In: Trends in Soliton Research
Editor: L.V. Chen, pp. 15–27
ISBN 1-59454-560-X

Chapter 2

LINEARIZATION OPERATORS OF SOLITONS IN THE DERIVATIVE NONLINEAR SCHRÖDINGER HIERARCHY

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Abstract

The derivative nonlinear Schrödinger (DNLS) equation arises as a physical model for ultra-short pulse propagation. In this article, linearization operators of solitons in the DNLS hierarchy are studied. It is shown that these operators commute with the recursion operator of the hierarchy. In addition, they can be factored into the recursion operator and the linearization operator of the DNLS equation. Consequently, the complete set of eigenfunctions for the linearization operators of the entire hierarchy are shown to be the same as those for the DNLS equation. These results lay the foundation for a unified soliton perturbation theory for the DNLS hierarchy. In addition, the derivation used in this paper is simpler than the one used before for other integrable hierarchies.

1 Introduction

In the theory of nonlinear waves, integrable equations play an important role because they can be solved analytically by the inverse scattering method ([1], [2], and references therein). Integrable equations support solitons which move stationarily and collide elastically. Many of these integrable equations are also physically significant as they govern various physical processes to the leading order of approximation. Notable examples include the KdV equation for shallow water waves, the nonlinear Schrödinger (NLS) equation for dispersive wave packets, the sine-Gordon equation for long Josephson junctions[3], and the derivative nonlinear Schrödinger (DNLS) equation for nonlinear Alfvén wave in space plasma physics[4] and ultra-short pulse propagation [5]. However, it is also recognized that in physical systems, various perturbations to the integrable equations such as damping and higher-order dispersion are also inevitable[5]. These perturbations will affect the evolution of solitons in an unknown way. Thus a soliton perturbation theory to integrable equations is also an important issue. In a soliton perturbation theory, based on either the inverse scattering method or a direct method (e.g., [6, 7, 8, 9, 10, 11, 12]), the key is to identify a complete set of the so-called "squared eigenfunctions", which are eigenfunctions of the linearization operator of the integrable equation. The "squared eigenfunctions" depend quadratically on the Jost functions of the spectral operator in the Lax pair. However, the exact form of this quadratic dependence differs for different equations. For instance, for the NLS equation, this dependence is a simple square [7, 9]. But for the KdV equation, the dependence is the spatial derivative of the square [6, 7]. For the massive Thirring model in lab coordinates [13], that dependence is more complicated. Because of this ever-changing quadratic dependence, we are forced to treat one equation at a time. The known exception is that for the KdV, NLS and modified KdV hierarchies, the same "squared eigenfunctions" are eigenfunctions of the linearization operators for the entire hierarchy [14, 15, 16, 17, 18]. It is expected that the same holds true for any integrable hierarchy, but a general proof is still not available at this time.

The DNLS equation and its variations (such as the modified NLS equation) are receiving more attention these days since these equations are relevant for ultra-short pulse propagation in fibers and other waveguides [5, 19, 20, 21, 22, 23, 24, 25, 26]. The current technology for generating femto-second pulses has stimulated theoretical studies of such pulses through the perturbed DNLS-type equations. With this motivation, here we study the DNLS hierarchy and lay the mathematical framework for a unified perturbation theory for the entire DNLS hierarchy.

The DNLS equation was first solved by the inverse scattering method in[27]. Its hierarchy and the recursion operator were obtained in [28]. A soliton perturbation theory for the DNLS equation and the modified NLS equation was developed recently[29, 12]. It was found that the eigenfunctions of the linearization operator are the derivatives of the squared Jost solutions, and these eigenfunctions form a complete set[12]. In this paper, we will show that the linearization operator of solitons for any member in the DNLS hierarchy can be factored into a function of the recursion operator and the linearization operator of the DNLS equation. In addition, the recursion operator and the linearization operator of the DNLS equation are commutable. Consequently, the complete eigenfunctions for the linearization operators of solitons for the entire DNLS hierarchy are the same as those obtained in[12] for the DNLS equation. Similar results for the adjoint linearization operators are also given. With these results, a unified soliton perturbation theory for the entire DNLS hierarchy can be formulated in a straightforward way. Such a perturbation theory should be useful in theoretical studies of ultra-short pulses.

It is noted that these results closely resemble those which have been obtained before for the KdV, NLS and modified KdV hierarchies[17]. However, our derivation in this paper, which is simply based on the commutability between the recursion operator and the linearization operator of the hierarchy, is simpler than that used in [17]. Thus it is easier to generalize to other integrable hierarchies. We hope that the results in this paper will shed light on a general proof that the linearization operators of solitons in any integrable hierarchy share the same complete set of eigenfunctions, and can be factored into the recursion operator and the linearization operator of the lowest member in the hierarchy.

2 The DNLS Hierarchy

The DNLS hierarchy was derived in Ref. [28]. Under certain restrictions it can be rewritten as

$$iU_t + [(-2i\Lambda)^{2n+1}U]_x = 0. \tag{1}$$

Here,

$$U = (\ u, \quad \bar{u}\ )^T, \tag{2}$$

the superscript "T" represents the transpose of a matrix, the bar denotes the complex conjugate, Λ is the recursion operator[28],

$$\Lambda = \frac{i}{2}(I + iI_+)\sigma_3 \frac{d}{dx}, \tag{3}$$

I is the unit matrix,

$$I_+ = -U\int_x^{\infty} dyU^{\dagger}\sigma_3 = -\begin{pmatrix} u \\ \bar{u} \end{pmatrix}\int_x^{\infty} dy(\ \bar{u}, \quad -u\ ), \tag{4}$$

σ_3 is the third Pauli spin matrix, "†" is the Hermitian, and n is a non-negative integer. When $n = 0$, Eq.(1) becomes the DNLS equation

$$iu_t + u_{xx} + i(|u|^2u)_x = 0. \tag{5}$$

When $n \geq 1$, it gives the other members in this hierarchy. It is noted that the power function in Eq. (1) can be generalized to any odd entire function [17]. But that generalization is trivial. For the clarity of presentation, we will not consider that slightly more general case.

We define an inner product between two arbitrary vectors as

$$\langle g|f\rangle = \int_{-\infty}^{+\infty} g^{\dagger} f dx.. \tag{6}$$

This definition of the inner product is more convenient than the one used in [12], as we will see later in this paper. We also define the adjoint operator O^+ for an arbitrary operator O as,

$$\langle g|Of\rangle = \langle O^+g|f\rangle. \tag{7}$$

Here f and g are arbitrary vectors with $f, g \to 0$, as $|x| \to \infty$. Then the adjoint operator of Λ is

$$\Lambda^+ = \frac{i}{2}\sigma_3 \frac{d}{dx}(I - iI_-), \tag{8}$$

where

$$I_- = -\sigma_3 U\int_{-\infty}^{x} dyU^{\dagger} = -\begin{pmatrix} u \\ -\bar{u} \end{pmatrix}\int_{-\infty}^{x} dy(\ \bar{u}, \quad u\ ). \tag{9}$$

Note that Λ^+ is different from that in [28] because of the different definition of the inner product.

Theorem 1. *The hierarchy equation (1) is equivalent to*

$$iU_t + (-2i\Lambda^+)^{2n+1}U_x = 0. \tag{10}$$

Proof. First we prove by induction that for any integer $n \geq 0$ and a localized vector U,

$$\langle U_x | \Lambda^n U \rangle = 0. \tag{11}$$

For this purpose, we note that for any vector $f = (\ f, \ \ \bar{f}\ )^T$,

$$\partial_x^{-1}\Lambda^+\partial_x f = \Lambda f - \frac{1}{2}U\langle U|\partial_x f\rangle = \Lambda f + \frac{1}{2}U\langle U_x|f\rangle. \tag{12}$$

In addition, Λ**f** is in the form of $(g, \bar{g})^T$, hence so is $\Lambda^n U$. Consequently, $\langle U_x|\Lambda^n U\rangle$ is a real quantity.

To prove Eq. (11) by induction, we can easily check that (11) holds for $n = 0$ and 1. Now, suppose it also holds for $0, 1, \ldots,$ up to $n-1$, then

$$\partial_x^{-1}(\Lambda^+)^n\partial_x U = \Lambda^n U, \tag{13}$$

and

$$\langle U_x|\Lambda^n U\rangle = \langle U_x|\partial_x^{-1}(\Lambda^+)^n\partial_x U\rangle = -\langle U|(\Lambda^+)^n\partial_x U\rangle. \tag{14}$$

However, from the definition of the adjoint operator (7), we have

$$\langle U_x|\Lambda^n U\rangle = \langle(\Lambda^+)^n U_x|U\rangle = \overline{\langle U|(\Lambda^+)^n U_x\rangle}. \tag{15}$$

Combining Eqs. (14) and (15) and recalling that $\langle U_x|\Lambda^n U\rangle$ is a real quantity, we conclude that

$$\langle U_x|\Lambda^n U\rangle = 0. \tag{16}$$

This completes the induction proof of Eq. (11). Because of Eq. (11) and relation (12), we quickly find that

$$\partial_x \Lambda^n U = (\Lambda^+)^n U_x. \tag{17}$$

Thus Eq. (10) is equivalent to Eq. (1).

One-soliton solutions of the hierarchy (1) or (10) are [27, 12]

$$u_{2n+1}(x,t) = -4\eta\frac{\bar{\zeta}_1 e^{2\theta} + \zeta_1 e^{-2\theta}}{(\zeta_1 e^{2\theta} + \bar{\zeta}_1 e^{-2\theta})^2}e^{-2i\varphi} = u_0(\theta)e^{-2i\hat{\varphi}}, \tag{18}$$

where $\zeta_1 = \Delta e^{i\gamma/2}$ is the discrete eigenvalue,

$$\theta = \eta(x - \hat{x}), \quad \hat{x} = c_{2n+1}t + x_0, \tag{19}$$

$$\varphi = \xi(x - \hat{x}) + \hat{\varphi}, \quad \hat{\varphi} = -\frac{1}{2}d_{2n+1}t + \varphi_0, \tag{20}$$

$$\xi = \mathrm{Re}(\zeta_1^2), \quad \eta = \mathrm{Im}(\zeta_1^2), \tag{21}$$

$$c_{2n+1} = (-1)^{n+1}2^{2n+1}\Delta^{4n+2}\frac{\sin(2n+2)\gamma}{\sin\gamma}, \tag{22}$$

and

$$d_{2n+1} = (-1)^n 2^{2n+2} \Delta^{4n+4} \frac{\sin(2n+1)\gamma}{\sin\gamma}. \tag{23}$$

When $n = 0$, these solitons become the DNLS soliton which has been obtained before [27]. Note that

$$(-2i\Lambda_0^+)^m U_{0x} = ic_m U_{0x} + d_m \sigma_3 U_0, \tag{24}$$

where

$$U_0 = [u_0(\theta),\ \bar{u}_0(\theta)]^T, \quad \Lambda_0^+ = \Lambda^+|_{U=U_0}, \tag{25}$$

and the recursion relation for coefficients c_m and d_m are

$$c_{m+1} = -4i\xi c_m - id_m, \quad d_{m+1} = 4i(\xi^2 + \eta^2)c_m, \tag{26}$$

and

$$c_0 = -i, \quad d_0 = 0. \tag{27}$$

3 Linearization Operators of the DNLS Hierarchy

In a coordinate system $(\tilde{x}, \tilde{t})$ moving with speed c_{2n+1},

$$\tilde{x} = x - c_{2n+1}t - x_0, \quad \tilde{t} = t, \tag{28}$$

we have

$$\partial_{\tilde{t}} = \partial_t - c_{2n+1}\partial_x, \quad \partial_{\tilde{x}} = \partial_x. \tag{29}$$

Let

$$\tilde{U} = U e^{-i\sigma_3 d_{2n+1} t}, \tag{30}$$

and drop the tildes, the hierarchy (10) becomes

$$iU_t - d_{2n+1}\sigma_3 U - ic_{2n+1}U_x + (-2i\Lambda^+)^{2n+1} U_x = 0. \tag{31}$$

Its stationary soliton solution is U_0. To linearize this equation around the soliton solution U_0, we write

$$U(x,t) = U_0(x) + \varepsilon q(x,t), \tag{32}$$

where $\varepsilon \ll 1$, and q is the perturbation. Simple calculations show that

$$\begin{aligned}(-i2\Lambda^+)^{2n+1}(U_{0x} + \varepsilon q_x) &= (-i2\Lambda_0^+)^{2n+1} U_{0x} + \varepsilon(-i2\Lambda_0^+)^{2n+1} q_x \\ &\quad + \varepsilon \sum_{i=1}^{2n+1} (-i2\Lambda_0^+)^{i-1} (ic_{2n+1-i}A + d_{2n+1-i}B)q + O(\varepsilon^2)\end{aligned} \tag{33}$$

where

$$A = i\partial_x [U_0 \int_{-\infty}^{x} dy U_{0y}^{\dagger}(y) + |u_0|^2], \tag{34}$$

and

$$B = -i\partial_x U_0 \int_{-\infty}^{x} dy U_0^{\dagger}(y) \sigma_3. \tag{35}$$

Thus the linearized equation of the DNLS hierarchy (10) [or equivalently, (1)] around a soliton $U_0(x)$ is

$$(i\partial_t - \mathcal{L}_{2n+1})q = 0, \tag{36}$$

where

$$\mathcal{L}_{2n+1} = ic_{2n+1}\partial_x + \sigma_3 d_{2n+1} - (-i2\Lambda_0^{+})^{2n+1}\partial_x - \sum_{i=1}^{2n+1}(-i2\Lambda_0^{+})^{i-1}(ic_{2n+1-i}A + d_{2n+1-i}B)$$

is the linearization operator. When $n=0$, $\mathcal{L}_1$ is the linearization operator for the DNLS equation. It can be rewritten as [12]

$$\mathcal{L}_1 = \begin{pmatrix} -\partial_{xx} - i2[|u_0|^2+2\xi]\partial_x - i2(|u_0|^2)_x + 4\Delta^4 & -iu_0^2\partial_x - i(u_0^2)_x \\ -i\bar{u}_0^2\partial_x - i(\bar{u}_0^2)_x & \partial_{xx} - i2[|u_0|^2+2\xi]\partial_x - i2(|u_0|^2)_x - 4\Delta^4 \end{pmatrix}. \tag{37}$$

4 Structure of Linearization Operators in the DNLS Hierarchy

In this section, we determine the structure as well as complete eigenfunctions of the DNLS hierarchy's linearization operator $\mathcal{L}_{2n+1}$. We present our results in a series of lemmas and a theorem.

Lemma 1. *$\mathcal{L}_1$ and $\Lambda_0^{\dagger}$ are commutable, i.e.,*

$$\mathcal{L}_1\Lambda_0^{\dagger} = \Lambda_0^{\dagger}\mathcal{L}_1.$$

This lemma can be verified directly.

Lemma 2. *$\mathcal{L}_{2n+1}$ and $\Lambda_0^{\dagger}$ are commutable for any non-negative integer n, i.e.,*

$$\mathcal{L}_{2n+1}\Lambda_0^{\dagger} = \Lambda_0^{\dagger}\mathcal{L}_{2n+1}, \quad n \geq 0. \tag{38}$$

Proof. We will use induction to prove it. When $n=0$, Eq. (38) is true in view of Lemma 1. Now suppose it is true for $n-1$, i.e.,

$$\mathcal{L}_{2n-1}\Lambda_0^{\dagger} = \Lambda_0^{\dagger}\mathcal{L}_{2n-1}, \tag{39}$$

we need to prove that (38) is true for n as well. Recalling equation (37) and utilizing assumption (39), it is easy to find that

$$\begin{aligned}
&(-2i\Lambda_0^{\dagger})\mathcal{L}_{2n+1} - \mathcal{L}_{2n+1}(-2i\Lambda_0^{\dagger}) = \\
&(-2i\Lambda_0^{\dagger})\left\{ic_{2n+1}\partial_x + d_{2n+1}\sigma_3 - (ic_{2n}A + d_{2n}B) - (-2i\Lambda_0^{\dagger})(ic_{2n-1}A + d_{2n-1}B)\right\} \\
&-\left\{ic_{2n+1}\partial_x + d_{2n+1}\sigma_3 - (ic_{2n}A + d_{2n}B) - (-2i\Lambda_0^{\dagger})(ic_{2n-1}A + d_{2n-1}B)\right\}(-2i\Lambda_0^{\dagger}) \\
&+(-2i\Lambda_0^{\dagger})^2(ic_{2n-1}\partial_x + d_{2n-1}\sigma_3)(-2i\Lambda_0^{\dagger}) - (-2i\Lambda_0^{\dagger})^3(ic_{2n-1}\partial_x + d_{2n-1}\sigma_3).
\end{aligned} \tag{40}$$

Now relating coefficients $c_{2n+1}, c_{2n}, d_{2n+1}$ and d_{2n} to c_{2n-1} and d_{2n-1} by the recursion relation (26), collecting all terms proportional to c_{2n-1} and d_{2n-1} respectively, and recalling the expressions for operators A and B, we can easily verify that the right hand side of the above equation vanishes identically. Thus $\mathcal{L}_{2n+1}$ and $\Lambda_0^\dagger$ are commutable for any n. This completes the induction proof.

Lemmas 1 and 2 have important consequences. Previously, we have found the DNLS linearization operator $\mathcal{L}_1$'s complete set of eigenfunctions as [12]

$$\left\{\Psi(x,\zeta), \tilde{\Psi}(x,\zeta), -\infty<\zeta<\infty; \quad \Psi(x,\zeta_1), \dot{\Psi}(x,\zeta_1), \tilde{\Psi}(x,\bar{\zeta}_1), \dot{\tilde{\Psi}}(x,\bar{\zeta}_1)\right\}. \tag{41}$$

Here

$$\Psi(x,\zeta)=\partial_x\left(\ \psi_1^2(x,\zeta),\ \ \psi_2^2(x,\zeta)\ \right)^T, \quad \tilde{\Psi}(x,\zeta)=\partial_x\left(\ \tilde{\psi}_1^2(x,\zeta),\ \ \tilde{\psi}_2^2(x,\zeta)\ \right)^T, \tag{42}$$

$$\dot{\Psi}(x,\zeta)=\partial_\zeta\Psi(x,\zeta)|_{\zeta=\zeta_1}, \quad \dot{\tilde{\Psi}}(x,\zeta)=\partial_\zeta\tilde{\Psi}(x,\zeta)|_{\zeta=\bar{\zeta}_1} \tag{43}$$

are squared eigenfunctions, and $\psi(x,\zeta)$ and $\tilde{\psi}(x,\zeta)$ are Jost solutions of the scattering equations for the hierarchy (1) and (10) which are given as [27, 12]

$$\psi(x,\zeta)=\left(\begin{array}{c}\Delta\frac{e^{-i2\zeta_1^2x+i\gamma}}{e^{4\eta x}+e^{-i\gamma}}\zeta^{-1}[1-a(\zeta)] \\ \frac{e^{4\eta x}+a(\zeta)e^{i\gamma}}{e^{4\eta x}+e^{i\gamma}}\end{array}\right)e^{i\zeta^2x}, \tag{44}$$

$$a(\zeta)=\exp(-i2\gamma)\frac{\zeta^2-\zeta_1^2}{\zeta^2-\bar{\zeta}_1^2}, \tag{45}$$

$$\tilde{\psi}(\zeta)=i\sigma_2\bar{\psi}(\bar{\zeta}). \tag{46}$$

The eigenrelations are

$$\mathcal{L}_1\Psi(x,\zeta)=-4(\zeta^2-\zeta_1^2)(\zeta^2-\bar{\zeta}_1^2)\Psi(x,\zeta), \tag{47}$$

$$\mathcal{L}_1\tilde{\Psi}(x,\zeta)=4(\zeta^2-\zeta_1^2)(\zeta^2-\bar{\zeta}_1^2)\tilde{\Psi}(x,\zeta), \tag{48}$$

$$\mathcal{L}_1\Psi(x,\zeta_1)=\mathcal{L}_1\tilde{\Psi}(x,\bar{\zeta}_1)=0, \tag{49}$$

$$\mathcal{L}_1\dot{\Psi}(x,\zeta_1)=-16i\eta\zeta_1\Psi(x,\zeta_1), \tag{50}$$

$$\mathcal{L}_1\dot{\tilde{\Psi}}(x,\bar{\zeta}_1)=-16i\eta\bar{\zeta}_1\tilde{\Psi}(x,\bar{\zeta}_1). \tag{51}$$

Because $\Lambda_0^\dagger$ and $\mathcal{L}_1$ as well as $\mathcal{L}_{2n+1}$ are commutable, the above eigenfunctions of $\mathcal{L}_1$ are also eigenfunctions of $\Lambda_0^\dagger$ and $\mathcal{L}_{2n+1}$. In addition, using the commutability relations as well as asymptotics of these operators and eigenfunctions at large $|x|$ values, we can easily obtain the eigenrelations of operators $\Lambda_0^\dagger$ and $\mathcal{L}_{2n+1}$ as follows:

$$\Lambda_0^\dagger\Psi(x,\zeta)=\zeta^2\Psi(x,\zeta), \tag{52}$$

$$\Lambda_0^\dagger\tilde{\Psi}(x,\zeta)=\zeta^2\tilde{\Psi}(x,\zeta), \tag{53}$$

$$\Lambda_0^\dagger\Psi(x,\zeta_1)=\zeta_1^2\Psi(x,\zeta_1), \tag{54}$$

$$\Lambda_0^\dagger\tilde{\Psi}(x,\bar{\zeta}_1)=\bar{\zeta}^2\tilde{\Psi}(x,\bar{\zeta}_1), \tag{55}$$

$$\Lambda_0^\dagger\dot{\Psi}(x,\zeta_1)=\zeta_1^2\dot{\Psi}(x,\zeta_1)+2\zeta_1\Psi(x,\zeta_1), \tag{56}$$

$$\Lambda_0^\dagger \dot{\tilde{\Psi}}(x,\bar{\zeta}_1) = \bar{\zeta}_1^2 \dot{\tilde{\Psi}}(x,\bar{\zeta}_1) + 2\bar{\zeta}_1 \tilde{\Psi}(x,\bar{\zeta}_1), \tag{57}$$

$$\mathcal{L}_{2n+1}\Psi(x,\zeta) = G_{2n+1}(-2i\zeta^2)\Psi(x,\zeta), \tag{58}$$

$$\mathcal{L}_{2n+1}\tilde{\Psi}(x,\zeta) = -G_{2n+1}(-2i\zeta^2)\tilde{\Psi}(x,\zeta), \tag{59}$$

$$\mathcal{L}_{2n+1}\Psi(x,\zeta_1) = \mathcal{L}_{2n+1}\tilde{\Psi}(x,\bar{\zeta}_1) = 0, \tag{60}$$

$$\mathcal{L}_{2n+1}\dot{\Psi}(x,\zeta_1) = -4i\zeta_1 G'_{2n+1}(-2i\zeta_1^2)\Psi(x,\zeta_1), \tag{61}$$

$$\mathcal{L}_{2n+1}\dot{\tilde{\Psi}}(x,\bar{\zeta}_1) = 4i\bar{\zeta}_1 G'_{2n+1}(-2i\bar{\zeta}_1^2)\tilde{\Psi}(x,\bar{\zeta}_1), \tag{62}$$

where

$$G_{2n+1}(z) = z^{2n+2} - ic_{2n+1}z - d_{2n+1}. \tag{63}$$

Eqs. (58)-(62) show that the complete eigenfunctions (41) of the DNLS linearization operator $\mathcal{L}_1$ are also the complete eigenfunctions of the linearization operator $\mathcal{L}_{2n+1}$ of the entire DNLS hierarchy. Below we further determine the structure of the linearization operators $\mathcal{L}_{2n+1}$ in the hierarchy.

Lemma 3. *For any non-negative integer n, the function defined by*

$$F_{2n+1}(z) = \frac{G_{2n+1}(z)}{(z+2i\zeta_1^2)(z+2i\bar{\zeta}_1^2)} \tag{64}$$

is a polynomial function of z. In particular, $F_1(z) = 1$.

Proof. First of all, we find from the recursion relations (26) and (27) that $c_1 = -4\xi$, $d_1 = 4(\xi^2+\eta^2)$. Also recall that $\zeta_1^2 = \xi + i\eta$. Thus it is easy to verify $F_1(z) = 1$. In order to prove that $F_{2n+1}(z)$ is a polynomial function for any $n \geq 1$, we just need to show that $G_{2n+1}(-2i\zeta_1^2)$ and $G_{2n+1}(-2i\bar{\zeta}_1^2)$ vanish. We will use induction to do that.

When $n=0$, $G_1(-2i\zeta_1^2)$ is obviously zero since $F_1(z) = 1$. Now suppose

$$G_{2n-1}(-2i\zeta_1^2) = (2i\zeta_1^2)^{2n} - 2c_{2n-1}\zeta_1^2 - d_{2n-1} = 0. \tag{65}$$

Utilizing the recursion relations (26), we have

$$\begin{aligned} G_{2n+1}(-2i\zeta_1^2) = \; & (2i\zeta_1^2)^{2n+2} - 2\zeta_1^2\left\{(4\eta^2 - 12\xi^2)c_{2n-1} - 4\xi d_{2n-1}\right\} \\ & - \left\{16\xi(\xi^2+\eta^2)c_{2n-1} + 4(\xi^2+\eta^2)d_{2n-1}\right\}. \end{aligned} \tag{66}$$

Then using the assumption (65), it is simple to verify that the right hand side of Eq. (66) is identically zero. Thus $G_{2n+1}(-2i\zeta_1^2)$ is zero for all non-negative integers n. Similarly, $G_{2n+1}(-2i\bar{\zeta}_1^2)$ is zero as well. Hence Lemma 3 is proved.

Next, we present the main result of this paper.

Theorem 2. *The linearization operator $\mathcal{L}_{2n+1}$ of the DNLS hierarchy has the following factorization:*

$$\mathcal{L}_{2n+1} = F_{2n+1}(-2i\Lambda_0^\dagger)\mathcal{L}_1, \tag{67}$$

where $F_{2n+1}(z)$ is the polynomial function defined in equation (64), $\mathcal{L}_1$ is the linearization operator of the DNLS equation, and n is any non-negative integer.

Proof. Define the operator

$$\mathcal{H}_{2n+1} \equiv \mathcal{L}_{2n+1} - F_{2n+1}(-2i\Lambda_0^\dagger)\mathcal{L}_1. \tag{68}$$

To prove that $\mathcal{H}_{2n+1} = 0$, we need to show that $\mathcal{H}_{2n+1}g(x) = 0$ for any function $g(x)$ in the L_2 space. Since the set of eigenfunctions (41) are complete in the L_2 space, thus we just need to show that $\mathcal{H}_{2n+1}$ acting on each function in this complete set is zero. This can be done by utilizing the eigenrelations of operators $\mathcal{L}_{2n+1}, \mathcal{L}_1$ and $\Lambda_0^\dagger$ presented above. With those relations, we can easily verify that $\mathcal{L}_{2n+1}$ acting on each of the functions in that complete set is zero. Thus $\mathcal{H}_{2n+1} = 0$, and the theorem is proved.

We note that the technique we used to prove the above factorization theorem is based almost entirely on the commutability relation between the recursion operator and the linearization operator of the DNLS hierarchy. This method is much simpler that the one used in [17] for the proof of similar factorization results for the KdV, NLS and modified NLS hierarchies. Thus, it will be the method of choice if one wants to determine the linearization-operator structure for a general integrable hierarchy.

The structure and complete eigenfunctions of the adjoint linearization operator $\mathcal{L}_{2n+1}^+$ can be obtained easily from the above results and Ref.[12]. In fact, because Λ_0^+ and $\mathcal{L}_{2n+1}$ are commutable, their adjoint operator Λ_0 and $\mathcal{L}_{2n+1}^+$ are commutable as well. The complete eigenfunctions of the adjoint DNLS linearization operator

$$\mathcal{L}_1^+ = \begin{pmatrix} -\partial_{xx} - i2[|u_0|^2 + 2\xi]\partial_x + 4\Delta^4 & -iu_0^2\partial_x \\ -i\bar{u}_0^2\partial_x & \partial_{xx} - i2[|u_0|^2 + 2\xi]\partial_x - 4\Delta^4 \end{pmatrix} \tag{69}$$

are

$$\left\{\Phi(x,\zeta), \tilde{\Phi}(x,\zeta), -\infty < \zeta < \infty; \quad \Phi(x,\zeta_1), \dot{\Phi}(x,\zeta_1), \tilde{\Phi}(x,\bar{\zeta}_1), \dot{\tilde{\Phi}}(x,\bar{\zeta}_1)\right\}, \tag{70}$$

where

$$\Phi(x,\zeta) = \left(\phi_1^2(x,\zeta), \ \ \phi_2^2(x,\zeta) \right)^T, \quad \tilde{\Phi}(x,\zeta) = \left(\tilde{\phi}_1^2(x,\zeta), \ \ \tilde{\phi}_2^2(x,\zeta) \right)^T, \tag{71}$$

$$\dot{\Phi}(x,\zeta) = \partial_\zeta \Phi(x,\zeta)|_{\zeta=\zeta_1}, \quad \dot{\tilde{\Phi}}(x,\zeta) = \partial_\zeta \tilde{\Phi}(x,\zeta)|_{\zeta=\bar{\zeta}_1} \tag{72}$$

are squared eigenfunctions, $\phi(x,\zeta)$ and $\tilde{\phi}(x,\zeta)$ are Jost solutions given by [27, 12]

$$\phi(x,\zeta) = \begin{pmatrix} \frac{e^{4\eta x} + a(\zeta)e^{-i\gamma}}{e^{4\eta x} + e^{-i\gamma}} \\ \Delta \frac{e^{i2\bar{\zeta}_1^2 x - i\gamma}}{e^{4\eta x} + e^{i\gamma}} \zeta^{-1}[1 - a(\zeta)] \end{pmatrix} e^{-i\zeta^2 x}, \tag{73}$$

$$\tilde{\phi}(\zeta) = -i\sigma_2 \bar{\phi}(\bar{\zeta}). \tag{74}$$

Then using the large-x asymptotics of the above eigenfunctions and the commutability between Λ_0 and $(\mathcal{L}_1^+, \mathcal{L}_{2n+1}^+)$, we can readily find that the set of functions (70) is also the complete set of eigenfunctions for the adjoint linearization operator $\mathcal{L}_{2n+1}^+$. The eigenrelations are the same as Eqs (58) to (62) except that $\mathcal{L}_{2n+1}$ is replaced by its adjoint $\mathcal{L}_{2n+1}^+$. The factorization formula for the adjoint operator $\mathcal{L}_{2n+1}^+$ is

$$\mathcal{L}_{2n+1}^+ = F_{2n+1}(i2\Lambda_0)\mathcal{L}_1^+. \tag{75}$$

Inner products between eigenfunctions (41) and adjoint eigenfunctions (70) are

$$\langle\tilde{\Phi}(\zeta')|\Psi(\zeta)\rangle = i2\pi\zeta^2 a^2(\zeta)\delta(\zeta^2-\zeta'^2), \tag{76}$$

$$\langle\Phi(\zeta')|\tilde{\Psi}(\zeta)\rangle = -i2\pi\zeta^2 \tilde{a}^2(\zeta)\delta(\zeta^2-\zeta'^2), \tag{77}$$

$$\langle\dot{\tilde{\Phi}}(\bar{\zeta}_1)|\Psi(\zeta_1)\rangle = \langle\tilde{\Phi}(\bar{\zeta}_1)|\dot{\Psi}(\zeta_1)\rangle = -\frac{\zeta_1}{2}\dot{a}^2(\zeta_1), \tag{78}$$

$$\langle\dot{\Phi}(\zeta_1)|\tilde{\Psi}(\bar{\zeta}_1)\rangle = \langle\Phi(\zeta_1)|\dot{\tilde{\Psi}}(\bar{\zeta}_1)\rangle = \frac{\bar{\zeta}_1}{2}\dot{\tilde{a}}^2(\bar{\zeta}_1), \tag{79}$$

$$\langle\dot{\tilde{\Phi}}(\bar{\zeta}_1)|\dot{\Psi}(\zeta_1)\rangle = -\frac{1}{2}\dot{a}^2(\zeta_1) - \frac{1}{2}\zeta_1\dot{a}(\zeta_1)\ddot{a}(\zeta_1), \tag{80}$$

$$\langle\dot{\Phi}(\zeta_1)|\dot{\tilde{\Psi}}(\bar{\zeta}_1)\rangle = \frac{1}{2}\dot{\tilde{a}}^2(\bar{\zeta}_1) + \frac{1}{2}\zeta_1\dot{\tilde{a}}(\bar{\zeta}_1)\ddot{\tilde{a}}(\bar{\zeta}_1). \tag{81}$$

The above results are sufficient to construct a direct soliton perturbation theory for the DNLS hierarchy.

5 Summary and Discussion

In summary, we have proved that the DNLS hierarchy is a new member in the club of hierarchies whose linearization operators around solitons can be factored into the recursion operator and the linearization operator of the first equation in the hierarchy, and the entire hierarchy share the same "squared eigenfunctions". With these results, it is straightforward to develop a direct soliton perturbation theory for any member in the DNLS hierarchy. Given the physical importance of the DNLS equation and the related modified NLS equation for modelling the propagation of ultra-short optical pulses [5], we expect that these results will be helpful for those physical applications. Furthermore, the technique we used to establish these results is simpler than the one used in [17] for other hierarchies. Thus the present technique is easier to be generalized when new hierarchies are encountered.

From a mathematical point of view, the results of this paper shed light on the structure of linearization operators in a general integrable hierarchy. It is known that similar results as those presented above have also been obtained for three other hierarchies: the KdV hierarchy, the NLS hierarchy, and the modified KdV hierarchy [17]. However, whether all integrable hierarchies share these properties is still unclear. Evidence suggests that this statement is true, but a general proof is still lacking. The derivation we used in this paper indicates that once the linearization operator of a hierarchy commutes with its recursion operator, then it naturally follows that the linearization operators of the entire hierarchy share the same set of eigenfunctions. It also follows that the linearization operators of the hierarchy linearized around solitons can be factored into the recursion operator and the linearization operator of the first equation in this hierarchy. Right now, several different methods (including the one used in this paper) exist for proving the commutability between the recursion operator and the linearization operator of a hierarchy [14, 16, 17, 18]. Even though this commutability for a general integrable hierarchy has not been established so far, we conjecture that it does hold. To prove this commutability, one needs to have a good understanding on how the recursion operator of an integrable hierarchy is constructed (see [30] and references therein for a discussion on this issue). Once this commutability is proved, then the structure of the linearization operators around solitons in a general hierarchy will immediately follow. At the same time, all linearization operators in the hierarchy will share the same complete set of eigenfunctions.

Acknowledgements

This work was supported in part by the Air Force Office of Scientific Research under Contract No. USAF F49620-99-1-0174, and by National Science Foundation under Grant No. DMS-9971712. The work of X.-J. Chen was also supported in part by the National Natural Science Foundation of China under Grant No. 10375027.

References

[1] S. Novikov, S. V. Manakov, L. P. Pitaevskij, V.E. Zakharov, *Theory of Solitons* (Consultants Bureau, 1984)

[2] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[3] D. W. McLaughlin and A. C. Scott, *Perturbation analysis of fluxon dynamics*, *Phys. Rev. A*, **18**, 1652(1978)

[4] E. Mjøhus and T. Hada, in *Nonlinear Waves and Chaos in Space Plasmas*, edited by T. Hada and H. Matsumoto (Terrapub, Tokyo, 1997), p. 121.

[5] Govind P. Agrawal, *Nonlinear Fiber Optics*, 3rd ed. (Academic Press, 2001)

[6] V. I. Karpman and E. M. Maslov, "Perturbation theory for solitons." *Sov. Phys. JETP*, **46**, 281(1978).

[7] D. J. Kaup and A. C. Newell, "Solitons as particles, oscillators, and in slowly changing media: a singular perturbation theory." *Proc. R. Soc. London, Ser. A*, **361**, 413(1978).

[8] J. P. Keener and D. W. McLaughlin, "Solitons under perturbations." *Phys. Rev. A*, **16**, 777(1977).

[9] D. J. Kaup, "Perturbation theory for solitons in optical fibers." *Phys. Rev. A*, **42**(9), 5689(1990)

[10] X. -J. Chen, Z. -D. Chen and N. -N. Huang, "A direct perturbation theory for dark solitons based on a complete set of the squared Jost solutions." *J. Phys. A*, **31**, 6929 (1998)

[11] J. Yang, J. and D. J., Kaup, "Stability and evolution of solitary waves in perturbed generalized nonlinear Schrödinger equations." *SIAM J. Appl. Math.* **60**, 967 (2000).

[12] X. -J. Chen and J. Yang, "Direct perturbation theory for solitons of the derivative nonlinear Schrödinger equation and the modified nonlinear Schrödinger equation", *Phys. Rev. E*, **65**(6), 066608(2002)

[13] D. J. Kaup and T. I. Lakoba, "The squared eigenfunctions of the massive Thirring model in laboratory coordinates", *J. Math. Phys*, **37**(1), 308-323 (1996)

[14] A. S. Fokas, "A symmetry approach to exactly solvable evolution equations", *J. Math. Phys*, **21**, 1318-1325(1980)

[15] P. J. Olver, *Applications of Lie groups to differential equations*, (Springer-Verlag, New York, 1993)

[16] R. G. Yordanov, "Why do soliton equations come in hierarchies?", *J. Math. Phys.* **34**(9), 4045(1993)

[17] J. Yang, "Complete eigenfunctions of linearized integrable equations expanded around a soliton solution." *J. Math. Phys.* **41**(9), 6614(2000).

[18] J. Yang, "Eigenfunctions of linearized integrable equations expanded around an arbitrary solution." *Stud. Appl. Math.* **108**, 145 (2002)

[19] N. Tzoar and M. Jain, "Self-phase modulation in long-geomerty optical waveguides." *Phys. Rev. A* **23**, 1266 (1981).

[20] D. Anderson and M. Lisak, "Nonlinear asymmetric self-phase modulation and self-steepening of pulses in long optical waveguides." *Phys. Rev. A* **27**, 1393 (1983).

[21] K. Ohkuma, Y.H. Ichikawa, and Y. Abe, "Soliton propagation along optical fibers." *Opt. Lett.* **12**, 516 (1987).

[22] M.J. Potasek, "Novel femtosecond solitons in optical fibers, photonic switching, and computing." *J. Appl. Phys.* **65**, 941(1989)

[23] A.A. Afanas'ev, E.V. Doktorov, R.A. Vlasov, and V.M. Volkov, "Evolution of femtosecond solitons in a cubic medium with a two-component relaxing nonlinearity." *Optics Comm.* **153**, 83 (1998)

[24] E.V. Doktorov and I.S. Kuten, "The Gordon-Haus effect for modified NLS solitons." *Europhys. Lett.* **53**), 22 (2001)

[25] V.S. Gerdjikov, E.V. Doktorov, and J. Yang, "Adiabatic interaction of N ultrashort solitons: Universality of the complex Toda chain model." *Phys. Rev. E* **64**, 056617 (2001)

[26] E.V. Doktorov, "The modified nonlinear Schrödinger equation: facts and artefacts." *Eur. Phys. J. B* **29**, 227 (2002).

[27] D. J. Kaup and A. C. Newell, "An exact solution for a derivative nonlinear Schrödinger equation." *J. Math. Phys.* **19**, 798(1978)

[28] V. S. Gerdjikov, M. I. Ivanov, P. P. Kulish, "Quadratic Bundle and Nonlinear Equations", *Theor. Math. Phys.* **44**, 342(1980). (In Russian)

[29] V. S. Shchesnovich and E. V. Doktorov, "Perturbation theory for the modified nonlinear Schrödinger solitons." *Physica D,* **129**, 115(1998)

[30] A.S. Fokas, "Symmetries and integrability." *Stud. Appl. Math.* **77**, 253 (1987).