Note on *Solving Solvable Quintics* by D.Dummit
communicated by Robin Chapman
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I write to fill what seems to be a slight gap in the proof of Theorem 1.

It does not seem clear that your polynomial f_{20} does not have repeated roots. There is no reason given why $\theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6$ is impossible. If you had defined your sextic resolvents as, say,

$$\phi_1 = x_1^3 x_2 x_3 + \text{corresponding terms, etc.}$$

then if $f(x) = x^5 - 2$, say, one finds that indeed $\phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_6 = 0$, so the question is not an idle one. If the discriminant of f_{20} were a power of that of f then clearly f_{20} could not have repeated roots but as you point out, this is not the case.

However one can argue as follows. If f_{20} has repeated roots then by the Galois action, $\theta_2 = \theta_3 \ldots = \theta_6$. Now

$$\theta_2 - \theta_3 = (x_1 - x_4)(x_3 - x_5)[(x_2 - x_1)(x_2 - x_4) - (x_2 - x_3)(x_2 - x_5)]$$

and we deduce that

$$(x_2 - x_1)(x_2 - x_4) = (x_2 - x_3)(x_2 - x_5).$$

Similarly, as $\theta_4 = \theta_5$, then

$$\theta_4 - \theta_5 = (x_3 - x_4)(x_1 - x_5)[(x_2 - x_3)(x_2 - x_4) - (x_2 - x_1)(x_2 - x_5)]$$

implies

$$(x_2 - x_3)(x_2 - x_4) = (x_2 - x_1)(x_2 - x_5).$$

As the differences $x_i - x_j$ are non zero one gets

$$(x_2 - x_1) = \pm(x_2 - x_3) \quad \text{and} \quad (x_2 - x_4) = \pm(x_2 - x_5),$$

where the two $\pm$ signs are the same. But $(x_2 - x_1) = (x_2 - x_3)$ implies that $x_1 = x_3$, which is not possible, so the $-$ signs are correct, and

$$2x_2 = x_1 + x_3 \quad \text{and} \quad 2x_2 = x_4 + x_5.$$

But then $5x_2 = x_1 + x_2 + x_3 + x_4 + x_5$ is rational, giving a contradiction.