagram.

1.2. **Knot diagrams.** The rightmost panel in Figure 1 is an example of a *knot diagram*. A knot diagram is simply a projection of a knot onto a plane containing information about how strands cross one another. In fact, this diagram is in what we call *regular position*. A diagram in regular position has the following properties: at no point in the diagram do more than two arcs cross at once, and no two arcs in the diagram are tangent to one another. Regular diagrams avoid any ambiguity in their representation of a knot. From this point onward, the term "diagram" will refer to one in regular position. Reidemeister's book [3] contains some of the foundational theory on knot diagrams, and allows for much of knot theory to be done in terms of knot diagrams alone.

1.3. **Organization of this paper.** The purpose of this paper is to provide a clear explanation of how to compute one presentation of the knot group (the fundamental group of the knot complement) of any tame knot from a knot diagram. The method presented here is a reinterpretation of lecture notes by Jae Choon Cha on the Wirtinger presentation [1]. Section 2 details a decomposition of a general knot into an open cover that is conducive to van Kampen's theorem. This decomposition is used in section 3 to show that a presentation for the knot group is computable for a general knot diagram, and we show how this computation can be performed directly on the diagram in question. We conclude with a few example calculations for small diagrams in Section 4.

2. DECOMPOSITION FROM A KNOT DIAGRAM

2.1. **The principal component.** In order to use the van Kampen theorem to compute the knot group, we will need a decomposition of the knot complement in the form of an open cover whose components have fundamental groups that we understand. This cover is constructed directly from a knot diagram as follows.

We assume the knot K with diagram M to exist in $\mathbb{R}^2 \times [-1, \infty)$ such that the strands in M lie entirely in $\mathbb{R}^2 \times \{0\}$ and a neighborhood of each crossing resembles Figure 2. The bottom of each undercrossing lies in $\mathbb{R}^2 \times \{-1\}$. With K in this position, we can define the principal component of our cover as $A := \mathbb{R}^2 \times [-1/2, \infty]$ so that each crossing inside A will locally resemble Figure 3.

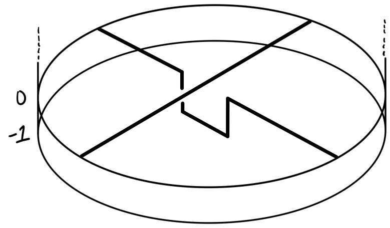


FIGURE 2.
Crossing of K

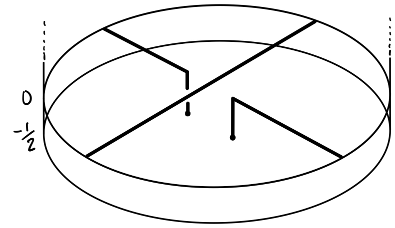


FIGURE 3.
Crossing in A

Notice that $K \cap A$ is a set of arcs corresponding with the strands in the original diagram M. In general, the number of strands and the number of crossings in a diagram are equal. So if the diagram M has n crossings, the complement $A \setminus K$ is homotopic to a wedge of n circles, which we know has a first fundamental group isomorphic to $\oplus_{i=1}^{n} \mathbb{Z}$.

2.2. **Taking care of the crossings.** Because of how we have constructed the principal component of the cover A, all that remains is to understand what each crossing contributes to the knot group. The following is a general solution to building the knot complement from A, one crossing at a time.

We first identify an annulus O in the plane $\mathbb{R}^2 \times \{-1/2\}$ that encircles the endpoints of the strands that meet this plane. To O we glue the $S^1 \times I$ face of a $D^2 \times I$ as in Figure 5. Note that $B := D^2 \times I$ is contractible and that O is homotopic to S^1. For each crossing in K we will glue B_i, a copy of B, as described above. The resultant cover $A \bigcup_i B_i$ is homotopic to $\mathbb{R}^2 \times [-1, \infty) \setminus K$.

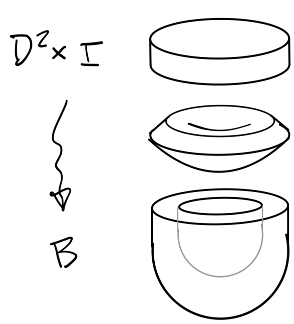


FIGURE 4.
$B \cong D^2 \times I$

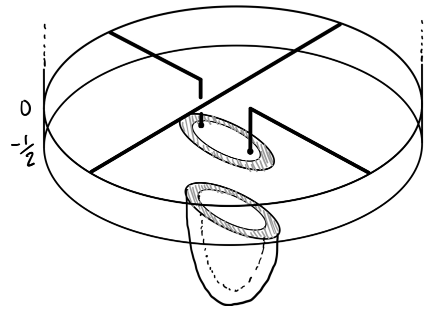


FIGURE 5.
Glueing B to A

3. Computation of the knot group

Now that we have a cover of the knot complement with components whose fundamental groups we understand, what is left is to apply Van Kampen's theorem to determine a presentation of the knot group. Since each crossing is identical in a neighborhood, it suffices to apply Van Kampen locally for each crossing.

3.1. **The crossing relator.** First, we choose an orientation for K and identify a point $p \in A$ that will be the common basepoint for each set in our cover. The generators of $\pi_1(A)$ can then be visualized as n loops, each winding around one of the n distinct strands of $K \cap A$. These loops are oriented according to the right-hand-rule with respect to the orientation of K.

We now turn our attention to the neighborhood of a general crossing (Figure 6). The annulus O is visualized as a homotopy-equivalent $S^1 \cup \xi$, where ξ is an arc in A from the S^1 to the basepoint p. We take the generator for the fundamental group of O to be the loop starting at p, going down ξ, once around the S^1 counterclockwise, and then back up ξ to p. Call this generator ω. It is now easy to see that ω is homotopy equivalent to the path composition $\gamma\alpha\gamma^{-1}\beta^{-1}$, which is precisely the relation we need on $A \cap B$.

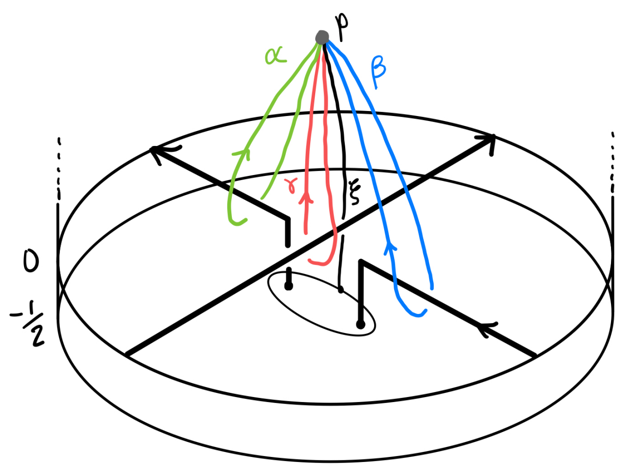


Figure 6. Generators near a crossing

3.2. **Computing from a diagram.** The relator we found in the previous section can be calculated directly from the knot diagram. Figure 7 represents the same picture as Figure 6 viewed from above. This view is conducive for computing the group directly from a knot diagram. The knot group for an arbitrary knot K can be calculated by the following steps.

(1) Choose an oriented knot diagram for K.
(2) Draw an arrow representing each generator of $\mathbb{R}^3 \setminus K$ with orientation determined by the right-hand-rule as in Figure 7.

(3) For each crossing, and starting with any generator for that crossing, list the generators in counterclockwise order. List the generator α as α^{-1} if and only if its arrow is pointing clockwise relative to the crossing. Add this word to the generators of $\pi_1(K)$.
(4) When all crossings have been accounted for, the group is a presentation of $\pi_1(K)$, called the *Wirtinger presentation.*

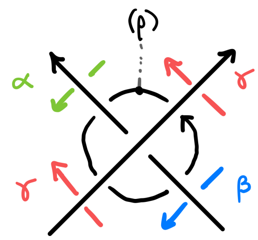
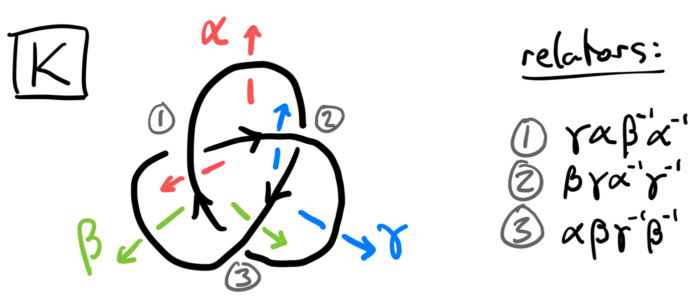


FIGURE 7.
Diagrammatic view of Figure 6

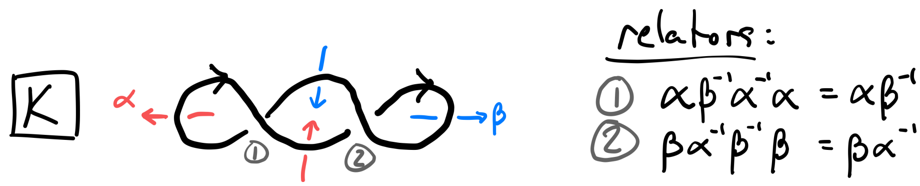
4. Example calculations

4.1. The trefoil.



$$\pi_1(K) \approx \left\langle \alpha, \beta, \gamma \,\middle|\, \gamma\alpha\beta^{-1}\alpha^{-1}, \beta\gamma\alpha^{-1}\gamma^{-1}, \alpha\beta\gamma^{-1}\beta^{-1} \right\rangle$$

4.2. A nontrivial diagram of the unknot.



Both relators imply that $\alpha = \beta$, so we have $\pi_1(K) \approx \langle \alpha, \beta \mid \alpha = \beta \rangle \approx \langle \alpha \mid \rangle \approx \mathbb{Z}$, as expected.

References

[1] Jae Choon Cha. Lecture note 11: Wirtinger presentation. Available at `https://gt.postech.ac.kr/~jccha/math-422-lecture-note-11-2018-spring/`, 2018.

[2] Richard H. Crowell and Ralph H. Fox. *Introduction to knot theory*. Springer-Verlag, New York-Heidelberg, 1977. Reprint of the 1963 original, Graduate Texts in Mathematics, No. 57.

[3] K. Reidemeister. *Knot theory*. BCS Associates, Moscow, Idaho, 1983. Translated from the German by Leo F. Boron, Charles O. Christenson and Bryan A. Smith.

Winooski, VT 05404

Email address: `cole.charbonneau@uvm.edu`