Roots of the Polynomial $x^6 - x^3 + x^2 + 2x - 1$

The polynomial $x^6 - x^3 + x^2 + 2x - 1$ is the product of the linear factor $x + 1$ and the irreducible polynomial $g(x) = x^5 - x^4 + x^3 - 2x^2 + 3x - 1$. The roots of $g(x)$ generate an extension of the rational numbers of degree 10 with dihedral Galois group. Let (all fifth roots are real fifth roots):

$$r_1 = \left[\frac{-823}{2} + 200\sqrt{5} - \frac{45}{2}\sqrt{\frac{395 + 79\sqrt{5}}{2}} + \frac{75}{2}\sqrt{\frac{395 - 79\sqrt{5}}{2}} \right]^{1/5} = 2.161178730214795139...$$

$$r_2 = \left[\frac{-823}{2} - 200\sqrt{5} - \frac{75}{2}\sqrt{\frac{395 + 79\sqrt{5}}{2}} - \frac{45}{2}\sqrt{\frac{395 - 79\sqrt{5}}{2}} \right]^{1/5} = -4.4411819562904313053...$$

$$r_3 = \left[\frac{-823}{2} - 200\sqrt{5} + \frac{75}{2}\sqrt{\frac{395 + 79\sqrt{5}}{2}} + \frac{45}{2}\sqrt{\frac{395 - 79\sqrt{5}}{2}} \right]^{1/5} = 1.596460134606070673...$$

$$r_4 = \left[\frac{-823}{2} + 200\sqrt{5} + \frac{45}{2}\sqrt{\frac{395 + 79\sqrt{5}}{2}} - \frac{75}{2}\sqrt{\frac{395 - 79\sqrt{5}}{2}} \right]^{1/5} = 1.892564407823392102...$$

Let ζ denote a fifth root of unity with $\zeta + \zeta^{-1} = (-1 + \sqrt{5})/2$ and set

$$\begin{aligned}
\alpha_1 &= r_1 + r_2 + r_3 + r_4 \\
\alpha_2 &= \zeta^4\, r_1 + \zeta^3\, r_2 + \zeta^2\, r_3 + \zeta\, r_4 \\
\alpha_3 &= \zeta^3\, r_1 + \zeta\, r_2 + \zeta^4\, r_3 + \zeta^2\, r_4 \\
\alpha_4 &= \zeta^2\, r_1 + \zeta^4\, r_2 + \zeta\, r_3 + \zeta^3\, r_4 \\
\alpha_5 &= \zeta\, r_1 + \zeta^2\, r_2 + \zeta^3\, r_3 + \zeta^4\, r_4.
\end{aligned}$$

Then the roots of $g(x)$ are the numbers

$$x_j = \frac{1}{5}(\alpha_j + 1) \qquad 1 \le j \le 5.$$

Choosing $\zeta = 0.3090169943749474241 + 0.9510565162951535721\, i$, the approximate numerical values of the roots are

$$\begin{aligned}
x_1 &= 0.4418042632707661 \\
x_2 &= 0.910820763699306960 + 0.658673915593740872\, i \\
x_3 &= -0.6317228953346896209 - 1.1800052781722433333\, i \\
x_4 &= -0.631722895334689621 + 1.180005278172243333\, i \\
x_5 &= 0.9108207636993069601 - 0.6586739155937408717\, i
\end{aligned}$$

Roots of the Polynomial $x^6 + x^5 - x^3 - x^2 - x + 1$

The polynomial $x^6 + x^5 - x^3 - x^2 - x + 1$ is the product of the linear factor $x - 1$ and the irreducible polynomial $f(x) = x^5 + 2x^4 + 2x^3 + x^2 - 1$. The roots of $f(x)$ generate an extension of the rational numbers of degree 10 with dihedral Galois group. Let (all fifth roots are real fifth roots):

$$r_1 = \left[\frac{597}{4} + \frac{215}{4}\sqrt{5} - \frac{35}{2}\sqrt{\frac{235 + 47\sqrt{5}}{2}} - 5\sqrt{\frac{235 - 47\sqrt{5}}{2}} \right]^{1/5} = 0.987302149971572432...$$
$$r_2 = \left[\frac{597}{4} - \frac{215}{4}\sqrt{5} + 5\sqrt{\frac{235 + 47\sqrt{5}}{2}} - \frac{35}{2}\sqrt{\frac{235 - 47\sqrt{5}}{2}} \right]^{1/5} = -2.1577613232119584055...$$
$$r_3 = \left[\frac{597}{4} - \frac{215}{4}\sqrt{5} - 5\sqrt{\frac{235 + 47\sqrt{5}}{2}} + \frac{35}{2}\sqrt{\frac{235 - 47\sqrt{5}}{2}} \right]^{1/5} = 2.5360246734119664479...$$
$$r_4 = \left[\frac{597}{4} + \frac{215}{4}\sqrt{5} + \frac{35}{2}\sqrt{\frac{235 + 47\sqrt{5}}{2}} + 5\sqrt{\frac{235 - 47\sqrt{5}}{2}} \right]^{1/5} = 3.5167916479261725785...$$

Let ζ denote a fifth root of unity with $\zeta + \zeta^{-1} = (-1 + \sqrt{5})/2$ and set

$$\begin{aligned}
\alpha_1 &= r_1 + r_2 + r_3 + r_4 \\
\alpha_2 &= \zeta^4\, r_1 + \zeta^3\, r_2 + \zeta^2\, r_3 + \zeta\, r_4 \\
\alpha_3 &= \zeta^3\, r_1 + \zeta\, r_2 + \zeta^4\, r_3 + \zeta^2\, r_4 \\
\alpha_4 &= \zeta^2\, r_1 + \zeta^4\, r_2 + \zeta\, r_3 + \zeta^3\, r_4 \\
\alpha_5 &= \zeta\, r_1 + \zeta^2\, r_2 + \zeta^3\, r_3 + \zeta^4\, r_4.
\end{aligned}$$

Then the roots of $f(x)$ are the numbers

$$x_j = \frac{1}{5}(\alpha_j - 2) \qquad 1 \le j \le 5.$$

Choosing $\zeta = 0.3090169943749474241 + 0.9510565162951535721\, i$, the approximate numerical values of the roots are

$$\begin{aligned}
x_1 &= 0.576471429619550610 \\
x_2 &= -0.182835990170361125 + 1.032925131232470020\, i \\
x_3 &= -1.105399724639414180 - 0.595451827091543919\, i \\
x_4 &= -1.105399724639414180 + 0.595451827091543919\, i \\
x_5 &= -0.182835990170361125 - 1.032925131232470020\, i
\end{aligned}$$