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The inverse scattering transform and squared eigenfunctions for a degenerate 3 × 3 operator

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Abstract
We present the covering set of the squared eigenfunctions for a degenerate 3×3 eigenvalue problem. The derivation follows the approach recently outlined by Yang and Kaup on this same equation (*J. Math. Phys.* **50** 023504 (2009)). This eigenvalue problem is important since it serves as the spectral problem for the inverse scattering transform (IST) of the vector NLS equation, the Sasa–Satsuma equation, and a degenerate two-level Maxwell–Bloch system. The use of this covering set would allow one to treat the linear perturbations of these equations in a common and systematic manner. Comparison with previous results on the perturbed continuous spectrum of the Sasa–Satsuma equation is made.

1. Introduction

Here we will consider the general, degenerate 3×3, eigenvalue problem

$$\partial_x V + \mathrm{i}\zeta J \cdot V = Q \cdot V, \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & Q_{13} \\ 0 & 0 & Q_{23} \\ Q_{31} & Q_{32} & 0 \end{bmatrix}, \tag{1}$$

on the interval $-\infty < x < +\infty$. By 'degenerate', we mean that two of the components of the matrix J are equal. $Q(x)$ is a potential matrix which vanishes like $Q(x \to \pm\infty) = o(1/x)$ for large x. The matrix $V(x, t)$ is a 3×3 solution matrix which contains the 'Jost' solutions as its columns. A 'Jost' solution is a standard and suitably normalized solution of an eigenvalue problem.

This problem has been studied as an inverse scattering transform (IST) since 1973 when Manakov used it [1] to create an IST for the vector nonlinear Schrödinger equation (VNLS). In addition, it also occurs in the IST for the Sasa–Satsuma equation [2] as well as for a two-level Maxwell–Bloch system with degeneracy [3]. The differences between these cases is how the components of the potential matrix are identified. They also differ in the form of the spectral evolution operator, which is the second element of the Lax pair. Here, we shall not be interested in the evolution, only in the scattering problem. In the Manakov case and the Maxwell–Bloch case, one pairs the components as $Q_{31} = \pm Q_{13}^*$, $Q_{32} = \pm Q_{23}^*$ while in the Sasa–Satsuma case, the components are paired as $Q_{32} = -Q_{13}$, $Q_{31} = -Q_{23}$, and $Q_{23} = Q_{13}^*$. Such a pairing of potential matrix components is called a 'reduction' since the number of independent components in the potential matrix has been reduced. In each of the above cases, there have been studies of their perturbation theory, their 'squared eigenfunctions' (SE) and the adjoints (ASE) of the SE [4, 5]. Here, we shall assume no reduction and shall take the four components of Q to be independent and uniquely different. Our goal will be to obtain the universal covering set for the SE and ASE of this eigenvalue problem [6].

This work has arisen out of a recent collaboration by these authors on perturbations of the Sasa–Satsuma equation [4] and long prior, on-going discussions with Professor Vladimir Gerdjikov. The results presented here are an outgrowth from [4, 6, 7]. In the work [4], a simplified approach to the treatment of perturbations of the Sata–Satsuma equation [2] was found. Following that, it then became possible to take a new look at the entire problem of soliton perturbation theory. In [4, 6] it was shown how reductions could give rise to SE composed of sums of products of Jost functions instead of simply a single product of Jost functions. Then in [7], this new approach was applied to the well-known AKNS (Ablowitz, Kaup, Newell, Segur) eigenvalue problem [8], re-obtaining the results of [9, 10], not only with a considerable reduction in the effort involved, but also an indication as to how this approach could be applied to higher order eigenvalue problems. Here a simplified approach to obtaining the SE and the ASE, their inner products and their closure relation had been detailed in a form which has not been adequately detailed before. It then became appropriate to return to the degenerate 3×3 eigenvalue problem of the Sasa–Satsuma equation and now include the bound state spectra, which had been omitted before due to technical difficulties at that time. For a basic reference to problems of this nature, the reader is referred to the above references and [11].

There are four steps which are common to all these problems and which we will detail in the following sections. The first step is to define and analyze what are known as Jost functions and the associated scattering matrix. This step is known as the 'direct scattering problem' and will be discussed in section 2. The second step is the 'inverse scattering problem' and is discussed in section 3, wherein one determines how to reconstruct the Jost functions and the potential matrix, given the scattering data. What is new here is that we can now include the bound state spectra. Next we take up the variations which can result from perturbations in the potential matrix or the scattering data. Here we take the results of the previous two sections and perturb them, following the approach in [4]. We determine the variations in the scattering data when the potential is perturbed in section 4 and also detail how to handle the variations in the bound state scattering data. Then in section 5 we will turn our attention to perturbing the inverse scattering problem and take up the opposite mapping: the variations in the potentials which arise when the scattering data, including bound state data, are perturbed. Here we will detail the SE (the definition of an SE is that it is an eigenstate of the linear equations for the perturbed potentials. Whence the perturbed potentials can be expanded in the SE.) From these results, in section 6, we can then construct the inner products of an SE with an ASE and the closure of the SE and ASE. As we proceed through these sections, appropriate comments will be made concerning the generalizablility of particular points. Finally, a summary will be given in section 7.

2. The direct scattering problem

In the direct scattering problem, one addresses the solutions of the eigenvalue problem, what their analytical properties are, what are the adjoint solutions and their properties, what is the scattering matrix and its properties, what are the features of the bound states, if any, what are the 'fundamental analytical solutions' and their adjoints, etc. We shall take up each of these topics below, sometimes only briefly. For (1), we shall define the Jost solutions by their asymptotics as $x \to \pm\infty$. For ζ real, there are two typical sets which are

$$\Phi(x \to -\infty) \to \mathrm{e}^{-\mathrm{i}\zeta Jx}, \qquad \Psi(x \to +\infty) = \mathrm{e}^{-\mathrm{i}\zeta Jx}. \tag{2}$$

For each of these solution matrices, we have three linearly independent solutions. Therefore, each of these two solution matrices must have their columns linearly dependent on the other's columns. This can be expressed as

$$\Phi = \Psi \cdot S, \qquad \Psi = \Phi \cdot R \tag{3}$$

where $S(\zeta)$ is called the scattering matrix and $R(\zeta)$ is its inverse.

With any problem such as (2), one can always construct an adjoint problem, which is an equivalent problem. To do so, one merely multiplies (1) by an adjoint matrix function, $V^A(x,t,\zeta)$, from the left, and then make use of the product rule of Calculus. Thus

$$V^A \cdot (\partial_x V + \mathrm{i}\zeta J \cdot V - Q \cdot V) = 0, \tag{4}$$

which is the same as

$$\partial_x(V^A \cdot V) + (-\partial_x V^A + \mathrm{i}\zeta V^A \cdot J - V^A \cdot Q) \cdot V = 0. \tag{5}$$

The adjoint problem is therefore

$$\partial_x V^A - \mathrm{i}\zeta V^A \cdot J = -V^A \cdot Q, \tag{6}$$

in which case (5) also gives

$$\partial_x(V^A \cdot V) = 0. \tag{7}$$

The solutions of the adjoint problem are called adjoint Jost functions, which are the rows of V^A. As Jost functions, they need to have a standard normalization which we take to be

$$\Phi^A(x \to -\infty) \to \mathrm{e}^{\mathrm{i}\zeta Jx}, \qquad \Psi^A(x \to +\infty) = \mathrm{e}^{\mathrm{i}\zeta Jx}. \tag{8}$$

Then from (4), (7) and (8), we have

$$\Phi^A \cdot \Phi = I_3 = \Psi^A \cdot \Psi, \qquad \forall\, x \text{ and for } \zeta = \text{real}, \tag{9}$$

where I_3 is the 3×3 identity matrix. In other words, the solution matrix for the adjoint Jost functions can be taken to be the inverse of the regular solution matrix. As a consequence of this and (5), we have

$$\Phi^A = R \cdot \Psi^A, \qquad \Psi^A = S \cdot \Phi^A. \tag{10}$$

As a consequence of (5), (7) and (10), we have that, given the Jost functions and their adjoints, one can obtain

$$\Psi^A \cdot \Phi = S \qquad \text{and} \qquad \partial_x S = 0. \tag{11}$$

Key to the solution of the inverse scattering problem will be the analytical properties of the Jost functions and their adjoints. How to determine these properties are now well documented and even detailed in textbooks [12]. The degenerate 3×3 case of (1) straightforwardly follows from the same techniques used in the 2×2 case [8, 13] and was first detailed by Manakov [1]. The Jost functions defined by (3) and (8) have the following analytical properties:

$$\Phi = \left[\Phi_1^+, \Phi_2^+, \Phi_3^-\right], \quad \Psi = \left[\Psi_1^-, \Psi_2^-, \Psi_3^+\right], \quad \Phi^A = \begin{bmatrix} \Phi_1^{A-} \\ \Phi_2^{A-} \\ \Phi_3^{A+} \end{bmatrix}, \quad \Psi^A = \begin{bmatrix} \Psi_1^{A+} \\ \Psi_2^{A+} \\ \Psi_3^{A-} \end{bmatrix}, \tag{12}$$

where the $\pm$ superscripts on the Jost functions and their adjoints refer to which half-plane the function is analytic in (up to a phase factor of $e^{\pm i\zeta x}$). The subscripts on the Jost functions refer to the column number of the appropriate solution vector while on the adjoints, it refers to the row number.

The analytical properties of S and R follow from that of the Jost functions and their adjoints;

$$S = \Psi^A \cdot \Phi = \begin{bmatrix} S_{11}^+ & S_{12}^+ & S_{13} \\ S_{21}^+ & S_{22}^+ & S_{23} \\ S_{31} & S_{32} & S_{33}^- \end{bmatrix}, \qquad R = S^{-1} = \Phi^A \cdot \Psi = \begin{bmatrix} R_{11}^- & R_{12}^- & R_{13} \\ R_{21}^- & R_{22}^- & R_{23} \\ R_{31} & R_{32} & R_{33}^+ \end{bmatrix}, \tag{13}$$

where those components without a superscript in general only exist on the real ζ-axis.

By collecting together those Jost functions which are analytic in the same half-plane, we can construct other matrix solutions of (1). We define χ^+, χ^-, χ^{A+} and χ^{A-} by

$$\chi^+ = \left[\Phi_1^+, \Phi_2^+, \Psi_3^+\right], \quad \chi^- = \left[\Psi_1^-, \Psi_2^-, \Phi_3^-\right], \quad \chi^{A+} = \begin{bmatrix} \Psi_1^{A+} \\ \Psi_2^{A+} \\ \Phi_3^{A+} \end{bmatrix}, \quad \chi^{A-} = \begin{bmatrix} \Phi_1^{A-} \\ \Phi_2^{A-} \\ \Psi_3^{A-} \end{bmatrix}. \tag{14}$$

Note that $\chi^{A\pm}$ is *not* the inverse of $\chi^{\pm}$, although $\Phi^A = \Phi^{-1}$ and $\Psi^A = \Psi^{-1}$ are defined to be so.

These χ's are related to the fundamental analytical solutions (FAS), $\Theta^{\pm}$ and , $\Theta^{A\pm}$ by a phase factor

$$\Theta^{\pm} = \chi^{\pm} \cdot e^{i\zeta Jx}, \qquad \Theta^{A\pm} = e^{-i\zeta Jx} \cdot \chi^{A\pm}. \tag{15}$$

Note that the χ's are essentially the FAS, except for the phase factor. (The advantage of using the χ's is that one can avoid having to carry along those phase factors in any algebraic manipulation.) Adding the appropriate phase factor to the χ's then gives the FAS corresponding to any Jost function. Θ^+ provides us with a set of three linearly independent and analytic functions in the upper-half-plane (UHP) and on the real ζ-axis while Θ^- provides us with another similar set on the real ζ-axis and in the lower-half-plane (LHP). Similarly for their adjoints, $\Theta^{A\pm}$.

On the real ζ axis, each $\chi^{\pm}$ and $\chi^{A\pm}$ can be related to the Φ and Φ^A Jost solutions by semi-triangular matrices:

$$\chi^{\pm} = \Phi \cdot A_{\pm}, \qquad A_+ = \begin{bmatrix} 1 & 0 & R_{13} \\ 0 & 1 & R_{23} \\ 0 & 0 & R_{33}^+ \end{bmatrix}, \qquad A_- = \begin{bmatrix} R_{11}^- & R_{12}^- & 0 \\ R_{21}^- & R_{22}^- & 0 \\ R_{31} & R_{32} & 1 \end{bmatrix}, \tag{16}$$

$$\chi^{A\pm} = A_{\pm}^A \cdot \Phi^A, \qquad A_+^A = \begin{bmatrix} S_{11}^+ & S_{12}^+ & S_{13} \\ S_{21}^+ & S_{22}^+ & S_{23} \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_-^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ S_{31} & S_{32} & S_{33}^- \end{bmatrix}. \tag{17}$$

Important matrix relations or products can be obtained from the above, which we shall need later. They are

$$\chi^+ = \chi^- \cdot T, \qquad \chi^- = \chi^+ \cdot T^{-1}, \tag{18}$$

where

$$T = \frac{1}{S_{33}^-}\begin{bmatrix} R_{22}^- & -R_{12}^- & -S_{13} \\ -R_{21}^- & R_{11}^- & -S_{23} \\ S_{31} & S_{32} & 1 \end{bmatrix}, \qquad T^{-1} = \frac{1}{R_{33}^+}\begin{bmatrix} S_{22}^+ & -S_{12}^+ & -R_{13} \\ -S_{21}^+ & S_{11}^+ & -R_{23} \\ R_{31} & R_{32} & 1 \end{bmatrix}. \tag{19}$$

One observes that T and T^{-1} decomposes into two parts, one part of which is analytic in a half-plane:

$$T = \frac{1}{S_{33}^-}P^- + \overline{\Gamma}, \qquad P^- = \begin{bmatrix} R_{22}^- & -R_{12}^- & 0 \\ -R_{21}^- & R_{11}^- & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \overline{\Gamma} = \begin{bmatrix} 0 & 0 & -\sigma_{13} \\ 0 & 0 & -\sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{bmatrix}, \tag{20}$$

$$T^{-1} = \frac{1}{R_{33}^+}P^+ + \Gamma, \qquad P^+ = \begin{bmatrix} S_{22}^+ & -S_{12}^+ & 0 \\ -S_{21}^+ & S_{11}^+ & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0 & 0 & -\rho_{13} \\ 0 & 0 & -\rho_{23} \\ \rho_{31} & \rho_{32} & 0 \end{bmatrix}, \tag{21}$$

where for $j = 1, 2$,

$$\sigma_{3j} = \frac{S_{3j}}{S_{33}^-}, \qquad \sigma_{j3} = \frac{S_{j3}}{S_{33}^-}, \qquad \rho_{3j} = \frac{R_{3j}}{R_{33}^+}, \qquad \rho_{j3} = \frac{R_{j3}}{R_{33}^+}. \tag{22}$$

A similar relation also exists between the χ^A's.

$$\begin{aligned} \chi^{A+} &= A_+^A \cdot \left(A_-^A\right)^{-1} \cdot \chi^{A-} = \left(\frac{1}{S_{33}^-}P^- - \overline{\Gamma}\right) \cdot \chi^{A-}, \\ \chi^{A-} &= A_-^A \cdot \left(A_+^A\right)^{-1} \cdot \chi^{A+} = \left(\frac{1}{R_{33}^+}P^+ - \Gamma\right) \cdot \chi^{A+}. \end{aligned} \tag{23}$$

Important products concerning the χ's are

$$\chi^{A+} \cdot \chi^+ = A^{A+} \cdot A^+ = \frac{1}{R_{33}^+}(P^+)^{-1}, \qquad \chi^{A-} \cdot \chi^- = A^{A-} \cdot A^- = \frac{1}{S_{33}^-}(P^-)^{-1}, \tag{24}$$

$$\chi^{A+} \cdot \chi^- = \begin{bmatrix} 1 & 0 & S_{13} \\ 0 & 1 & S_{23} \\ R_{31} & R_{32} & 1 \end{bmatrix}, \qquad \chi^{A-} \cdot \chi^+ = \begin{bmatrix} 1 & 0 & R_{13} \\ 0 & 1 & R_{23} \\ S_{31} & S_{32} & 1 \end{bmatrix}. \tag{25}$$

Equations (24) allow us to construct the inverses of the χ's in terms of other fundamental matrices while those in (25) will be useful in a later section.

The remaining part of the direct scattering problem is to detail the asymptotics of the Jost functions as one approaches any essential singularity on the boundary of the region of analyticity. There is only one essential singularity at $|\zeta| = \infty$ in this problem. Taking ζ to be real, then Φ and Ψ have a common asymptotic expansion which is

$$\Phi, \Psi = \left(I_3 + \frac{\mathrm{i}}{2\zeta}\mathcal{B}^{(1)} + \frac{1}{4\zeta^2}\mathcal{B}^{(2)} + \cdots\right) \cdot \mathrm{e}^{-\mathrm{i}\zeta Jx} \quad \text{as} \quad |\zeta| \to \infty, \tag{26}$$

one finds that $\mathcal{B}^{(1)}$ can be given by

$$[\mathcal{B}^{(1)}, J] = 2Q \qquad \text{and} \qquad \partial_x\mathcal{B}^{(1)} - Q \cdot \mathcal{B}^{(1)} = \tfrac{1}{2}[\mathcal{B}^{(2)}, J]. \tag{27}$$

The first equation will determine those parts of $\mathcal{B}^{(1)}$ which do not commute with J (and are linear in the components of Q) while the second equation will determine those parts of $\mathcal{B}^{(1)}$ that do commute with J. These latter parts will be the spatial integrals of quadratic products of the components of Q. Solving the first equation gives

$$\mathcal{B}^{(1)} = \begin{bmatrix} X & X & -Q_{13} \\ X & X & -Q_{23} \\ Q_{31} & Q_{32} & X \end{bmatrix}, \tag{28}$$

where the X's represent the part of the solution which would be obtained from the second equation in (27), which we shall not need and which, as noted above, are integrals over the quadratic products of the components of the potential matrix, Q.

In the above, we have discussed the direct scattering problem for the 3×3 degenerate eigenvalue problem, (1). This eigenvalue problem is perhaps the simplest extension beyond the AKNS eigenvalue problem. As in the AKNS case, every Jost function is analytic in a sector, there are only two sectors, and the scattering matrix decomposes into a block-like structure. The construction of the χ's and FAS are straightforward as well as the T matrix. The main additional difference is the existence of the 'polarization matrices', $P^{\pm}$. Consequently in this and higher order systems, one could expect these matrices to occur anytime two or more of the eigenvalues of J were equal. As one would expect, these polarization matrices will appear over and over again when we look at the perturbations.

Further details concerning localization conditions which Q should satisfy in order for the assumed solutions of S and R to exist can be found elsewhere (see e.g. [12, 14]). Next we shall take up the inverse scattering problem, wherein we detail what the scattering data is and how it is related to the scattering matrix.

3. The inverse scattering problem

Let us now consider the inverse scattering problem. Equation (18) can be viewed as a Riemann–Hilbert problem upon using (20) and (21). Consider Cauchy's integral theorem applied to Θ^+ in the UHP where Θ^+ is analytic. Its asymptotics are given by (26) upon deleting the phase factor at the end. It is obvious that for ζ in the UHP,

$$\Theta^+(\zeta) = \frac{1}{2} I_3 + \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{R}} \frac{\mathrm{d}\zeta'}{\zeta' - \zeta} \Theta^+(\zeta'), \tag{29}$$

where $\mathcal{R}$ indicates that the path of the integral is along the real axis. Now, along the real axis, from (20), we have

$$\Theta^+ = \frac{1}{S_{33}^-} \Theta^- \cdot P^- + \Theta^- \cdot \mathrm{e}^{-\mathrm{i}\zeta J x} \cdot \overline{\Gamma} \cdot \mathrm{e}^{\mathrm{i}\zeta J x}, \tag{30}$$

where we have used the fact that

$$[J, P^{\pm}] = 0. \tag{31}$$

Since S_{33}^- is analytic in the LHP (see (13)), the first term in (30) can be extended into the LHP. That term will have a pole wherever $S_{33}^-(\zeta)$ has a zero in the LHP. Whence

$$\int_{\mathcal{R}} \frac{\mathrm{d}\zeta'}{(\zeta' - \zeta) S_{33}^-(\zeta')} \Theta^-(\zeta') \cdot P^-(\zeta') = -2\pi\,\mathrm{i} \sum_{k=1}^{N^-} \frac{1}{(\zeta_k^- - \zeta)\left(S_{33}^-\right)_k'} \Theta^-(\zeta_k^-) \cdot P_k^- + \pi \mathrm{i} I_3, \tag{32}$$

where N^- is the number of zeros of $S_{33}^-(\zeta)$ in the LHP (assumed finite), ζ_k^- is the kth zero of $S_{33}^-(\zeta)$, $\left(S_{33}^-\right)_k'$ is $\mathrm{d}S_{33}^-/\mathrm{d}\zeta$ evaluated at $\zeta = \zeta_k^-$, P_k^- is P^- evaluated at the zero, and the last term comes from the integral along an infinite semi-circle in the LHP. Putting (29), (30) and (32) together gives

$$\Theta^+(\zeta) = I_3 + \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{R}} \frac{\mathrm{d}\zeta'}{\zeta' - \zeta} \Theta^-(\zeta') \cdot \mathrm{e}^{-\mathrm{i}\zeta' J x} \cdot \overline{\Gamma}(\zeta')\, \mathrm{e}^{\mathrm{i}\zeta' J x} - \sum_{k=1}^{N^-} \frac{1}{(\zeta_k^- - \zeta)\left(S_{33}^-\right)_k'} \Theta^-(\zeta_k^-) \cdot P_k^-. \tag{33}$$

Thus we have Θ^+ given in terms of Θ^- on the real axis and at each zero of S_{33}^- in the LHP. Similarly, starting from the second part of (18), we obtain, for ζ in the LHP,

$$\Theta^-(\zeta) = I_3 - \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{R}} \frac{\mathrm{d}\zeta'}{\zeta'-\zeta}\Theta^+(\zeta')\cdot \mathrm{e}^{-\mathrm{i}\zeta' Jx}\cdot \Gamma(\zeta')\cdot \mathrm{e}^{\mathrm{i}\zeta' Jx} + \sum_{k=1}^{N^+} \frac{1}{(\zeta_k^+ - \zeta)\left(R_{33}^+\right)_k'}\Theta^+\left(\zeta_k^+\right)\cdot P_k^+, \tag{34}$$

where N^+ is the number of zeros of $R_{33}^+(\zeta)$ in the UHP, ζ_k^+ is such a zero and $\left(R_{33}^+\right)_k' = \mathrm{d}R_{33}^+/\mathrm{d}\zeta$ evaluated at this zero. As one can see from the above and as remarked earlier, the separation of T and T^{-1} into the Γ and P matrices is a natural separation of T and its inverse into their contributions to the continuous spectra (Γ's) and the discrete spectra.

In the absence of bound states, the six columns in (33) and (34) naturally decouple into two sets of equations: three equations for inversion about $+\infty$ and another set of three equations for inversion about $-\infty$. Either set could be used to reconstruct the Jost functions and thereby the potential matrix. Thus among these six coupled equations, there can only be three equations which are actually independent. In hindsight, this interdependence of the six columns is not surprising, since the two equations in (18), which were used to obtain (33) and (34), were not independent in the first place.

When there are bound states present, in general all six columns in (33) and (34) will be coupled due to the diagonal nature of the P's. The solution of the inverse scattering problem now reduces to one of recovering from these six columns, when bound states are present, three independent equations from these six columns, which may be used to reconstruct the Θ's.

There is a general way to go about obtaining this separation which was illustrated in [7]. Basically what one does is to replace the P_k's by off-diagonal components from the Γ's. To see how this can be done, let us assume compact support, which will give us the same result. When there is compact support, χ^+ and χ^- are both analytic in the entire complex plane. Then by (19) and (20), we see that the residues of $\chi^-\cdot T$ at the zeros of S_{33}^- in the LHP must exactly vanish. The same can be said for $\chi^+\cdot T$ at the zeros of R_{33}^+ in the UHP. Consequently we have

$$\chi_k^-\cdot P_k^- = -\chi_k^-\cdot\left(S_{33}^-\overline{\Gamma}\right)_k, \qquad \chi_k^+\cdot P_k^+ = -\chi_k^+\cdot\left(R_{33}^+\Gamma\right)_k. \tag{35}$$

These identities represent the fact that at one of these zeros, the three solutions contained in the appropriate χ are no longer linearly independent. Let us now define the two matrices

$$\overline{C}_k = \frac{1}{\left(S_{33}^-\right)_k'}\left(S_{33}^-\overline{\Gamma}\right)_k = \frac{1}{\left(S_{33}^-\right)_k'}\begin{bmatrix} 0 & 0 & -S_{13,k} \\ 0 & 0 & -S_{23,k} \\ S_{31,k} & S_{32,k} & 0 \end{bmatrix}, \tag{36}$$

$$C_k = \frac{1}{\left(R_{33}^+\right)_k'}\left(R_{33}^+\Gamma\right)_k = \frac{1}{\left(R_{33}^+\right)_k'}\begin{bmatrix} 0 & 0 & -R_{13,k} \\ 0 & 0 & -R_{23,k} \\ R_{31,k} & R_{32,k} & 0 \end{bmatrix}, \tag{37}$$

where the subscripts k on S_{ij} and R_{ij} indicate that these quantities are to be evaluated at the appropriate zeros in the appropriate half-plane. Equations (33) and (34) then become

$$\Theta^+(\zeta) = I_3 + \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{R}} \frac{\mathrm{d}\zeta'}{\zeta'-\zeta}\Theta^-(\zeta')\cdot \mathrm{e}^{-\mathrm{i}\zeta' Jx}\cdot \overline{\Gamma}(\zeta')\,\mathrm{e}^{\mathrm{i}\zeta' Jx} + \sum_{k=1}^{N^-} \frac{1}{(\zeta_k^- - \zeta)}\Theta^-\left(\zeta_k^-\right)\cdot \mathrm{e}^{-\mathrm{i}\zeta_k^- Jx}\cdot \overline{C}_k\cdot \mathrm{e}^{\mathrm{i}\zeta_k^- Jx}, \tag{38}$$

$$\Theta^{-}(\zeta) = I_3 - \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{R}} \frac{\mathrm{d}\zeta'}{\zeta' - \zeta} \Theta^{+}(\zeta') \cdot \mathrm{e}^{-\mathrm{i}\zeta' J x} \cdot \Gamma(\zeta') \cdot \mathrm{e}^{\mathrm{i}\zeta' J x} - \sum_{k=1}^{N^{+}} \frac{1}{\left(\zeta_k^{+} - \zeta\right)} \Theta^{+}\left(\zeta_k^{+}\right) \cdot \mathrm{e}^{-\mathrm{i}\zeta_k^{+} J x} \cdot C_k \cdot \mathrm{e}^{\mathrm{i}\zeta_k^{+} J x}, \tag{39}$$

where ζ is in the UHP for (38) and in the LHP for (39). Due to the non-diagonal nature of C_k's, one now will find that the columns of these equations will directly separate into two separate sets of equations: one set for inversion about $+\infty$, involving only the Ψ's, and one other set for inversion about $-\infty$, involving only the Φ's. As given, these equations are simply identities between the Θ's, and are referred to as the 'linear dispersion relations' (LDR), in analogy with similar equations found in optics. Evaluation of (38) for ζ on the real axis and at the zeros of R_{33}^{+} in the UHP, and evaluation of (39) for ζ on the real axis and at the zeros of S_{33}^{-} in the LHP, will give a set of coupled algebro-singular integral equations for the FAS.

From the first two columns of (38) and the last column of (39), one obtains three column vector equations involving only the Φs, and therefore are the equations for inversion about $-\infty$. The quantities in these equations are (for $j = 1, 2$)

- the reflection coefficients σ_{3j} and ρ_{j3} on the real axis,
- the zeros of $S_{33}^{-}(\zeta)$ in the LHP $\left(\zeta_k^{-}; k = 1, 2, \ldots, N^{-}\right)$ and the values of $(\overline{C}_k)_{3j}$ at each such zero,
- the zeros of $R_{33}^{+}(\zeta)$ in the UHP $\left(\zeta_k^{+}; k = 1, 2, \ldots, N^{+}\right)$ and the values of $(C_k)_{j3}$ at each such zero.

This is the miminal set of information, scattering data, required for inversion about $-\infty$.

From the remaining columns, we obtain the LDRs for the Ψ's, which provide inversion about $+\infty$. In those equations, we find that we must specify the following quantities:

- the reflection coefficients σ_{j3} and ρ_{3j} on the real axis,
- the zeros of $S_{33}^{-}(\zeta)$ in the LHP $\left(\zeta_k^{-}; k = 1, 2, \ldots, N^{-}\right)$ and the values of $(\overline{C}_k)_{j3}$ at each such zero,
- the zeros of $R_{33}^{+}(\zeta)$ in the UHP $\left(\zeta_k^{+}; k = 1, 2, \ldots, N^{+}\right)$ and the values of $(C_k)_{3j}$ at each zero.

This is the miminal set of information, scattering data, required for inversion about $+\infty$.

From either one of these two sets of equations, one can then reconstruct the potentials. This will be accomplished by solving the appropriate columns in (38) and (39) for the appropriate columns of $\Theta^{\pm}$. Once these solutions are found, then using the asymptotic relations (26) and (28), one can obtain the potentials, which then completes the solution of the inverse scattering problem. For further details and proofs of the existence of solutions, see [1, 15].

Now for some closing comments on the inverse scattering problem. First, the key to the inverse scattering equations, (38) and (39), is the FAS and the relationship shown in (18). Also important is the manner in which the matrix T and its inverse decompose into different analytic parts in (20) and (21). Such a decomposition is essential if one is to close the contour in the opposite half-plane. Second, if one decomposes (38) and (39) into its various columns, one finds that these equations contain *both* inversion procedures and therefore one-half of these equations are redundant. The form of these equations in (36) and (37) have all six columns coupled when bound states are present and thus cannot be used to reconstruct the Θ's, since the independent specification of the scattering data for both sets would lead to inconsistencies and nonexistence of solutions. However once the form of these equations in (38) and (39) is achieved, one is then able to select out a minimal set of equations which require a minimal set of independent information (the scattering data), from which one could obtain solutions for the Θ's.

In hindsight, the method for solving this 3×3 inverse scattering problem has really not varied significantly from that used in the AKNS case [8, 13]. The only complexity which has occurred has been a doubling of the potential components, reflection coefficients and normalization coefficients. Since J has degenerate diagonal elements, we have had the appearance of the 'polarization' matrices, $P^{\pm}$. Of course there are technical details such as signs, numerical coefficients and an increased complexity of the expressions. But once relationship (18) has been established, then the remainder of the problem becomes essentially technical details. Let us now turn to a consideration of perturbations and the SE.

4. Variations in scattering data

In the previous sections, we have described the solution of the direct scattering problem and then the inverse scattering problem. In the direct scattering problem, given potentials which are suitably restricted, then in general the scattering coefficients exist and are unique. In the inverse scattering problem, given a set of scattering data (which was itemized following equation (39)), then there is a potential which can be recovered from these scattering data. We shall now assume that such is true in both cases, and that for the given potential and its associated scattering data, there will be finite neighborhoods surrounding such sets where the same will be found to occur. Whence for any linear perturbation of the potential, there will exist a unique linear variation in the scattering data and vice versa. So the first task here will be to determine these relationships between the variations and the perturbations, such that the variation may be given in terms of the perturbation. It is from these two relationships that one can obtain what is known as the SE and their adjoints, ASE, along with their inner products and their closure relation.

The approach presented here will differ from that in the original approach [9, 10] which we will briefly outline now. In that approach, one first found the variations in the scattering data in terms of perturbations of the potentials. The coefficients of these were squares of the Jost functions and were the *adjoints* of the SE. It was then shown that these adjoints were eigenfunctions of an integro-differential operator. Then one found the adjoint of that operator, which would be the eigenvalue operator for the SE. Now by guess and trial, one found that its eigenfunctions were also products of Jost functions and their adjoints. These products were what are called the SE. Then inner products between the SE and ASE were defined and explicitly evaluated from the various asymptotics of the Jost solutions. Once these inner products were known, then one could construct what should be the closure relation, by simply expanding an arbitrary function in these SE. But to prove closure, a rather long process was used, which required the use of the Marchenko equations, which are obtained from the LDR. A much neater proof of closure was given later by Gerdjikov and Khristov [16].

Here we shall take a different approach for finding the inner products and closure relation [4, 17]. As a prelude to this, let us make the following remarks. Determining the scattering data is accomplished by solving the eigenvalue problem. Thus applying perturbations to the potentials in the eigenvalue problem will allow one to compute the resulting variations in the scattering data. Similarly, one solves the inverse scattering problem by starting from (18). From that, one obtains the LDR, (38) and (39), by which one takes the scattering data and reconstructs the potential. Whence in going from perturbations of the scattering data to the resulting variations of the potential, we should likewise expect to start from (18), perturb it by applying arbitrary perturbations to the scattering data and then obtain the resulting variations in the potentials. This is the basis for the approach which we shall now use.

In this section, we will calculate the variations in the scattering data due to the perturbations of the potential. For an eigenvalue problem like (2), this is well known. One perturbs (2), and then uses the method of variation of parameters to solve the resulting differential equation. One obtains

$$\partial_x (V^A \cdot \delta V) = V^A \cdot \delta Q \cdot V \tag{40}$$

where V is any solution and δV is the perturbed V resulting from perturbations in δQ. V^A is any adjoint solution. Integrating this and letting $V = \Phi$ and $V^A = \Psi^A$, then since $\delta V(x \to -\infty) = 0$ and $\delta V(x \to +\infty) = \mathrm{e}^{\mathrm{i}\zeta J x} \cdot \delta S$, we have

$$\delta S = \int_{-\infty}^{\infty} \Psi^A \cdot \delta Q \cdot \Phi \, \mathrm{d}x. \tag{41}$$

Similarly if we take $V = \Psi$ and $V^A = \Phi^A$, we obtain

$$\delta R = -\int_{-\infty}^{\infty} \Phi^A \cdot \delta Q \cdot \Psi \, \mathrm{d}x \tag{42}$$

from which one may proceed to calculate all perturbations in the scattering data. Observe that the integrand contains a product of an adjoint Jost solution and a regular Jost solution. This product is the ASE.

Continuing, for the reflection coefficients, we obtain (for $j = 1, 2$)

$$\delta\sigma_{j3} = \frac{1}{\left(S_{33}^-\right)^2} \int_{-\infty}^{\infty} \mathrm{d}x [P^-(\zeta) \cdot \chi^{A-}(x,\zeta) \cdot \delta Q(x) \cdot \chi^-(x,\zeta)]_{j3}, \tag{43}$$

$$\delta\sigma_{3j} = \frac{1}{\left(S_{33}^-\right)^2} \int_{-\infty}^{\infty} \mathrm{d}x [\chi^{A-}(x,\zeta) \cdot \delta Q(x) \cdot \chi^-(x,\zeta) \cdot P^-(\zeta)]_{3j}, \tag{44}$$

$$\delta\rho_{j3} = \frac{-1}{\left(R_{33}^+\right)^2} \int_{-\infty}^{\infty} \mathrm{d}x [P^+(\zeta) \cdot \chi^{A+}(x,\zeta) \cdot \delta Q(x) \cdot \chi^+(x,\zeta)]_{j3}, \tag{45}$$

$$\delta\rho_{3j} = \frac{-1}{\left(R_{33}^+\right)^2} \int_{-\infty}^{\infty} \mathrm{d}x [\chi^{A+}(x,\zeta) \cdot \delta Q(x) \cdot \chi^+(x,\zeta) \cdot P^+(\zeta)]_{3j}. \tag{46}$$

As mentioned earlier, (43) and (46) apply to inversion about $+\infty$ while (44) and (45) apply to inversion about $-\infty$.

For the bound state eigenvalues and normalization coefficients, one works with Taylor expansions about a zero of R_{33}^+ or S_{33}^-. Consider a general function of ζ, $g(\zeta)$. Expanding it in a Taylor series about a zero, ζ_k, we have

$$g(\zeta) = g_k + g_k'(\zeta - \zeta_k) + \tfrac{1}{2} g_k''(\zeta - \zeta_k)^2 + \cdots, \tag{47}$$

where the primes indicate differentiation with respect to ζ and the subscript k's indicate evaluation at $\zeta = \zeta_k$. Varying the quantities in the above expression give us

$$\delta g(\zeta) = \delta(g_k) - g_k' \delta\zeta_k + [\delta(g_k') - g_k'' \delta\zeta_k](\zeta - \zeta_k) + \cdots, \tag{48}$$

from which we have

$$\delta(g_k) = [\delta g(\zeta)]_k + g_k' \delta\zeta_k, \qquad \delta(g_k') = \{\partial_\zeta [\delta g(\zeta)]\}_k + g_k'' \delta\zeta_k, \ldots. \tag{49}$$

In other words, since there is also a shift in the eigenvalues, one has to also shift where the quantity is evaluated at. To get the shift in the eigenvalues, one applies the above to $S_{33}^-(\zeta)$ and $R_{33}^+(\zeta)$, which gives

$$\delta\zeta_k^+ = -\frac{\left(\delta R_{33}^+\right)_k}{\left(R_{33}^{+\prime}\right)_k} = \frac{1}{\left(R_{33}^{+\prime}\right)_k}\int_{-\infty}^{\infty} \mathrm{d}x[\chi^{A+}(x,\zeta)\cdot\delta Q(x)\cdot\chi^+(x,\zeta)]_{33,k}, \tag{50}$$

$$\delta\zeta_k^- = -\frac{\left(\delta S_{33}^-\right)_k}{\left(S_{33}^{-\prime}\right)_k} = \frac{-1}{\left(S_{33}^{-\prime}\right)_k}\int_{-\infty}^{\infty} \mathrm{d}x[\chi^{A-}(x,\zeta)\cdot\delta Q(x)\cdot\chi^-(x,\zeta)]_{33,k}, \tag{51}$$

where the subscripts on the far right indicate the component numbers of the resulting matrix inside the brackets.

The variations in the normalization coefficients are a bit more difficult to obtain since we have to move off the real ζ-axis. A general procedure for determining the variations in these coefficients has been outlined in [7]. The basic idea is that if the normalization coefficient is associated with a zero in the UHP, then one expresses the variations in terms of Jost solutions which are analytically extendible into that half-plane. Let us consider normalization coefficients in the scattering data for inversion about $+\infty$. To start with, assuming compact support initially, we need the variations of S_{j3} in the LHP and the variations of R_{3j} in the UHP (for $j = 1, 2$). Now by (41), we have

$$\delta S_{j3} = \int_{-\infty}^{\infty} \Psi_j^A\cdot\delta Q\cdot\Phi_3\,\mathrm{d}x. \tag{52}$$

By (14), Ψ_j^A is analytic in the UHP while Φ_3 is analytic in the LHP. Thus we want to express Ψ_j^A in terms of Jost functions analytic in the LHP. Using (23), one finds that this combination is

$$\Psi_j^{A+} = \frac{S_{j3}}{S_{33}}\Psi_3^{A-} + \sum_{\ell=1}^{2} P_{j\ell}^-\Phi_\ell^{A-}. \tag{53}$$

Using this in (52), from the definition of P^- in (20) and then replacing the Jost functions by the appropriate χ's, one obtains

$$\delta S_{j3} = \frac{1}{S_{33}^-}\int_{-\infty}^{\infty}[S_{j3}(\chi^{A-}\cdot\delta Q\cdot\chi^-)_{33} + (P^-\cdot\chi^{A-}\cdot\delta Q\cdot\chi^-)_{j3}]\,\mathrm{d}x. \tag{54}$$

Due to (23), the integrand vanishes identically at any zero of S_{33}^-, whence evaluation of this at a zero of S_{33}^- requires the use of l'Hopital's rule. From (48) for $g = S_{33}^-$, upon replacing the Jost functions with the appropriate χ's, we have

$$\delta\left(S_{33,k}^{-\prime}\right) = S_{33,k}^{-\prime\prime}\delta\zeta_k^- + \left(\partial_\zeta\int_{-\infty}^{\infty}\chi^{A-}\cdot\delta Q\cdot\chi^-\,\mathrm{d}x\right)_{33,k}. \tag{55}$$

From (54) and (55), upon using (49) and (51) as needed, we then have

$$\begin{aligned}\delta(\overline{C}_k)_{j3} &= \delta\left(\frac{-S_{j3,k}}{S_{33,k}^{-\prime}}\right)\\ &= \frac{-1}{S_{33,k}^{-\prime}}\left\{\int_{-\infty}^{\infty}\mathrm{d}x\,\partial_\zeta\left[\frac{1}{S_{33}^{-\prime}(\zeta)}P^-(\zeta)\cdot\chi^{A-}(x,\zeta)\cdot\delta Q(x)\cdot\chi^-(x,\zeta)\right]_{j3}\right\}_k.\end{aligned} \tag{56}$$

Similarly for the other two normalization constants, for inversion about $+\infty$, we have

$$\delta R_{3j} = -\int_{-\infty}^{\infty}\Phi_3^{A+}\cdot\delta Q\cdot\Psi_j^-. \tag{57}$$

We now need Ψ_j^- in terms of functions analytic in the UHP. This follows from (3):

$$\Psi_j^- = \Psi_3^+\frac{R_{3j}}{R_{33}^+} + \sum_{\ell=1}^{2}\Phi_\ell^+ P_{\ell j}^+. \tag{58}$$

With this, (57) becomes

$$\delta R_{3j} = \frac{-1}{R_{33}^{+}} \int_{-\infty}^{\infty} [(\chi^{A+} \cdot \delta Q \cdot \chi^{+})_{33} R_{3j} + (\chi^{A+} \cdot \delta Q \cdot \chi^{+} \cdot P^{+})_{3j}]\, \mathrm{d}x. \tag{59}$$

And after more calculations, one finally obtains

$$\begin{aligned}\delta(C_k)_{3j} &= \delta\left(\frac{R_{3j,k}}{R_{33,k}^{+\prime}}\right) \\ &= \frac{-1}{R_{33,k}^{+\prime}} \left\{ \int_{-\infty}^{\infty} \mathrm{d}x\, \partial_\zeta \left[\frac{1}{R_{33}^{+\prime}(\zeta)} \chi^{A+}(x,\zeta) \cdot \delta Q(x) \cdot \chi^{+}(x,\zeta) \cdot P^{+}(\zeta) \right]_{3j} \right\}_k .\end{aligned} \tag{60}$$

With this, we have completed the project to determine the variations in the scattering data (for inversion about $+\infty$). Let us now turn to determining the variations in the potentials when one perturbs the scattering data.

5. Variations of potentials

To obtain the inverse of the above relationships, we will follow the approach used in [4, 17], which is based on equation (18). Actually, to obtain these variations, it becomes more productive if one modifies this equation so as to more directly obtain the needed answer. One starts this modification with either of the matrix products $\chi^{A+} \cdot \chi^{-}$ or $\chi^{A-} \cdot \chi^{+}$. (Note that the rows of $\chi^{A\pm}$ are a linear combination of the rows in $(\chi^{A\pm})^{-1}$. Whence this form will contain all the conditions already in $\chi^{+} = \chi^{-} \cdot T$.) As noted earlier, the full inverse scattering equations, (38) and (39), actually separate into two independent sets of equations and one only requires one-half of these equations to have a full solution. Thus to find the variations in Q due to any perturbations of the scattering data, one only needs to use one of these two sets of scattering data. As we did in the previous section, we shall choose to use the set for inversion about $+\infty$, which is listed at the end of section 3.

First from (25), observe that the products of $\chi^{A+} \cdot \chi^{-}$ and $\chi^{A-} \cdot \chi^{+}$ consist only of the off-diagonal components of S and R. In $\chi^{A+} \cdot \chi^{-}$, the components of S and R which appear are exactly those needed to construct the reflection coefficients for inversion about $+\infty$. So once we divide these components by S_{33}^{-} or R_{33}^{+}, as appropriate, we will have the appropriate reflection coefficients in terms of the Jost functions. Thus to this end, we construct the matrix $G(\zeta)$,

$$G(\zeta) = M^{+} \cdot \chi^{A+} \cdot \chi^{-} \cdot M^{-} = \begin{bmatrix} 1 & 0 & \sigma_{13} \\ 0 & 1 & \sigma_{23} \\ \rho_{31} & \rho_{32} & \dfrac{1}{R_{33}^{+} S_{33}^{-}} \end{bmatrix}, \tag{61}$$

where we have taken

$$M^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/R_{33}^{+} \end{bmatrix}, \qquad M^{-} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/S_{33}^{-} \end{bmatrix}. \tag{62}$$

It is interesting to note that, in contrast to (19), the determinant of G in (61) is unity.

Let us now define

$$F^{+} = M^{+} \cdot \Theta^{A+}, \qquad F^{-} = \Theta^{-} \cdot M^{-}, \tag{63}$$

each of which is analytic in the appropriate half-plane, except for poles at the zeros of R_{33}^{+} or S_{33}^{-}, according to their superscripts. With this, from (61), we obtain

$$F^{+} \cdot F^{-} = \mathrm{e}^{-\mathrm{i}\zeta J x} \cdot G \cdot \mathrm{e}^{\mathrm{i}\zeta J x}. \tag{64}$$

If we now vary the quantities in (64), we can obtain, for ζ real,

$$(F^+)^{-1}\cdot\delta F^+ + (\delta F^-)\cdot(F^-)^{-1} = \Pi \tag{65}$$

where

$$\Pi = (F^+)^{-1}\cdot \mathrm{e}^{-\mathrm{i}\zeta Jx}\cdot\delta G\cdot \mathrm{e}^{\mathrm{i}\zeta Jx}\cdot(F^-)^{-1}. \tag{66}$$

Now (65) can be viewed as a Riemann–Hilbert problem. Whence with the use of Cauchy's theorem, we can extend $(F^+)^{-1}\cdot\delta F^+$ into the UHP. Note that the F's are basically Jost functions or their adjoints. G basically consists of reflection coefficients. Once we have extended $(F^+)^{-1}\cdot\delta F^+$ into the UHP, we may address its asymptotics. From the asymptotics of δF for large $|\zeta|$, we can then obtain δQ, which will then be related to the perturbations of the reflection coefficients contained in δG.

Consider the equation $f^+(\zeta)+f^-(\zeta) = \Pi(\zeta)$ where $f^+ = (F^+)^{-1}\cdot\delta F^+$ and $f^- = (\delta F^-)\cdot(F^-)^{-1}$ are analytic in the appropriate half-plane, except for a possible finite number of simple poles, with each vanishing like $O(1/\zeta)$ as $|\zeta|\to\infty$. Also assume that the integral of $|\Pi(\zeta)|$, along the real ζ-axis, is finite. Then we have that for ζ in the UHP,

$$f^+(\zeta) = (F^+)^{-1}\cdot\delta F^+ = \frac{1}{2\pi\,\mathrm{i}}\int_{\mathcal{R}}\frac{\Pi(\zeta')\,\mathrm{d}\zeta'}{\zeta'-\zeta} - \sum_{k=1}^{N^+}\frac{f_k^+}{\zeta_k^+-\zeta} + \sum_{k=1}^{N^-}\frac{f_k^-}{\zeta_k^--\zeta}, \tag{67}$$

where f_k^+ are the residues of $f^+(\zeta)$ at the kth zero of R_{33}^+ in the UHP while f_k^- are the residues of $f^-(\zeta)$ at the kth zero of S_{33}^- in the LHP.

Now consider this expression for $|\zeta|\to\infty$. From (15), (26), (27) and (63), we have

$$F^+ = I_3 + O(1/\zeta), \qquad \delta F^+ = \frac{\mathrm{i}}{2\zeta}\delta\mathcal{B}^{(1)} + \cdots \quad \text{as } |\zeta|\to\infty. \tag{68}$$

where $\mathcal{B}^{(1)}$ is given by (28), from which we have

$$\delta\mathcal{B}^{(1)} = \begin{bmatrix} X & X & -\delta Q_{13} \\ X & X & -\delta Q_{23} \\ \delta Q_{31} & \delta Q_{32} & X \end{bmatrix}. \tag{69}$$

In the limit of $|\zeta|\to\infty$ in the UHP, (67) becomes

$$\delta\mathcal{B}^{(1)} = \int_{\mathcal{R}}\frac{\mathrm{d}\zeta}{\pi}\Pi(\zeta) - 2\mathrm{i}\sum_{k=1}^{N^+} f_k^+ + 2\mathrm{i}\sum_{k=1}^{N^-} f_k^-, \tag{70}$$

which relates the variations in Q to the variations in G, f_k^+ and f_k^-, the latter three of which we can relate to perturbations in the scattering data.

First we calculate Π. From (24) and (63), we have

$$(F^+)^{-1} = \frac{1}{R_{33}^+}\Theta^+\cdot P^+\cdot(M^+)^{-1}, \qquad (F^-)^{-1} = \frac{1}{S_{33}^-}(M^-)^{-1}\cdot P^-\cdot\Theta^{A-}. \tag{71}$$

There are two formulas which are key to the following result. These are, for ζ real:

$$\chi^+\cdot P^+ = \left[R_{33}\chi_1^- - R_{31}\chi_3^+,\, R_{33}\chi_2^- - R_{32}\chi_3^+,\, \chi_3^+\right], \quad P^-\cdot\chi^{A-} = \begin{bmatrix} S_{33}\chi_1^{A+} - S_{13}\chi_3^{A-} \\ S_{33}\chi_2^{A+} - S_{23}\chi_3^{A-} \\ \chi_3^{A-} \end{bmatrix}, \tag{72}$$

by which, (61) and (66) give

$$\Pi = \sum_{\ell=1}^{2}\chi_\ell^-\chi_3^{A-}\delta\sigma_{\ell 3} + \sum_{\ell=1}^{2}\chi_3^+\chi_\ell^{A+}\delta\rho_{3\ell}. \tag{73}$$

Now we need to address the bound states and evaluate the residues, $f_k^{\pm}$. From (62), (63) and (71), we obtain

$$f_k^+ = \frac{1}{R_{33,k}^{+\prime}} \chi_k^+ \cdot P_k^+ \cdot (\delta\chi^{A+})_k + \frac{R_{33,k}^{+\prime\prime}}{\left(R_{33,k}^{+\prime}\right)^3} \left(\delta R_{33}^+\right)_k \left(\chi_3^+ \chi_3^{A+}\right)_k - \frac{1}{\left(R_{33,k}^{+\prime}\right)^2} \left[\partial_\zeta \left(\chi_3^+ \chi_3^{A+} \delta R_{33}^+\right)\right]_k . \tag{74}$$

We apply (49) as needed to evaluate the variations and derivatives at $\zeta = \zeta_k^+$, obtaining

$$f_k^+ = \frac{1}{R_{33,k}^{+\prime}} \chi_k^+ \cdot P_k^+ \cdot \left[\delta\left(\chi_k^{A+}\right) - \chi_k^{A+\prime} \delta\zeta_k^+\right] + \left(\chi_3^+ \chi_3^{A+}\right)_k \delta\left(\frac{1}{R_{33,k}^{+\prime}}\right) + \frac{1}{R_{33,k}^{+\prime}} \left[\partial_\zeta \left(\chi_3^+ \chi_3^{A+}\right)\right]_k \delta\zeta_k^+ . \tag{75}$$

At a zero of R_{33}^+, (18) and (23) become doubly degenerate, each giving only one condition, which are

$$\chi_{3,k}^+ = \chi_{1,k}^+ R_{13,k}^+ + \chi_{2,k}^+ R_{23,k}^+, \qquad \chi_{3,k}^{A+} = R_{31,k}^+ \chi_{1,k}^{A+} + R_{32,k}^+ \chi_{2,k}^{A+}. \tag{76}$$

Using the first equation in (72) to eliminate P_k^+ and also applying both equations in (76) to (75), all variations of the Jost functions magically cancel, and we are left with

$$f_k^+ = \sum_{\ell=1}^{2} (C_k)_{3\ell} \left[\partial_\zeta \left(\chi_3^+ \chi_\ell^{A+}\right)\right]_k \delta\zeta_k^+ + \sum_{\ell=1}^{2} \left(\chi_3^+ \chi_\ell^{A+}\right)_k (\delta C_k)_{3\ell} . \tag{77}$$

Carrying out the same steps for f_k^-, we first have

$$\begin{aligned} f_k^- = {} & \frac{1}{S_{33,k}^{-\prime}} (\delta\chi^-)_k \cdot P_k^- \cdot \chi_k^{A-} + \frac{S_{33,k}^{-\prime\prime}}{\left(S_{33,k}^{-\prime}\right)^3} \left(\delta S_{33}^-\right)_k \left(\chi_3^- \chi_3^{A-}\right)_k \\ & - \frac{1}{\left(S_{33,k}^{-\prime}\right)^2} \left[\partial_\zeta \left(\chi_3^- \chi_3^{A-} \delta S_{33}^-\right)\right]_k , \end{aligned} \tag{78}$$

and then upon expanding the derivatives and variations at $\zeta = \zeta_k^-$,

$$\begin{aligned} f_k^- = {} & \frac{1}{S_{33,k}^{-\prime}} \left[\delta\left(\chi_k^-\right) - \chi_k^{-\prime} \delta\zeta_k^-\right] \cdot P_k^- \cdot \chi_k^{A-} + \left(\chi_3^- \chi_3^{A-}\right) \delta\left(\frac{1}{S_{33,k}^{-\prime}}\right) \\ & + \frac{1}{S_{33,k}^{-\prime}} \left[\partial_\zeta \left(\chi_3^- \chi_3^{A-}\right)\right]_k \delta\zeta_k . \end{aligned} \tag{79}$$

The analogy of (76), at a zero of S_{33}^-, is

$$\chi_{3,k}^- = \chi_{1,k}^- S_{13,k}^- + \chi_{2,k}^- S_{23,k}^-, \qquad \chi_{3,k}^{A-} = S_{31,k}^- \chi_{1,k}^{A-} + S_{32,k}^- \chi_{2,k}^{A-\,\prime}, \tag{80}$$

which then with the second equation in (72) gives us

$$f_k^- = -\sum_{\ell=1}^{2} (\overline{C}_k)_{\ell 3} \left[\partial_\zeta \left(\chi_\ell^- \chi_3^{A-}\right)\right]_k \delta\zeta_k^- - \sum_{\ell=1}^{2} \left(\chi_\ell^- \chi_3^{A-}\right)_k (\delta\overline{C}_k)_{\ell 3}. \tag{81}$$

With this, we have completed the calculations necessary to obtain the variations in the potentials due to perturbations in the scattering data for inversion about $+\infty$. In the next section, we shall combine the results in this and the previous section to obtain the inner products of the SE and ASE as well as the closure relation.

6. The SE, ASE, their inner products and closure

The SE are the coefficients of $\delta\sigma_{\ell 3}$, $\delta\rho_{3\ell}$, $\delta\zeta_k^+$, δC_k and $\delta\overline{C}_k$ in the above expressions. Looking at the various components of the SE in the above expressions, it becomes clear that a simple labeling according to the components of the Jost functions is quite impractical. The origin of this 'irregular' indexing is due to the occurrence of zeros in the potential matrix Q. So before proceeding further, let us devise a more simple labeling system for this set which is associated with inversion about $+\infty$. First, as per (69) and (70), we only have four nontrivial potentials. And as can be seen from (69), we require only four of the nine components in the 3×3 matrix (70), which are the 13, 23, 31, 32 components. These we could just stack into a column vector, in an increasing order as per their components, as

$$\delta Q(x) = \begin{bmatrix} \delta Q_{13}(x) \\ \delta Q_{23}(x) \\ \delta Q_{31}(x) \\ \delta Q_{32}(x) \end{bmatrix}. \tag{82}$$

Once we have this order determined, then we can set up each SE as a column matrix, as in (82). From (69), (70) and (73), we can now identify the individual SE which belong to the continuous spectrum and from (77) and (81), we can identify those which belong to the bound state spectra.

For $\ell = 1, 2$ we can construct

$$Z_\ell^-(x,\zeta) = \begin{bmatrix} \chi_{1\ell}^-\chi_{33}^{A-} \\ \chi_{2\ell}^-\chi_{33}^{A-} \\ \chi_{3\ell}^-\chi_{31}^{A-} \\ \chi_{3\ell}^-\chi_{32}^{A-} \end{bmatrix}, \quad Z_{\ell,k}^-(x) = Z_\ell^-\left(x,\zeta_k^-\right), \quad Z_{d,k}^-(x) = \sum_{\ell=1}^{2} (\overline{C}_k)_{\ell 3} \left[\partial_\zeta Z_\ell^-(x,\zeta)\right]_k, \tag{83}$$

$$Z_\ell^+(x,\zeta) = \begin{bmatrix} \chi_{13}^+\chi_{\ell 3}^{A+} \\ \chi_{23}^+\chi_{\ell 3}^{A+} \\ \chi_{33}^+\chi_{\ell 1}^{A+} \\ \chi_{33}^+\chi_{\ell 2}^{A+} \end{bmatrix}, \quad Z_{\ell,k}^+(x) = Z_\ell^+\left(x,\zeta_k^+\right), \quad Z_{d,k}^+(x) = \sum_{\ell=1}^{2} (C_k)_{3\ell} \left[\partial_\zeta Z_\ell^+(x,\zeta)\right]_k. \tag{84}$$

Then we have

$$\begin{aligned} \Sigma \cdot \delta Q(x) = &-\sum_{\ell=1}^{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\pi} \left[Z_\ell^-(x,\zeta)\delta\sigma_{\ell 3}(\zeta) + Z_\ell^+(x,\zeta)\delta\rho_{3\ell}(\zeta)\right] \\ &+ 2\mathrm{i}\sum_{k=1}^{N^-} \left[Z_{d,k}^-(x)\delta\zeta_k^- + \sum_{\ell=1}^{2} Z_{\ell,k}^-(x)(\delta\overline{C}_k)_{\ell 3}\right] \\ &+ 2\mathrm{i}\sum_{k=1}^{N^+} \left[Z_{d,k}^+(x)\delta\zeta_k^+ + \sum_{\ell=1}^{2} Z_{\ell,k}^+(x)(\delta C_k)_{3\ell}\right], \end{aligned} \tag{85}$$

where the matrix Σ is

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{86}$$

Looking at the above, except for the existence of the polarization and the dimension of the system, these results are not unexpected or surprising, since they agree and correlate with what we know from the AKNS case [9, 10]. The continuous part also agrees with what was found earlier in [4] once the proper reductions are applied. This is detailed in the appendix.

Similarly we can define the ASE as row vectors ($j = 1, 2$)

$$
\begin{aligned}
Z_j^{A-}(x,\zeta) &= \sum_{n=1}^{2} P_{jn}^{-}\left[\chi_{n1}^{A-}\chi_{33}^{-},\,\chi_{n2}^{A-}\chi_{33}^{-},\,\chi_{n3}^{A-}\chi_{13}^{-},\,\chi_{n3}^{A-}\chi_{23}^{-}\right](x,\zeta),\\
Z_{d,j,k}^{A-}(x) &= \left\{\partial_\zeta\left[\frac{1}{S_{33,k}^{-\prime}}Z_j^{A-}(x,\zeta)\right]\right\}_k,\\
Z_k^{A-}(x) &= \left[\chi_{31}^{A-}\chi_{33}^{-},\,\chi_{32}^{A-}\chi_{33}^{-},\,\chi_{33}^{A-}\chi_{13}^{-},\,\chi_{33}^{A-}\chi_{23}^{-}\right]\left(x,\zeta_k^{-}\right),
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
Z_j^{A+}(x,\zeta) &= \sum_{n=1}^{2}\left[\chi_{31}^{A+}\chi_{3n}^{+},\,\chi_{32}^{A+}\chi_{3n}^{+},\,\chi_{33}^{A+}\chi_{1n}^{+},\,\chi_{33}^{A+}\chi_{2n}^{+}\right]P_{nj}^{+},\\
Z_{d,j,k}^{A+}(x) &= \left\{\partial_\zeta\left[\frac{1}{R_{33,k}^{+\prime}}Z_j^{A+}(x,\zeta)\right]\right\}_k\\
Z_k^{A+}(x) &= \left[\chi_{31}^{A+}\chi_{33}^{+},\,\chi_{32}^{A+}\chi_{33}^{+},\,\chi_{33}^{A+}\chi_{13}^{+},\,\chi_{33}^{A+}\chi_{23}^{+}\right]\left(x,\zeta_k^{+}\right).
\end{aligned}
\tag{88}
$$

Using (72), one may also show that (for $j = 1, 2$)

$$
Z_j^{A-}(x,\zeta_k^{-}) = (\overline{C}_k)_{j3}\left(S_{33}^{-\prime}\right)_k Z_k^{A-}(x), \qquad Z_j^{A+}\left(x,\zeta_k^{+}\right) = -(C_k)_{3j}\left(R_{33}^{+\prime}\right)_k Z_k^{A+}(x). \tag{89}
$$

Now, using the above-defined adjoint states, (43), (46), (50), (51), (56) and (60) become (for $j = 1, 2$)

$$
\delta\sigma_{j3}(\zeta) = \frac{1}{\left(S_{33}^{-}\right)^2}\int_{-\infty}^{\infty} Z_j^{A-}(x,\zeta)\cdot\delta Q(x)\,\mathrm{d}x, \tag{90}
$$

$$
\delta\rho_{3j}(\zeta) = \frac{-1}{\left(R_{33}^{+}\right)^2}\int_{-\infty}^{\infty} Z_j^{A+}(x,\zeta)\cdot\delta Q(x)\,\mathrm{d}x, \tag{91}
$$

$$
\delta\zeta_k^{+}(\zeta) = \frac{1}{R_{33,k}^{+\prime}}\int_{-\infty}^{\infty} Z_k^{A+}(x)\cdot\delta Q(x)\,\mathrm{d}x, \tag{92}
$$

$$
\delta\zeta_k^{-}(\zeta) = \frac{-1}{S_{33,k}^{-\prime}}\int_{-\infty}^{\infty} Z_k^{A-}(x)\cdot\delta Q(x)\,\mathrm{d}x, \tag{93}
$$

$$
\delta\left(C_k\right)_{3j}(\zeta) = \frac{-1}{R_{33,k}^{+\prime}}\int_{-\infty}^{\infty} Z_{d,j,k}^{A+}(x)\cdot\delta Q(x)\,\mathrm{d}x, \tag{94}
$$

$$
\delta(\overline{C}_k)_{j3}(\zeta) = \frac{-1}{S_{33,k}^{-\prime}}\int_{-\infty}^{\infty} Z_{d,j,k}^{A-}(x)\cdot\delta Q(x)\,\mathrm{d}x. \tag{95}
$$

Given the above results, we can obtain the inner products between the SE and the ASE. Note that by (85), given any linear perturbations in the scattering data, one can obtain the variations in the potential. Furthermore, all perturbations of the components of the scattering data are linearly independent. On the opposite side, per (90)–(95), given any linear perturbations in the potentials, one can obtain the resulting variations in the scattering data. Again, these four potential components are each taken to be linearly independent. So if we take (85) and insert it into (90)–(95), then we must come back to identically the same thing. Requiring the result of this substitution to be of this form, this gives us, for $j, \ell = 1$ or 2, that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, Z_j^{A-}(x,\zeta)\cdot\Sigma\cdot Z_\ell^-(x,\zeta') = -\pi\left(S_{33}^-\right)^2\delta_\ell^j\delta(\zeta-\zeta'), \tag{96}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, Z_j^{A+}(x,\zeta)\cdot\Sigma\cdot Z_\ell^+(x,\zeta') = \pi\left(R_{33}^+\right)^2\delta_\ell^j\delta(\zeta-\zeta'), \tag{97}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, Z_k^{A+}(x)\cdot\Sigma\cdot Z_{d,k'}^+(x) = \frac{-\mathrm{i}}{2}R_{33,k}^{+\prime}\delta_{k'}^k, \qquad (k,k'=1,2,\ldots,N^+), \tag{98}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, Z_k^{A-}(x)\cdot\Sigma\cdot Z_{d,k'}^-(x) = \frac{\mathrm{i}}{2}S_{33,k}^{-\prime}\delta_{k'}^k, \qquad (k,k'=1,2,\ldots,N^-), \tag{99}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, Z_{d,\ell,k}^{A+}(x)\cdot\Sigma\cdot Z_{\ell,k'}^+(x) = \frac{\mathrm{i}}{2}R_{33,k}^{+\prime}\delta_{k'}^k\delta_\ell^j, \qquad (k,k'=1,2,\ldots,N^+), \tag{100}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, Z_{d,\ell,k}^{A-}(x)\cdot\Sigma\cdot Z_{j,k'}^-(x) = \frac{\mathrm{i}}{2}S_{33,k}^{-\prime}\delta_{k'}^k\delta_\ell^j, \qquad (k,k'=1,2,\ldots,N^-), \tag{101}$$

where $\delta_k^{k'}$ is the Kronecker delta and $\delta(x)$ is the Dirac delta function. All other possible inner products vanish.

Now, let us do the opposite, take (90)–(95) and insert these equations into (85). Then since all the δQ's are linearly independent and arbitrary for all x, it follows that $(\ell, j = 1, 2)$

$$\begin{aligned}\sum_{\ell=1}^{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\pi}&\left\{\frac{1}{\left[R_{33}^+(\zeta)\right]^2}Z_\ell^+(x,\zeta)\cdot Z_\ell^{A+}(y,\zeta) - \frac{1}{\left[S_{33}^-(\zeta)\right]^2}Z_\ell^-(x,\zeta)\cdot Z_\ell^{A-}(y,\zeta)\right\}\\ &-2i\sum_{k=1}^{N^-}\frac{1}{S_{33,k}^{-\prime}}\left[Z_{d,k}^-(x)\cdot Z_k^{A-}(y) + \sum_{\ell=1}^{2}Z_{\ell,k}^-(x)\cdot Z_{d,\ell,k}^{A-}(y)\right]\\ &+2\mathrm{i}\sum_{k=1}^{N^+}\frac{1}{R_{33,k}^{+\prime}}\left[Z_{d,k}^+(x)\cdot Z_k^{A+}(y) - \sum_{\ell=1}^{2}Z_{\ell,k}^+(x)\cdot Z_{d,\ell,k}^{A+}(y)\right] = \Sigma\delta(x-y),\end{aligned} \tag{102}$$

which is the closure relation.

Using (89) and the above definitions, one may readily show that for compact support, this closure relation also has the following representation:

$$\begin{aligned}\sum_{\ell=1}^{2}\int_C\frac{\mathrm{d}\zeta}{\pi}\frac{1}{\left[R_{33}^+(\zeta)\right]^2}Z_\ell^+(x,\zeta)\cdot Z_\ell^{A+}(y,\zeta) - \sum_{\ell=1}^{2}\int_{\overline{C}}\frac{\mathrm{d}\zeta}{\pi}\frac{1}{\left[S_{33}^-(\zeta)\right]^2}Z_\ell^-(x,\zeta)\cdot Z_\ell^{A-}(y,\zeta)\\ = \Sigma\delta(x-y),\end{aligned} \tag{103}$$

where $\overline{C}$ is the standard AKNS contour [8] in the LHP, going from $\zeta = -\infty$ on the real ζ-axis to $\zeta = +\infty$ on the real ζ-axis, while going under all zeros of $S_{33}^-(\zeta)$. The contour C is the standard AKNS contour in the UHP, going from $\zeta = -\infty$ on the real ζ-axis to $\zeta = +\infty$ on the real ζ-axis, while going above all zeros of $R_{33}^+(\zeta)$.

7. Summary

What we have done here is to take the procedure outline in [4, 6, 7] and obtained the covering set of SE and ASE for the degenerate 3×3 eigenvalue problem given in (1). We see that this covering set is a set of products of the Jost solutions and the adjoint Jost solutions. From this covering set, upon applying the proper reductions to the potential matrix, Q, and the resulting symmetries of the scattering matrix, S, and its inverse, R, one can then obtain the SE and ASE for the Sasa–Satsuma equation [4]. Let us note that this covering set, as it is, is also the SE and ASE for the VNLS, the continuous spectra part of which has already been detailed in the appendix of [4]. Note that because the potential elements in the VNLS have been already treated as being distinct, since q^* is treated as being different from q, these SE and ASE are not sums of products. However, there is still a reduction here, wherein symmetries become imposed on the scattering matrix, S, and its inverse, R. These symmetries then lead to the proper complex conjugation relations between the potential components.

On the other hand, for the Sasa–Satsuma equation, since the required reductions actually identify certain potential components as being the negative of others (i.e. $Q_{13} = -Q_{32}$ and $Q_{23} = -Q_{31}$), one has the option of using the covering set of SE and ASE given here to calculate the variations in, say Q_{13}, and also to calculate the variations in Q_{32}. However upon taking into account the symmetries in scattering data and the Jost functions, one would then find that the variations in Q_{32} would be exactly the negative of the previous variation. Thus even in cases where there are reductions in the potential matrix and corresponding symmetries in the scattering data, one has the option of also using the covering set of SE and ASE. This is illustrated in the appendix, where it is shown that upon using the symmetries of the scattering data corresponding to the reductions for the Sasa–Satsuma equation, from the ones given in this paper, one may obtain the expressions for SE and ASE given in [4].

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Appendix. Squared eigenfunctions of the Sasa–Satsuma equation through symmetry reduction

In this appendix, we show how to obtain the SE and ASE of the Sasa–Satsuma equation from the results in the main text through symmetry reduction. In the Sasa–Satsuma equation, the potential matrix Q is

$$Q = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix}, \tag{A.1}$$

i.e.

$$Q_{13} = -Q_{32} = Q_{23}^* = -Q_{31}^*. \tag{A.2}$$

This matrix has two symmetries:

$$Q^\dagger = -Q, \tag{A.3}$$

and

$$\sigma Q \sigma = Q^*, \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.4}$$

Here the superscript '†' represents the Hermitian of a matrix. Due to these two symmetries, it is easy to show that the Jost functions and the scattering matrix have the following symmetries [4]:

$$\Phi^{\dagger}(\zeta^*) = \Phi^{-1}(\zeta), \qquad \Psi^{\dagger}(\zeta^*) = \Psi^{-1}(\zeta), \tag{A.5}$$

$$\Phi_{\pm}(\zeta) = \sigma\Phi_{\pm}^*(-\zeta^*)\sigma, \qquad \Psi_{\pm}(\zeta) = \sigma\Psi_{\pm}^*(-\zeta^*)\sigma, \tag{A.6}$$

$$S^{\dagger}(\zeta^*) = S^{-1}(\zeta), \tag{A.7}$$

and

$$S(\zeta) = \sigma S^*(-\zeta^*)\sigma. \tag{A.8}$$

Consequently,

$$\rho_{31}(\zeta) = \sigma_{13}^*(\zeta), \quad \rho_{32}(\zeta) = \sigma_{23}^*(\zeta), \quad \zeta \in \mathbb{R}, \tag{A.9}$$

$$\rho_{31}(\zeta) = \sigma_{23}(-\zeta), \quad \rho_{32}(\zeta) = \sigma_{13}(-\zeta), \quad \zeta \in \mathbb{R}, \tag{A.10}$$

$$\sigma\Psi_1^-(\zeta) = \Psi_2^{A+}(-\zeta), \quad \sigma\Psi_2^-(\zeta) = \Psi_1^{A+}(-\zeta), \quad \sigma\Psi_3^+(\zeta) = \Psi_3^{A-}(-\zeta), \quad \zeta \in \mathbb{R}. \tag{A.11}$$

Now we use these symmetries to simplify expansions (85) and (90)–(95). For simplicity, we consider this reduction in the absence of bound states (the case with bound states proceeds similarly). In the absence of bound states, expansion (85) becomes

$$\Sigma \cdot \delta Q(x) = -\sum_{\ell=1}^{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\pi}\left[Z_\ell^-(x,\zeta)\delta\sigma_{\ell 3}(\zeta) + Z_\ell^+(x,\zeta)\delta\rho_{3\ell}(\zeta)\right]. \tag{A.12}$$

Using the symmetries of reflection coefficients (A.9) and (A.10), this expansion reduces to

$$\begin{aligned}\Sigma \cdot \delta Q(x) = -\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\pi}\big\{&\left[Z_1^-(x,\zeta) + Z_2^+(x,-\zeta)\right]\delta\sigma_{13}(\zeta)\\ &+\left[Z_1^+(x,\zeta) + Z_2^-(x,-\zeta)\right]\delta\rho_{31}(\zeta)\big\}.\end{aligned} \tag{A.13}$$

Using the symmetries of Jost functions (A.11) as well as definitions (83)–(84), we get

$$Z_1^-(x,\zeta) + Z_2^+(x,-\zeta) = \begin{bmatrix}\chi_{11}^-\chi_{33}^{A-} + \chi_{31}^-\chi_{32}^{A-}\\ \chi_{21}^-\chi_{33}^{A-} + \chi_{31}^-\chi_{31}^{A-}\\ \chi_{31}^-\chi_{31}^{A-} + \chi_{21}^-\chi_{33}^{A-}\\ \chi_{31}^-\chi_{32}^{A-} + \chi_{11}^-\chi_{33}^{A-}\end{bmatrix}(x,\zeta), \tag{A.14}$$

and

$$Z_1^+(x,\zeta) + Z_2^-(x,-\zeta) = \begin{bmatrix}\chi_{13}^+\chi_{13}^{A+} + \chi_{33}^+\chi_{12}^{A+}\\ \chi_{23}^+\chi_{13}^{A+} + \chi_{33}^+\chi_{11}^{A+}\\ \chi_{33}^+\chi_{11}^{A+} + \chi_{23}^+\chi_{13}^{A+}\\ \chi_{33}^+\chi_{12}^{A+} + \chi_{13}^+\chi_{13}^{A+}\end{bmatrix}(x,\zeta). \tag{A.15}$$

Due to the above-mentioned symmetries, the last two equations in expansion (A.13) are redundant and can be dropped. Thus that expansion becomes

$$\begin{bmatrix}\delta Q_{13}(x)\\ \delta Q_{23}(x)\end{bmatrix} = -\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{\pi}[Z^-(x,\zeta)\delta\sigma_{13}(\zeta) + Z^+(x,\zeta)\delta\rho_{31}(\zeta)], \tag{A.16}$$

where

$$Z^{-}(x,\zeta)=\begin{bmatrix}\chi_{11}^{-}\chi_{33}^{A-}+\chi_{31}^{-}\chi_{32}^{A-}\\ \chi_{21}^{-}\chi_{33}^{A-}+\chi_{31}^{-}\chi_{31}^{A-}\end{bmatrix}(x,\zeta),\qquad Z^{+}(x,\zeta)=\begin{bmatrix}\chi_{13}^{+}\chi_{13}^{A+}+\chi_{33}^{+}\chi_{12}^{A+}\\ \chi_{23}^{+}\chi_{13}^{A+}+\chi_{33}^{+}\chi_{11}^{A+}\end{bmatrix}(x,\zeta). \tag{A.17}$$

These functions $Z^{\pm}(x,\zeta)$ are the SE of the Sasa–Satsuma equation. These SE are different from the ones derived in [4] because expansion (A.16) above is in terms of $(\delta\sigma_{13},\delta\rho_{31})$, while the potential expansion in [4] was in terms of $(\delta\sigma_{31},\delta\rho_{13})$. This difference corresponds to the fact that the variation calculations in this paper are for inversion about $x=+\infty$, while those in [4] were for inversion about $x=-\infty$.

Expansions (90)–(95) can be similarly reduced by symmetry. Using relations (A.2), we see that equations (90)–(91) become

$$\delta\sigma_{13}(\zeta)=\frac{1}{\left(S_{33}^{-}\right)^{2}}\int_{-\infty}^{\infty}Z^{A-}(x,\zeta)\cdot\begin{bmatrix}\delta Q_{13}(x)\\ \delta Q_{23}(x)\end{bmatrix}\mathrm{d}x, \tag{A.18}$$

$$\delta\rho_{31}(\zeta)=\frac{-1}{\left(R_{33}^{+}\right)^{2}}\int_{-\infty}^{\infty}Z^{A+}(x,\zeta)\cdot\begin{bmatrix}\delta Q_{13}(x)\\ \delta Q_{23}(x)\end{bmatrix}\mathrm{d}x, \tag{A.19}$$

where

$$Z^{A-}=\sum_{n=1}^{2}P_{1n}^{-}\left[\chi_{n1}^{A-}\chi_{33}^{-}-\chi_{n3}^{A-}\chi_{23}^{-},\,\chi_{n2}^{A-}\chi_{33}^{-}-\chi_{n3}^{A-}\chi_{13}^{-}\right], \tag{A.20}$$

and

$$Z^{A+}=\sum_{n=1}^{2}\left[\chi_{31}^{A+}\chi_{3n}^{+}-\chi_{33}^{A+}\chi_{2n}^{+},\,\chi_{32}^{A+}\chi_{3n}^{+}-\chi_{33}^{A+}\chi_{1n}^{+}\right]P_{n1}^{+}. \tag{A.21}$$

These functions $Z^{A\pm}(x,\zeta)$ are the ASE of the Sasa–Satsuma equation. They are the counterparts of the ASE derived in [4], except that the ASE above are for the formulas $(\delta\sigma_{13},\delta\rho_{31})$, while the ASE in [4] were for the formulas $(\delta\sigma_{31},\delta\rho_{13})$.

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