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Complete eigenfunctions of linearized integrable equations expanded around a soliton solution

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Complete eigenfunctions for an integrable equation linearized around a soliton solution are the key to the development of a direct soliton perturbation theory. In this article, we explicitly construct such eigenfunctions for a large class of integrable equations including the KdV, NLS and mKdV hierarchies. We establish the striking result that the linearization operators of all equations in the same integrable hierarchy share the *same* complete set of eigenfunctions. Furthermore, these eigenfunctions are precisely the squared eigenfunctions of the associated eigenvalue problem. The key step in our derivation is to show that the linearization operator of an integrable equation can be factored into a function of the integro-differential operator which generates the integrable equation, and the linearization operator of the lowest-order integrable equation in the same hierarchy. We also obtain similar results for the adjoint linearization operator of an integrable equation. Even though our analysis is conducted only for the KdV, NLS and mKdV hierarchies, similar results are expected for other integrable hierarchies as well. We further explicitly present the complete eigenfunctions for the KdV, NLS and mKdV hierarchy equations and give their inner products, thus they can be readily used to develop a direct soliton perturbation theory for any of those hierarchy equations. © *2000 American Institute of Physics.* [S0022-2488(00)02709-2]

I. INTRODUCTION

Many physical wave systems are governed by nonlinear integrable equations at the lowest order of approximation. For instance, pulse transmission in optical fibers and wave propagation in deep water are described by the nonlinear Schrödinger (NLS) equation.[1,2] Evolution of shallow water waves is described by the Korteweg–de Vries (KdV) equation,[3,4] and internal waves at the interface of two layers of equal depth are described by the modified Korteweg–de Vries (mKdV) equation.[5] Integrable equations support soliton solutions which travel stationarily and collide elastically. They also possess many other remarkable properties such as infinite conservation laws and Painlevé property.[5–7] When perturbations such as damping, higher order dispersion and nonlinearity are brought into consideration, a physical system is then better modeled by perturbed integrable equations.[1,2,8,9] In a perturbed system, solitons in general will not remain stationary anymore. To study their evolution and subsequent excitation of radiation, one would need to develop a soliton perturbation theory. Several such theories have been developed in the past. One is the inverse-scattering-based soliton perturbation theory, which was developed in the 1970s.[10–14] This method is intimately related to the inverse scattering technique. The second one, also developed in the 1970s, is based on the Green's function for the linearized integrable equation expanded around solitons.[15] The third one, also originated in the 1970s[16,17] and further developed in the 1990s,[18–23] is the direct soliton perturbation theory. It is based on the complete set of eigenfunctions for the linearized equation expanded around solitons. In essence, this theory shares the same ideas as the second one, but it is conceptually simpler and has a wider appeal. Several other

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approaches such as the adiabatic method[24] and the variational method[25] have also been developed. But these methods cannot capture radiation modes, and thus are mathematically incomplete.

The key in the direct soliton perturbation theory is to find a complete set of bounded eigenfunctions for the linearized equation around a soliton solution. This set allows one to solve the linear inhomogeneous equations at various orders of the perturbation expansion. Suppression of secular growth in those solutions then results in the dynamical equations for soliton parameters and radiation coefficients. At the moment, such a complete set of eigenfunctions has been identified only for the sine-Gordon, Benjamin–Ono, NLS and KdV equations.[16–22,26] But some general ideas have also been hinted or put forward. In Ref. 18, Kaup obtained the complete sets of eigenfunctions for the linearized NLS equation around a soliton solution from his observation that these functions are related to the squared Zakharov–Shabat eigenstates. Indeed, the connection between eigenfunctions of linearized integrable equations and squared eigenstates of the associated eigenvalue problem has been hinted by inverse-scattering-based soliton perturbation theory.[10–13] But it has never been clearly articulated and demonstrated for the general case. In Ref. 19, Herman proposed to use the Lax pair of an integrable equation to find the complete eigenfunctions of linearized equations. In this approach, one first determines the time evolution of the squared eigenfunctions of the associated eigenvalue problem from the Lax pair. Then one tries to find the correct combination of squared eigenfunctions to satisfy the linearized integrable equation. Herman applied this method to the KdV and NLS equations and successfully obtained the complete eigenstates. The procedure proposed by Herman is suggestive, and it can also work for linearization of integrable equations around time-dependent solutions such as multi-soliton solutions. However, its disadvantage is that, for every integrable equation, one has to verify that squared eigenfunctions of the eigenvalue problem solve the linearized equation around a soliton solution (or a general solution). It is not clear yet whether this will always be the case. From these previous works, we see that, although some interesting ideas have been proposed to construct complete eigenfunctions of linearized integrable equations, what these eigenfunctions must be for a general integrable equation is still unknown.

In this article, we construct complete eigenfunctions for a large class of integrable equations linearized around a single-soliton solution. This class includes the KdV, NLS and mKdV hierarchies. The striking result which we will establish is that linearization operators of all integrable equations in the same hierarchy share the *same* complete set of eigenfunctions (the corresponding eigenvalues differ from one equation to another). Furthermore, these eigenfunctions are also eigenstates of the integro-differential operator which generates the hierarchy, thus they are directly related to the squared eigenfunctions of the eigenvalue problem associated with the hierarchy. In fact, our results are even stronger. We will show that the linearization operator for any equation in a hierarchy can be factored into the integro-differential operator which generates the hierarchy, and the linearization operator of the lowest-order equation in this hierarchy. All the other results cited above are simple consequences of this factorization result. Our findings confirm that, for a broad class of integrable equations, squared eigenstates of the eigenvalue problem also solve the linearized equation around a soliton solution. Thus, squared eigenfunctions of the eigenvalue problem are the natural basis of expansion in a direct soliton perturbation theory. They also indicate that, unlike Herman's approach,[19] only the eigenvalue operator of the Lax pair is relevant in the construction of complete eigenfunctions for the linearized equation around single-soliton solutions. The time evolution operator of the Lax pair can be neglected. This is why an entire hierarchy can share the same complete set of eigenfunctions, since they are all associated with the same eigenvalue operator. Although our focus of this article is on the KdV, mKdV and NLS hierarchies, the ideas and basic results should hold for other integrable hierarchies as well. Based on these results, we then explicitly give the complete sets of eigenfunctions for linearization operators of the KdV, NLS and mKdV hierarchies. We also give similar results for the adjoint linearization operators, and explicitly obtain the common adjoint eigenstates for each hierarchy. With these complete eigenstates and adjoint eigenstates available, it is now a simple matter to develop a direct soliton perturbation theory for all the KdV, NLS and mKdV hierarchy equations. We note that another application of these complete eigenfunctions is in the study of eigenvalue bifurcation of solitary waves from the edge of the continuous spectrum in a perturbed integrable equation.[27–29] Lastly, we comment that our analysis is independent of the inverse scattering theory, even though connections to inverse scattering are still visible.

II. COMPLETE EIGENFUNCTIONS OF LINEARIZATION OPERATORS FOR THE KdV HIERARCHY

We start by considering the eigenmodes of linearization operators for the KdV hierarchy. This hierarchy is of the form[30]

$$q_t + C(4L_s^+) q_x = 0, \tag{2.1}$$

where $C(k^2)$ is the phase velocity of the linearized equations, and the integro-differential operator L_s^+ is

$$L_s^+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2} q_x \int_x^\infty dy. \tag{2.2}$$

Here the subscript ''s'' in L_s^+ refers to ''Schrödinger,'' as the associated eigenvalue problem for the KdV hierarchy (2.1) is the Schrödinger equation.[30,31] The adjoint operator of L_s^+ is

$$L_s = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}\int_{-\infty}^x dy\, q_y. \tag{2.3}$$

Notice that

$$L_s^+ f_x = (L_s f)_x \tag{2.4}$$

for any function $f(x)$ which vanishes at infinity. Thus Eq. (2.1) can be recast in terms of the adjoint operator L_s as

$$q_t + [C(4L_s)q]_x = 0. \tag{2.5}$$

In this section, we require the phase velocity function $C(z)$ to be entire. When $C(z) = -z$, Eq. (2.1) becomes the KdV equation:

$$q_t + 6qq_x + q_{xxx} = 0. \tag{2.6}$$

When $C(z) = z^2$, Eq. (2.1) is the fifth-order KdV hierarchy:

$$q_t + q_{xxxxx} + 10qq_{xxx} + 20q_xq_{xx} + 30q^2q_x = 0. \tag{2.7}$$

Other members in this hierarchy can be obtained by choosing different functions for the phase velocity $C(z)$.

In the rest of this section, occasions will arise where we want to apply the operator L_s^+ [and L_0^+ to be defined in Eq. (2.13)] on a function $g'(x)$, where $g(x)$ is related to continuous eigenfunctions and is oscillatory at infinity. In such cases, we adopt the following convention for the integral term involved:

$$\int_x^\infty g'(y)\, dy \equiv -g(x). \tag{2.8}$$

This convention echoes the fact that, when we obtain a particular KdV hierarchy equation from (2.1), terms such as $\int_x^\infty q'(y)\, dy$ are always evaluated as $-q(x)$ and so on. This convention applies notably to the commutability relation (2.20) and the factorization formula (2.22) when they operate on continuous eigenfunctions. It applies to the eigenfunction relation (2.61) as well. We emphasize that this convention is only a technical issue. It does not cause any controversy or ambiguity in our main results expressed in Theorems 1 and 2. In fact, we could have chosen to work with the operator L_s and avoid operators L_s^+ (and L_0^+) altogether. The way to do it is to start with the KdV hierarchy (2.5) instead of (2.1). The results of course would be the same, but the derivations would be a little cumbersome.

We now consider soliton solutions in the KdV hierarchy (2.1) and linearization of (2.1) around solitons. One can check that the soliton family

$$q(x,t)=2\eta^2 \operatorname{sech}^2 \eta\{x-C(-4\eta^2)t\} \tag{2.9}$$

satisfies Eq. (2.1), where η is a free amplitude parameter. By rescaling the variables x and q by η and η^2, respectively, and by denoting $C(\eta^2 z)$ as $C(z)$, we can normalize $\eta=1$ in the soliton solution (2.9) while keeping the evolution equation (2.1) intact. We also adopt the coordinate system moving with speed $C(-4)$, i.e.,

$$\overline{x}=x-C(-4)t, \quad \overline{t}=t. \tag{2.10}$$

When the bars are dropped, Eq. (2.1) finally becomes

$$q_t+[C(4L_s^+)-C(-4)]q_x=0, \tag{2.11}$$

where

$$q_0(x)=2\operatorname{sech}^2 x \tag{2.12}$$

is its normalized soliton solution.

Two operators, L_0^+ and L_0, will be used frequently in the rest of this section. They are defined as L_s^+ and L_s with $q(x,t)$ replaced by $q_0(x)$, i.e.,

$$L_0^+=-\frac{1}{4}\frac{\partial^2}{\partial x^2}-q_0+\frac{1}{2}q_{0x}\int_x^\infty dy, \tag{2.13}$$

and

$$L_0=-\frac{1}{4}\frac{\partial^2}{\partial x^2}-q_0+\frac{1}{2}\int_{-\infty}^x dy\, q_{0y}. \tag{2.14}$$

Naturally, L_0 is the adjoint operator of L_0^+, just as L_s is the adjoint operator of L_s^+. Note that

$$L_0^+ q_{0x}=-q_{0x}. \tag{2.15}$$

This relation will be used in the proof of Theorem 1.

We now linearize the evolution equation (2.11) around its soliton solution (2.12). We set

$$q(x,t)=q_0(x)+\tilde{q}(x,t), \tag{2.16}$$

where $\tilde{q}\ll 1$. When it is substituted into Eq. (2.11) and higher order terms are discarded, the linearized equation of (2.11) is

$$\tilde{q}_t+L_{kh}\tilde{q}=0, \tag{2.17}$$

where L_{kh} is the linearization operator. Here the subscript ''kh'' is the abbreviation of ''KdV hierarchy.'' We also denote the adjoint operator of L_{kh} as L_{kh}^A. For the KdV equation, $C(z)=-z$. In this case, linearization of Eq. (2.11) around the soliton (2.12) shows that the linearization operator is

$$L_{\rm kdv}=\frac{\partial^3}{\partial x^3}+(6q_0-4)\frac{\partial}{\partial x}+6q_{0x}. \tag{2.18}$$

Its adjoint operator is

$$L_{\rm kdv}^A=-\frac{\partial^3}{\partial x^3}-(6q_0-4)\frac{\partial}{\partial x}. \tag{2.19}$$

An important property is that, L_0^+ and $L_{\rm kdv}$ are commutable, and L_0 and $L_{\rm kdv}^A$ are commutable, i.e.,

$$L_0^+L_{\rm kdv}=L_{\rm kdv}L_0^+, \tag{2.20}$$

and

$$L_0L_{\rm kdv}^A=L_{\rm kdv}^AL_0. \tag{2.21}$$

These facts can be verified by direct calculations.

The first important result of this section is the following theorem which gives the simple expressions for the linearization operator L_{kh} and its adjoint operator L_{kh}^A for any KdV hierarchy equation.

Theorem 1: *For any KdV hierarchy equation (2.11) where $C(z)$ is an entire function, the linearization operator L_{kh} and its adjoint operator L_{kh}^A have the following factorizations*:

$$L_{kh}=M(4L_0^+)L_{\rm kdv}, \tag{2.22}$$

$$L_{kh}^A=M(4L_0)L_{\rm kdv}^A, \tag{2.23}$$

where the function $M(z)$ is defined as

$$M(z)\equiv\frac{C(-4)-C(z)}{4+z}. \tag{2.24}$$

Before proving this theorem, we present an example first. Let us take the fifth-order KdV equation (2.11), where $C(z)=z^2$. Then $M(z)=4-z$. Straightforward calculations show that

$$M(4L_0^+)L_{\rm kdv}=\frac{\partial^5}{\partial x^5}+10q_0\frac{\partial^3}{\partial x^3}+20q_{0x}\frac{\partial^2}{\partial x^2}+(-16+80q_0-30q_0^2)\frac{\partial}{\partial x}+40q_{0x}. \tag{2.25}$$

This is exactly the linearization operator L_{kh} when one linearizes Eq. (2.11) directly. The L_{kh}^A factorization formula (2.23) can be similarly verified in this special case.

Proof: It suffices to prove this theorem for $C(z)$ as a power function, $C(z)=z^n$, where n is any positive integer, as any entire function can be expanded into a power series. In this case, Eq. (2.11) becomes

$$q_t+[(4L_s^+)^n-(-4)^n]q_x=0, \tag{2.26}$$

and

$$M(z)=-\sum_{i=1}^{n}z^{i-1}(-4)^{n-i}. \tag{2.27}$$

When Eq. (2.16) is substituted into the operator $4L_s^+$, linearization of $4L_s^+$ is

$$4L_s^+ = 4L_0^+ - 4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy + O(\tilde{q}^2). \tag{2.28}$$

Thus, linearization of $(4L_s^+)^n$ is

$$(4L_s^+)^n = (4L_0^+)^n + \sum_{i=1}^{n} (4L_0^+)^{i-1}\left[-4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy\right](4L_0^+)^{n-i} + O(\tilde{q}^2). \tag{2.29}$$

When this equation is utilized, we find the linearization operator L_{kh} of the evolution equation (2.26) to be

$$L_{kh}\tilde{q} = [(4L_0^+)^n - (-4)^n]\frac{\partial \tilde{q}}{\partial x} + \sum_{i=1}^{n} (4L_0^+)^{i-1}\left[-4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy\right](4L_0^+)^{n-i} q_{0x}. \tag{2.30}$$

Recalling Eqs. (2.15), (2.18) and (2.27), the above equation becomes

$$\begin{aligned} L_{kh}\tilde{q} &= \sum_{i=1}^{n} (4L_0^+)^{i-1}(-4)^{n-i}\left[(4L_0^+ + 4)\frac{\partial \tilde{q}}{\partial x} - 4q_{0x}\tilde{q} - 2q_0\tilde{q}_x\right] \\ &= \sum_{i=1}^{n} (4L_0^+)^{i-1}(-4)^{n-i}[-\tilde{q}_{xxx} - (6q_0 - 4)\tilde{q}_x - 6q_{0x}\tilde{q}] \\ &= -\sum_{i=1}^{n} (4L_0^+)^{i-1}(-4)^{n-i} L_{\text{kdv}}\tilde{q} = M(4L_0^+)L_{\text{kdv}}\tilde{q}. \end{aligned} \tag{2.31}$$

Thus the factorization formula (2.22) is proved. To prove L_{kh}^A factorization formula (2.23), we recall the fact that, for any two operators P and Q, $(PQ)^A = Q^A P^A$, where the superscript ''A'' represents the adjoint operator. Since L_0 is the adjoint operator of L_0^+, thus from Eq. (2.31), we have

$$L_{kh}^A = -\sum_{i=1}^{n} (-4)^{n-i} L_{\text{kdv}}^A (4L_0)^{i-1}. \tag{2.32}$$

But L_0 and L_{kdv}^A are commutable [see Eq. (2.21)]. So

$$L_{kh}^A = -\sum_{i=1}^{n} (4L_0)^{i-1}(-4)^{n-i} L_{\text{kdv}}^A = M(4L_0)L_{\text{kdv}}^A. \tag{2.33}$$

This proves the L_{kh}^A factorization formula (2.23).

Remark: The only piece of information we used to prove the L_{kh} factorization formula (2.22) is the simple relation (2.15) for q_0, and the only information we used to prove the L_{kh}^A factorization formula (2.23) is (2.22) and the commutability relation between operators L_0 and L_{kdv}^A.

Theorem 1 is an elegant and important result of this section. It relates the linearization operators L_{kh} and L_{kh}^A of an arbitrary KdV hierarchy equation to the integro-differential operators L_0^+, L_0, and the KdV linearization operators L_{kdv} and L_{kdv}^A. Another important fact, which we will establish later, is that L_0^+ (L_0) and L_{kdv} (L_{kdv}^A) share the same complete set of eigenstates. This fact, together with the factorization formulas (2.22) and (2.23), will immediately result in the *same* complete set of eigenstates for L_{kh} and L_{kh}^A of all KdV hierarchy equations.

We first write down the complete sets of eigenfunctions for L_{kdv} and L_{kdv}^A, which have been worked out before.[19,20,26] The complete eigenfunctions and generalized eigenfunctions of L_{kdv} are

$$\Psi(x,k)=\frac{1}{(k+2i)^2}[ik(k^2+4)-4(k^2+2)\tanh x-8ik\tanh^2 x+8\tanh^3 x]e^{ikx}, \tag{2.34}$$

$$\Psi_1(x)=\mathrm{sech}^2 x\tanh x, \quad \Psi_2(x)=\mathrm{sech}^2 x(1-x\tanh x), \tag{2.35}$$

where

$$L_{\mathrm{kdv}}\Psi=-ik(k^2+4)\Psi, \tag{2.36}$$

$$L_{\mathrm{kdv}}\Psi_1=0, \quad L_{\mathrm{kdv}}\Psi_2=-8\Psi_1, \tag{2.37}$$

and $-\infty<k<\infty$ in (2.34) and (2.36). We note that the discrete states Ψ_1 and Ψ_2 are obtained by taking variations to the free amplitude and position parameters in a KdV soliton. Eigenstates for the adjoint operator L_{kdv}^A are

$$\Phi(x,k)=\frac{1}{(k+2i)^2}[k^2-4ik\tanh x-4\tanh^2 x]e^{-ikx}, \tag{2.38}$$

$$\Phi_1(x)=\mathrm{sech}^2 x, \quad \Phi_2(x)=\tanh x+x\,\mathrm{sech}^2 x, \tag{2.39}$$

where

$$L_{\mathrm{kdv}}^A\Phi=-ik(k^2+4)\Phi, \tag{2.40}$$

$$L_{\mathrm{kdv}}^A\Phi_1=0, \quad L_{\mathrm{kdv}}^A\Phi_2=-8\Phi_1, \tag{2.41}$$

and $-\infty<k<\infty$ in (2.38) and (2.40) as well.

The nonzero inner products between eigenstates and their adjoint eigenstates are

$$\langle\Psi(x,k),\Phi(x,k')\rangle=2\pi ika_0^2\delta(k-k') \tag{2.42}$$

and

$$\langle\Psi_1(x),\Phi_2(x)\rangle=\langle\Psi_2(x),\Phi_1(x)\rangle=1, \tag{2.43}$$

where $a_0=(k-2i)/(k+2i)$. Here the inner products between two (vector) functions $f(x)$ and $g(x)$ are defined as

$$\langle f(x),g(x)\rangle\equiv\int_{-\infty}^{\infty}f(x)^T g(x)\,dx, \tag{2.44}$$

and the superscript ''T'' represents the transverse of a vector or matrix. The closure relation is

$$\int_{-\infty}^{\infty}\frac{1}{2\pi ika_0^2}\Psi(x,k)\Phi(x',k)dk+\sum_{j=1}^{2}\Psi_j(x)\Phi_j(x')=\delta(x-x'). \tag{2.45}$$

A critical fact is that the above eigenstates of L_{kdv} and L_{kdv}^A are *also* the eigenstates of L_0^+ and L_0, respectively. More specifically, we have

$$L_0^+\Psi=\frac{k^2}{4}\Psi, \quad L_0^+\Psi_1=-\Psi_1, \quad L_0^+\Psi_2=-\Psi_2-\Psi_1, \tag{2.46}$$

and

$$L_0\Phi = \frac{k^2}{4}\Phi, \quad L_0\Phi_1 = -\Phi_1, \quad L_0\Phi_2 = -\Phi_2 - 1. \tag{2.47}$$

These results can be proved in several different ways. One way is to verify them directly by substituting the eigenstates (2.34), (2.35), (2.38), and (2.39) into Eqs. (2.46) and (2.47). A better way is to recall the crucial fact that L_0^+ (L_0) and L_{kdv} (L_{kdv}) are commutable. Thus

$$L_{\text{kdv}}L_0^+\Psi(x,k) = L_0^+L_{\text{kdv}}\Psi(x,k) = -ik(k^2+4)L_0^+\Psi(x,k). \tag{2.48}$$

This shows that $L_0^+\Psi$ is an eigenstate of L_{kdv} with eigenvalue $-ik(k^2+4)$. But we know from (2.36) that the only eigenstate of L_{kdv} with eigenvalue $-ik(k^2+4)$ is $\Psi(x,k)$. Thus $L_0^+\Psi = \lambda\Psi$, where λ is a constant. In other words, $\Psi(x,k)$ is also an eigenstate of L_0^+. By taking the limit $x\to\infty$, we can easily find that the eigenvalue $\lambda = k^2/4$. Other relations in (2.46) and (2.47) can be proved similarly.

The third way of proving (2.46) and (2.47) is probably the most stimulating. This proof makes use of the important relationship between eigenfunctions (2.34), (2.35), (2.38), and (2.39) of the KdV linearization operators and squared eigenstates of the Schrödinger equation with a soliton potential (2.12):

$$v_{xx} + (\zeta^2 + q_0(x))v = 0. \tag{2.49}$$

Using conventional notation, we define the eigenstates $\psi(x,\zeta)$ and $\phi(x,\zeta)$ of (2.49) as

$$\psi \to e^{i\zeta x}, \quad x\to\infty, \tag{2.50}$$

$$\phi \to e^{-i\zeta x}, \quad x\to -\infty. \tag{2.51}$$

Then it is easy to check that

$$\psi(x,\zeta) = \frac{\zeta + i\tanh x}{\zeta + i}e^{i\zeta x} \tag{2.52}$$

and

$$\phi(x,\zeta) = \frac{\zeta - i\tanh x}{\zeta + i}e^{-i\zeta x}. \tag{2.53}$$

For real values of ζ, Eqs. (2.52) and (2.53) give the continuous eigenstates of the Schrödinger operator. When $\zeta = i$, they produce the same discrete eigenstate

$$\psi_1 = \phi_1 = \tfrac{1}{2}\operatorname{sech} x. \tag{2.54}$$

It can be directly verified that eigenstates (2.34), (2.35), (2.38), and (2.39) of the KdV linearization operators are related to the the squared eigenstates of the Schrödinger operator as follows:

$$\Psi(x,k) = \frac{\partial\psi^2(x,k/2)}{\partial x}, \tag{2.55}$$

$$\Psi_1(x) = -2\frac{\partial\psi_1^2}{\partial x}, \quad \Psi_2(x) = -\left(i\frac{\partial^2\psi^2}{\partial x\partial\zeta} + \frac{\partial\psi^2}{\partial x}\right)\Bigg|_{\zeta=i}, \tag{2.56}$$

$$\Phi(x,k) = \phi^2(x,k/2), \tag{2.57}$$

$$\Phi_1(x)=4\phi_1^2, \quad \Phi_2(x)=2\left(i\frac{\partial\phi^2}{\partial\zeta}+\phi^2\right)\bigg|_{\zeta=i}-1. \tag{2.58}$$

Now we need to recall the fact that squared eigenstates of the Schrödinger equation are eigenstates of the integro-differential operators L_0^+ and L_0. Specifically, we have

$$L_0\phi^2=\zeta^2\phi^2, \tag{2.59}$$

$$L_0\phi_1^2=-\phi_1^2, \quad L_0\frac{\partial\phi^2}{\partial\zeta}\bigg|_{\zeta=i}=\left(-\frac{\partial\phi^2}{\partial\zeta}+2i\phi^2\right)\bigg|_{\zeta=i}, \tag{2.60}$$

$$L_0^+(\psi^2)_x=\zeta^2(\psi^2)_x, \tag{2.61}$$

$$L_0^+(\psi_1^2)_x=-(\psi_1^2)_x, \quad L_0^+\frac{\partial^2\psi^2}{\partial\zeta\partial x}\bigg|_{\zeta=i}=\left\{-\frac{\partial^2\psi^2}{\partial\zeta\partial x}+2i(\psi^2)_x\right\}\bigg|_{\zeta=i}. \tag{2.62}$$

Proof of Eq. (2.59) for general potential $q(x,t)$ can be found in Ref. 30. By taking $\zeta=i$ in (2.59), the first equation in (2.60) can be obtained. By taking the derivative of Eq. (2.59) with respect to ζ, and then taking $\zeta=i$, the second equation in (2.59) is proved. Equations (2.61) and (2.62) can be derived similarly. We note that Eq. (2.61) is valid for general potential $q(x,t)$, not just the soliton potential (2.12). We also remind the reader that in Eq. (2.61), the convention (2.8) has been applied. When all the relations (2.55)–(2.58) and (2.59)–(2.62) are utilized, Eqs. (2.46) and (2.47) are proved again.

Now since the eigenstates of L_{kdv} (L_{kdv}^A) are also the eigenstates of L_0^+ (L_0), the L_{kh} and L_{kh}^A factorization formulas (2.22) and (2.23) quickly show that the eigenstates of the KdV linearization operator L_{kdv} (L_{kdv}^A) are also the eigenstates of the linearization operators L_{kh} (L_{kh}^A) of all KdV hierarchy equations. This is the major result of this section. We summarize it in the following theorem.

Theorem 2: *The linearization operators L_{kh} of all KdV hierarchy equations (2.11) expanded around the soliton (2.12) share the same complete set of L_{kdv}-eigenstates (2.34) and (2.35), and the adjoint linearization operators L_{kh}^A of all KdV hierarchy equations (2.11) share the same complete set of L_{kdv}^A-eigenstates (2.38) and (2.39). Furthermore,*

$$L_{kh}\Psi(x,k)=ik\{C(k^2)-C(-4)\}\Psi(x,k), \tag{2.63}$$

$$L_{kh}\Psi_1(x)=0, \quad L_{kh}\Psi_2(x)=-8M(-4)\Psi_1(x), \tag{2.64}$$

$$L_{kh}^A\Phi(x,k)=ik\{C(k^2)-C(-4)\}\Phi(x,k), \tag{2.65}$$

$$L_{kh}^A\Phi_1(x)=0, \quad L_{kh}^A\Phi_2(x)=-8M(-4)\Phi_1(x). \tag{2.66}$$

The proof of this theorem follows readily from the factorization formulas (2.22) and (2.23), the L_{kdv} and L_{kdv}^A eigenfunction relations (2.36), (2.37), (2.40), and (2.41), and L_0^+ and L_0 eigenfunction relations (2.46) and (2.47). With the results of Theorem 2, one can now develop a direct soliton perturbation theory for any KdV hierarchy equation.[6,14,19,20]

An interesting fact which was not appreciated in the past is that, for any constant λ, all the linearly independent solutions to the linearization operator equation

$$L_{kh}u(x)=\lambda u(x) \tag{2.67}$$

are given by the function $\Psi(x,k)$ [see Eq. (2.34)], where

$$\lambda=ik\{C(k^2)-C(-4)\}, \tag{2.68}$$

and k is allowed to be a complex number. This fact follows directly from Theorem 2. To see this, we take $C(z)$ as a power function, $C(z)=z^n$, where n is a positive integer. Notice that operator L_{kh}, as well as Eq. (2.67), is $(2n+1)$-th order. Thus Eq. (2.67) should have $(2n+1)$ linearly independent solutions. But Eq. (2.68) has exactly $(2n+1)$ roots, and each root k would give one solution $\Psi(x,k)$ for (2.67). Thus we do get $(2n+1)$ solutions altogether for (2.67), all of which are in the same form $\Psi(x,k)$ with just different k values. If k is real, $\Psi(x,k)$ is bounded at infinity, and is thus an eigenfunction of operator L_{kh}. If k is truly complex, then the corresponding $\Psi(x,k)$ solution becomes unbounded. Thus we see that, for the integrable KdV hierarchy, the linearization operator equation (2.67) is also completely solvable. Similar facts go to the adjoint operator equation $L_{kh}^{A}\overline{u}=\overline{\lambda}\overline{u}$, where all its solutions are given by $\Phi(x,\overline{k})$ with $\overline{\lambda}=i\overline{k}[C(\overline{k}^2)-C(-4)]$. These facts are additional manifestations of magic associated with integrable equations.

In the next two sections, we will derive similar results for the NLS and mKdV hierarchies. More specifically, we will show that the linearization operators of all NLS (mKdV) hierarchy equations share the same complete sets of eigenfunctions. In each case, we will present the common eigenfunctions explicitly, work out their inner products, and relate them to squared eigenstates of the associated eigenvalue problem.

III. COMPLETE EIGENFUNCTIONS OF LINEARIZATION OPERATORS FOR THE NLS HIERARCHY

The integrable equations associated with the Zakharov–Shabat eigenvalue problem can be written as[30]

$$i\begin{bmatrix} r_t \\ -q_t \end{bmatrix} - \omega(2L_z^+)\begin{bmatrix} r \\ q \end{bmatrix} = 0, \tag{3.1}$$

where the integro-differential operator L_z^+ is

$$L_z^+ = \frac{1}{2i}\begin{bmatrix} \frac{\partial}{\partial x} - 2r\int_{-\infty}^{x} dy\, q & 2r\int_{-\infty}^{x} dy\, r \\ -2q\int_{-\infty}^{x} dy\, q & -\frac{\partial}{\partial x} + 2q\int_{-\infty}^{x} dy\, r \end{bmatrix}, \tag{3.2}$$

and $\omega(k)$ is the dispersion relation of the linearization equation in the r-component. The adjoint operator of L_z^+ is

$$L_z = \frac{1}{2i}\begin{bmatrix} -\frac{\partial}{\partial x} - 2q\int_{x}^{\infty} dy\, r & -2q\int_{x}^{\infty} dy\, q \\ 2r\int_{x}^{\infty} dy\, r & \frac{\partial}{\partial x} + 2r\int_{x}^{\infty} dy\, q \end{bmatrix}. \tag{3.3}$$

For the NLS hierarchy, $\omega(k)$ must be an even function of k, and $q=-r^*$. In this section, we require $\omega(k)$ to be an entire function of k. Then $\omega(k)$ can be expanded into a Taylor series of even powers of k. Thus, we can rewrite the NLS hierarchy (3.1) as

$$i\begin{bmatrix} r_t \\ -q_t \end{bmatrix} - \Omega(\hat{L}^+)\begin{bmatrix} r \\ q \end{bmatrix} = 0, \tag{3.4}$$

where the operator

$$\hat{L}^+ = 4L_z^{+2}, \tag{3.5}$$

or specifically,

$$\hat{L}^{+}=-\begin{bmatrix} \frac{\partial^2}{\partial x^2}-4qr-2r_x\int_{-\infty}^{x}dyq+2r\int_{-\infty}^{x}dyq_y & 2r_x\int_{-\infty}^{x}dyr+2r\int_{-\infty}^{x}dyr_y \\ 2q_x\int_{-\infty}^{x}dyq+2q\int_{-\infty}^{x}dyq_y & \frac{\partial^2}{\partial x^2}-4qr-2q_x\int_{-\infty}^{x}dyr+2q\int_{-\infty}^{x}dyr_y \end{bmatrix}, \tag{3.6}$$

$\Omega(z)$ is an entire function, and $q=-r^*$. The adjoint operator $\hat{L}$ of $\hat{L}^+$ is $4L_z^2$, i.e.,

$$\hat{L}=-\begin{bmatrix} \frac{\partial^2}{\partial x^2}-4qr+2q_x\int_{x}^{\infty}dyr-2q\int_{x}^{\infty}dyr_y & 2q_x\int_{x}^{\infty}dyq+2q\int_{x}^{\infty}dyq_y \\ 2r_x\int_{x}^{\infty}dyr+2r\int_{x}^{\infty}dyr_y & \frac{\partial^2}{\partial x^2}-4qr+2r_x\int_{x}^{\infty}dyq-2r\int_{x}^{\infty}dyq_y \end{bmatrix}. \tag{3.7}$$

When $\Omega(z)=z$, Eq. (3.4) becomes the NLS equation

$$ir_t+r_{xx}+2|r|^2r=0. \tag{3.8}$$

When $\Omega(z)=z^2$, Eq. (3.4) gives the fourth-order NLS hierarchy equation:

$$ir_t-[r_{xxxx}+6(|r|^2r_x)_x+2|r|^2r_{xx}+2r^2r_{xx}^*-2rr_xr_x^*+6|r|^4r]=0. \tag{3.9}$$

Higher order NLS hierarchy equations can be obtained similarly.

The NLS hierarchy (3.4) allows soliton solutions whose amplitude and velocities are free parameters, just like the NLS equation. We can normalize the velocity to be zero by a Galilean transformation, and amplitude to be 1 by a rescaling of variables. Then the normalized soliton simply becomes

$$\begin{bmatrix} r \\ q \end{bmatrix}=\begin{bmatrix} \operatorname{sech} x\, e^{-i\Omega(-1)t} \\ -\operatorname{sech} x\, e^{i\Omega(-1)t} \end{bmatrix}. \tag{3.10}$$

With a change of variables

$$\overline{r}=re^{-i\Omega(-1)t}, \quad \overline{q}=qe^{i\Omega(-1)t}, \tag{3.11}$$

and the bars dropped, the NLS hierarchy (3.4) becomes

$$i\begin{bmatrix} r_t \\ -q_t \end{bmatrix}+[\Omega(-1)-\Omega(\hat{L}^+)]\begin{bmatrix} r \\ q \end{bmatrix}=0, \tag{3.12}$$

and

$$r_0=-q_0=\operatorname{sech} x \tag{3.13}$$

is its soliton solution. We define operators $\hat{L}_0^+$ and $\hat{L}_0$ as $\hat{L}^+$ and $\hat{L}$ with (r,q) replaced by (r_0,q_0), Then one can verify that

$$\hat{L}_0^+\begin{bmatrix} r_0 \\ q_0 \end{bmatrix}=-\begin{bmatrix} r_0 \\ q_0 \end{bmatrix}. \tag{3.14}$$

This relation will be used to prove Theorem 3.

Next, we linearize the NLS hierarchy (3.12) around its soliton (3.13). We write

$$\begin{bmatrix} r \\ q \end{bmatrix} = \begin{bmatrix} r_0 + \tilde{r} \\ q_0 - \tilde{q} \end{bmatrix}, \tag{3.15}$$

where $\tilde{r}, \tilde{q} \ll 1$. When (3.15) is substituted into the NLS hierarchy (3.12), linearization of this equation is

$$i \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}_t + L_{nh} \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix} = 0, \tag{3.16}$$

where L_{nh} is the linearization operator, and the subscript ''nh'' is abbreviation for ''NLS hierarchy.'' The adjoint operator of L_{nh} will be denoted as L_{nh}^A. For the NLS equation, $\Omega(z) = z$. Then linearization of Eq. (3.12) shows that

$$L_{\mathrm{NLS}} = \begin{bmatrix} \dfrac{\partial^2}{\partial x^2} - 1 + 4 \operatorname{sech}^2 x & 2 \operatorname{sech}^2 x \\ -2 \operatorname{sech}^2 x & -\dfrac{\partial^2}{\partial x^2} + 1 - 4 \operatorname{sech}^2 x \end{bmatrix}. \tag{3.17}$$

Its adjoint operator L_{NLS}^A is

$$L_{\mathrm{NLS}}^A = L_{\mathrm{NLS}}^T. \tag{3.18}$$

We introduce the Pauli spin matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.19}$$

which we will use below. Then

$$L_{\mathrm{NLS}} \sigma_3 = \sigma_3 L_{\mathrm{NLS}}^A, \tag{3.20}$$

as $L_{\mathrm{NLS}} \sigma_3$ is a self-adjoint operator.

Similar to the KdV hierarchy, here we also have the important property that L_{NLS} and $\hat{L}_0^+$ are commutable, and L_{NLS}^A and $\hat{L}_0$ are commutable, i.e.,

$$L_{\mathrm{NLS}} \hat{L}_0^+ = \hat{L}_0^+ L_{\mathrm{NLS}}, \tag{3.21}$$

and

$$L_{\mathrm{NLS}}^A \hat{L}_0 = \hat{L}_0 L_{\mathrm{NLS}}^A. \tag{3.22}$$

In addition, the following factorization theorem for L_{nh} and L_{nh}^A holds.

Theorem 3: *For any NLS hierarchy equation (3.12) where $\Omega(z)$ is an entire function, the linearization operator L_{nh} and its adjoint operator L_{nh}^A have the following factorizations*:

$$L_{nh} = \hat{M}(\hat{L}_0^+) L_{\mathrm{NLS}}, \tag{3.23}$$

$$L_{nh}^A = \hat{M}(\hat{L}_0) L_{\mathrm{NLS}}^A, \tag{3.24}$$

where the function $\hat{M}(z)$ is defined as

$$\hat{M}(z) \equiv \frac{\Omega(z) - \Omega(-1)}{z+1}. \tag{3.25}$$

Proof: Recall that an entire function can be expanded into a power series. Thus it suffices to prove the above theorem for $\hat{M}(z)=z^n$, where n is a positive integer. In this case, the NLS hierarchy (3.12) becomes

$$i\begin{bmatrix} r_t \\ -q_t \end{bmatrix}+[(-1)^n-(\hat{L}^+)^n]\begin{bmatrix} r \\ q \end{bmatrix}=0, \tag{3.26}$$

and $\hat{M}(z)=\Sigma_{i=1}^{n} z^{i-1}(-1)^{n-i}$. We first derive the linearization of operator $(\hat{L}^+)^n$. Substituting Eq. (3.15) into (3.6), we find that

$$\hat{L}^+=\hat{L}_0^+ +\mathcal{F}+O(\tilde{r}^2,\tilde{r}\tilde{q},\tilde{q}^2), \tag{3.27}$$

where the operator $\mathcal{F}$ contains all terms which are first order in $\tilde{r}$ and $\tilde{q}$. Even though the expression for $\mathcal{F}$ can be readily obtained, it is not really needed in this proof. From the $\hat{L}^+$ linearization (3.27), we then find linearization of $(\hat{L}^+)^n$ as

$$(\hat{L}^+)^n=(\hat{L}_0^+)^n+\sum_{i=1}^{n} (\hat{L}_0^+)^{i-1}\mathcal{F}(\hat{L}_0^+)^{n-i}+O(\tilde{r}^2,\tilde{r}\tilde{q},\tilde{q}^2). \tag{3.28}$$

When the above equation and (3.15) are substituted into the NLS hierarchy (3.26), we find that the linearization operator L_{nh} as defined in Eq. (3.16) is

$$L_{nh}\begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}=[(-1)^n-(\hat{L}_0^+)^n]\sigma_3\begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}-\sum_{i=1}^{n} (\hat{L}_0^+)^{i-1}\mathcal{F}(\hat{L}_0^+)^{n-i}\begin{bmatrix} r_0 \\ q_0 \end{bmatrix}. \tag{3.29}$$

Recalling the relation (3.14), we can simplify Eq. (3.29) as

$$L_{nh}\begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}=-\hat{M}(\hat{L}_0^+)\left\{(\hat{L}_0^+ +1)\sigma_3\begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}+\mathcal{F}\begin{bmatrix} r_0 \\ q_0 \end{bmatrix}\right\}. \tag{3.30}$$

This equation holds for any positive integers of n. When $n=1$, the NLS hierarchy (3.26) becomes the NLS equation, and $\hat{M}(z)=1$. Thus Eq. (3.30) leads to the relation

$$L_{\mathrm{NLS}}\begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}=-\left\{(\hat{L}_0^+ +1)\sigma_3\begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix}+\mathcal{F}\begin{bmatrix} r_0 \\ q_0 \end{bmatrix}\right\}. \tag{3.31}$$

Of course, this relation can also be checked directly when one derives the specific expression for $\mathcal{F}$ from the linearization of $\hat{L}^+$, and substitutes it into the above equation. Finally, when Eq. (3.31) is inserted into Eq. (3.30), the L_{nh} factorization formula (3.23) is then proved. To prove the L_{nh}^A factorization formula (3.24), we note that $\hat{L}_0$ is the adjoint operator of $\hat{L}_0^+$. Thus, from (3.23), we immediately have

$$L_{nh}^A=L_{\mathrm{NLS}}^A\hat{M}(\hat{L}_0). \tag{3.32}$$

But $\hat{L}_0$ and L_{NLS}^A are commutable [see Eq. (3.22)], thus formula (3.24) is obtained.

Next, we use the factorization formulas (3.23) and (3.24) to construct complete sets of eigenstates for L_{nh} and L_{nh}^A of an arbitrary NLS hierarchy equation. The complete sets of eigenstates for the NLS linearization operators L_{NLS} and L_{NLS}^A have been worked out by Kaup[18] by his observation that these eigenstates were related to the squared Zakharov–Shabat eigenfunctions. We reformulate his results as follows. For the operator L_{NLS}, the continuous and discrete eigenstates are

$$\Psi(x,k)=\frac{1}{(k+i)^2}\begin{bmatrix} -\mathrm{sech}^2 x \\ (\tanh x+ik)^2 \end{bmatrix}e^{-ikx}, \quad -\infty<k<\infty, \tag{3.33}$$

$$\overline{\Psi}(x,k)=\sigma_1\Psi(x,k), \tag{3.34}$$

$$\Psi_1(x)=\operatorname{sech}x\begin{bmatrix}1\\-1\end{bmatrix},\quad \Psi_2(x)=\operatorname{sech}x\tanh x\begin{bmatrix}1\\1\end{bmatrix}, \tag{3.35}$$

$$\Psi_3(x)=\operatorname{sech}x(x\tanh x-1)\begin{bmatrix}1\\1\end{bmatrix},\quad \Psi_4(x)=x\operatorname{sech}x\begin{bmatrix}1\\-1\end{bmatrix}, \tag{3.36}$$

where

$$L_{\mathrm{NLS}}\Psi=(1+k^2)\Psi,\quad L_{\mathrm{NLS}}\overline{\Psi}=-(1+k^2)\overline{\Psi}, \tag{3.37}$$

$$L_{\mathrm{NLS}}\Psi_1=L_{\mathrm{NLS}}\Psi_2=0, \tag{3.38}$$

$$L_{\mathrm{NLS}}\Psi_3=-2\Psi_1,\quad L_{\mathrm{NLS}}\Psi_4=-2\Psi_2. \tag{3.39}$$

Note that these four discrete eigenstates are derived from variations of the NLS soliton with respect to its four free parameters: phase, position, amplitude and velocity. The continuous and discrete eigenstates for the adjoint operator L_{NLS}^A are

$$\Phi(x,k)=-\frac{1}{(k+i)^2}\begin{bmatrix}\operatorname{sech}^2 x\\(\tanh x-ik)^2\end{bmatrix}e^{ikx},\quad -\infty<k<\infty, \tag{3.40}$$

$$\overline{\Phi}(x,k)=\sigma_1\Phi(x,k), \tag{3.41}$$

$$\Phi_j(x)=\sigma_3\Psi_j(x),\quad j=1,2,3,4, \tag{3.42}$$

where

$$L_{\mathrm{NLS}}^A\Phi=(1+k^2)\Phi,\quad L_{\mathrm{NLS}}^A\overline{\Phi}=-(1+k^2)\overline{\Phi}, \tag{3.43}$$

$$L_{\mathrm{NLS}}^A\Phi_1=L_{\mathrm{NLS}}^A\Phi_2=0, \tag{3.44}$$

$$L_{\mathrm{NLS}}^A\Phi_3=-2\Phi_1,\quad L_{\mathrm{NLS}}^A\Phi_4=-2\Phi_2. \tag{3.45}$$

The nonzero inner products between the eigenstates and adjoint eigenstates are

$$\langle\Psi(x,k),\Phi(x,k')\rangle=-2\pi a^2\delta(k-k'), \tag{3.46}$$

$$\langle\overline{\Psi}(x,k),\overline{\Phi}(x,k')\rangle=2\pi a^2\delta(k-k'), \tag{3.47}$$

$$\langle\Psi_1,\Phi_3\rangle=-2=\langle\Psi_3,\Phi_1\rangle, \tag{3.48}$$

$$\langle\Psi_2,\Phi_4\rangle=2=\langle\Psi_4,\Phi_2\rangle. \tag{3.49}$$

Here the inner product $\langle,\rangle$ is as defined in Eq. (2.44), and

$$a=(k-i)/(k+i). \tag{3.50}$$

The closure relation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \delta(x-x') = \int_{-\infty}^{\infty} \frac{1}{2\pi a^2} [\overline{\Psi}(x,k)\overline{\Phi}(x',k) - \Psi(x,k)\Phi(x',k)]dk$$
$$+ \frac{1}{2}[\Psi_2(x)\Phi_4(x') + \Psi_4(x)\Phi_2(x') - \Psi_1(x)\Phi_3(x') - \Psi_3(x)\Phi_1(x')]. \tag{3.51}$$

Commutability relations (3.21) and (3.22) between operators $\hat{L}_0^+$ ($\hat{L}_0$) and $L_{\rm NLS}$ ($L_{\rm NLS}^A$) imply that the eigenstates of $L_{\rm NLS}$ ($L_{\rm NLS}^A$) are also eigenstates of $\hat{L}_0^+$ ($\hat{L}_0$). Indeed, we can show that

$$\hat{L}_0^+ \Psi(x,k) = k^2 \Psi(x,k), \tag{3.52}$$

$$\hat{L}_0^+ \overline{\Psi}(x,k) = k^2 \overline{\Psi}(x,k), \tag{3.53}$$

$$\hat{L}_0^+ \Psi_j(x) = -\Psi_j(x), \quad j = 1,2, \tag{3.54}$$

$$\hat{L}_0^+ \Psi_3(x) = -\Psi_3(x) - 2\Psi_2(x), \tag{3.55}$$

$$\hat{L}_0^+ \Psi_4(x) = -\Psi_4(x) - 2\Psi_1(x), \tag{3.56}$$

and

$$\hat{L}_0 \Phi(x,k) = k^2 \Phi(x,k), \tag{3.57}$$

$$\hat{L}_0 \overline{\Phi}(x,k) = k^2 \overline{\Phi}(x,k), \tag{3.58}$$

$$\hat{L}_0 \Phi_j(x) = -\Phi_j(x), \quad j = 1,2, \tag{3.59}$$

$$\hat{L}_0 \Phi_3(x) = -\Phi_3(x) + 2\Phi_2(x), \tag{3.60}$$

$$\hat{L}_0 \Phi_4(x) = -\Phi_4(x) + 2\Phi_1(x). \tag{3.61}$$

This fact, together with the L_{nh} and L_{nh}^A factorization formulas (3.23) and (3.24), immediately leads to the conclusion that the $L_{\rm NLS}$ ($L_{\rm NLS}^A$) eigenstates are eigenstates of linearization operators L_{nh} (L_{nh}^A) of all NLS hierarchy equations. This result is summarized in the following theorem.

Theorem 4: *The linearization operators L_{nh} of all NLS hierarchy equations (3.12) expanded around the soliton (3.13) share the same complete set of $L_{\rm NLS}$ eigenstates (3.33) to (3.36), and the adjoint operators L_{nh}^A of all NLS hierarchy equations share the same complete set of $L_{\rm NLS}^A$ eigenstates (3.40)–(3.42). Furthermore,*

$$L_{nh}\Psi(x,k) = [\Omega(k^2) - \Omega(-1)]\Psi(x,k), \tag{3.62}$$

$$L_{nh}\overline{\Psi}(x,k) = -[\Omega(k^2) - \Omega(-1)]\overline{\Psi}(x,k), \tag{3.63}$$

$$L_{nh}\Psi_1(x,k) = L_{nh}\Psi_2(x,k) = 0, \tag{3.64}$$

$$L_{nh}\Psi_3(x,k) = -2\hat{M}(-1)\Psi_1, \quad L_{nh}\Psi_4(x,k) = -2\hat{M}(-1)\Psi_2, \tag{3.65}$$

and

$$L_{nh}^A \Phi(x,k) = [\Omega(k^2) - \Omega(-1)]\Phi(x,k), \tag{3.66}$$

$$L_{nh}^{A}\overline{\Phi}(x,k)=-[\Omega(k^2)-\Omega(-1)]\overline{\Phi}(x,k), \tag{3.67}$$

$$L_{nh}^{A}\Phi_1(x,k)=L_{nh}^{A}\Phi_2(x,k)=0, \tag{3.68}$$

$$L_{nh}^{A}\Phi_3(x,k)=-2\hat{M}(-1)\Phi_1,\quad L_{nh}^{A}\Phi_4(x,k)=-2\hat{M}(-1)\Phi_2. \tag{3.69}$$

The equations (3.62)–(3.69) come directly from Eqs. (3.23), (3.24), (3.33)–(3.36), (3.40)–(3.42), and (3.52)–(3.61). A by-product of this theorem is that, for any constants λ and $\overline{\lambda}$, the linear equation $L_{nh}u=\lambda u$ and its adjoint equation $L_{nh}^{A}\overline{u}=\overline{\lambda}\overline{u}$ are completely solvable, and their solutions all have the form $\Psi(x,k)$ and $\Phi(x,\overline{k})$, respectively (k and $\overline{k}$ are now allowed to be complex numbers). This is similar to the KdV hierarchy case (see the end of Sec. II).

Lastly, we would like to draw the reader's attention to the close relationship between eigenstates (3.33)–(3.36) and (3.40)–(3.42) of linearized NLS hierarchy equations (3.12) and squared eigenstates of the Zakharov–Shabat eigenvalue problem with a soliton potential:

$$v_{1x}+i\zeta v_1=q_0v_2, \tag{3.70}$$

$$v_{2x}-i\zeta v_2=r_0v_1, \tag{3.71}$$

where r_0 and q_0 are given in Eq. (3.13). This connection was first mentioned in Ref. 18. It is also hinted by the result that the NLS hierarchy eigenstates (3.33)–(3.36) and (3.40)–(3.42) are also eigenstates of the integro-differential operators $\hat{L}_0^+$ and $\hat{L}_0$. Using standard notations, we define Jost functions of Eqs. (3.70) and (3.71) as

$$\psi(x,\zeta)=\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\rightarrow\begin{bmatrix}0\\ 1\end{bmatrix}e^{i\zeta x},\quad x\rightarrow\infty, \tag{3.72}$$

$$\overline{\psi}(x,\zeta)=\begin{bmatrix}\overline{\psi}_1\\ \overline{\psi}_2\end{bmatrix}\rightarrow\begin{bmatrix}1\\ 0\end{bmatrix}e^{-i\zeta x},\quad x\rightarrow\infty, \tag{3.73}$$

$$\phi(x,\zeta)=\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\rightarrow\begin{bmatrix}1\\ 0\end{bmatrix}e^{-i\zeta x},\quad x\rightarrow-\infty, \tag{3.74}$$

$$\overline{\phi}(x,\zeta)=\begin{bmatrix}\overline{\phi}_1\\ \overline{\phi}_2\end{bmatrix}\rightarrow\begin{bmatrix}0\\ -1\end{bmatrix}e^{i\zeta x},\quad x\rightarrow-\infty. \tag{3.75}$$

For the soliton potential (3.13), these Jost functions have the following simple expressions:

$$\psi(x,\zeta)=\frac{1}{1-2i\zeta}\begin{bmatrix}\operatorname{sech}x\\ \tanh x-2i\zeta\end{bmatrix}e^{i\zeta x}, \tag{3.76}$$

$$\overline{\psi}(x,\zeta)=\frac{1}{1+2i\zeta}\begin{bmatrix}\tanh x+2i\zeta\\ -\operatorname{sech}x\end{bmatrix}e^{-i\zeta x}, \tag{3.77}$$

$$\phi(x,\zeta)=\frac{2i\zeta+1}{2i\zeta-1}\overline{\psi}(x,\zeta),\quad \overline{\phi}(x,\zeta)=\frac{1-2i\zeta}{1+2i\zeta}\psi(x,\zeta). \tag{3.78}$$

An important property is that the squared eigenstates of the Zakharov–Shabat system are eigenfunctions of operators L_z^+ and L_z.[30] Specifically,

$$L_z^+\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}=\zeta\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix},\quad L_z^+\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}=\zeta\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix},\tag{3.79}$$

$$L_z\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}=\zeta\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix},\quad L_z\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}=\zeta\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}.\tag{3.80}$$

These relations hold for general potentials, not just for soliton ones.

Now it is a simple matter to recognize that eigenstates (3.33)–(3.36) and (3.40)–(3.42) are related to the squares of the Zakharov–Shabat eigenstates (3.76)–(3.78) as follows:

$$\Psi(x,k)=\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=k/2},\quad \overline{\Psi}(x,k)=\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-k/2},\tag{3.81}$$

$$\Psi_1(x)=2\left\{\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2}+\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.82}$$

$$\Psi_2(x)=2\left\{\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2}-\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.83}$$

$$\Psi_3(x)=i\left\{\frac{\partial}{\partial\zeta}\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2}+\frac{\partial}{\partial\zeta}\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2}\right\}+2\left\{\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2}-\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.84}$$

$$\Psi_4(x)=i\left\{\frac{\partial}{\partial\zeta}\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2}-\frac{\partial}{\partial\zeta}\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.85}$$

$$\Phi(x,k)=\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=k/2},\quad \overline{\Phi}(x,k)=\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-k/2},\tag{3.86}$$

$$\Phi_1(x)=2\left\{\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2}+\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.87}$$

$$\Phi_2(x)=-2\left\{\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2}-\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.88}$$

$$\Phi_3(x)=i\left\{\frac{\partial}{\partial\zeta}\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2}+\frac{\partial}{\partial\zeta}\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2}\right\}+2\left\{\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2}-\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2}\right\},\tag{3.89}$$

$$\Phi_4(x)=-i\left\{\frac{\partial}{\partial\zeta}\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2}-\frac{\partial}{\partial\zeta}\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2}\right\}.\tag{3.90}$$

It has been shown by Kaup[10] that the sets of squared Zakharov–Shabat eigenstates

$$\left\{\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix},\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix},\zeta\ \text{real},\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2},\frac{\partial}{\partial\zeta}\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}_{\zeta=i/2},\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2},\frac{\partial}{\partial\zeta}\begin{bmatrix}\overline{\phi}_2^2\\ -\overline{\phi}_1^2\end{bmatrix}_{\zeta=-i/2}\right\}\tag{3.91}$$

and

$$\left\{\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix},\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix},\zeta \ \text{real},\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2},\frac{\partial}{\partial\zeta}\begin{bmatrix}\psi_1^2\\ \psi_2^2\end{bmatrix}_{\zeta=i/2},\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2},\frac{\partial}{\partial\zeta}\begin{bmatrix}\overline{\psi}_1^2\\ \overline{\psi}_2^2\end{bmatrix}_{\zeta=-i/2}\right\} \tag{3.92}$$

are each complete (Kaup's proof was actually made for general potentials). Thus, in view of the above relationship between NLS hierarchy eigenstates and squared Zakharov–Shabat eigenstates, the respective completeness of the NLS hierarchy eigenstates (3.33)–(3.36) and adjoint eigenstates (3.40)–(3.42) naturally follows. In fact, this is how Kaup[18] established the completeness of the NLS eigenstates (3.33)–(3.36) in the first place. We also note that Eqs. (3.79) and (3.80), together with the relations (3.81)–(3.90), reproduce Eqs. (3.52)–(3.61) again.

IV. COMPLETE EIGENFUNCTIONS OF LINEARIZATION OPERATORS FOR THE mKdV HIERARCHY

In this section, we extend our results above to the mKdV hierarchy. Similar to the NLS hierarchy, the mKdV hierarchy is also a special case of the more general class of integrable equations (3.1). It can be written as

$$\begin{bmatrix} r_t\\ -q_t\end{bmatrix}+\overline{C}(2L_z^+)\begin{bmatrix} r_x\\ -q_x\end{bmatrix}=0, \tag{4.1}$$

where $q=-r$, L_z^+ is the operator defined in Eq. (3.2), and $\overline{C}(k)$ is the phase velocity of both variables and must be an even function. In this section, we require $\overline{C}(k)$ to be entire. The vector form (4.1) of the mKdV hierarchy is convenient as it then becomes a special class of the general integrable equations (3.1), on which a wealth of information has been obtained.[30] However, in applications, one usually works with only one variable, and the equation is scaler, just like the mKdV equation. It is often awkward to work with the vector form (4.1) of this hierarchy and translate the results into the scaler form in the end. Thus one is motivated to obtain a scaler form for the mKdV hierarchy and work with it from the very beginning. We will do this and construct complete eigenstates of linearization operators for the scaler mKdV hierarchy below.

We first derive the scaler form of the mKdV hierarchy (4.1). Note that $\overline{C}(k)$ is an even and entire function. Thus we can write $\overline{C}(2L_z^+)$ as $C(4L_s^{+2})$, where $C(k)$ is entire. Recalling $q=-r$, the operator $4L_s^{+2}$ becomes

$$4L_s^{+2}=\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B} & \mathcal{A}\end{bmatrix}, \tag{4.2}$$

where operators $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathcal{A}=-\frac{\partial^2}{\partial x^2}-4r^2-2r_x\int_{-\infty}^x dy\, r+2r\int_{-\infty}^x dy\, r_y\,, \tag{4.3}$$

$$\mathcal{B}=-2r_x\int_{-\infty}^x dy\, r-2r\int_{-\infty}^x dy\, r_y\,. \tag{4.4}$$

One can verify that, for any positive integer n,

$$(4L_s^{+2})^n=\begin{bmatrix}\mathcal{A}_n & \mathcal{B}_n\\ \mathcal{B}_n & \mathcal{A}_n\end{bmatrix}, \tag{4.5}$$

where

$$\mathcal{A}_n+\mathcal{B}_n=(\mathcal{A}+\mathcal{B})^n. \tag{4.6}$$

Thus, if we define the operator

$$L_m^+=\tfrac{1}{2}(\mathcal{A}+\mathcal{B}), \tag{4.7}$$

i.e.,

$$L_m^+=-\frac{1}{4}\frac{\partial^2}{\partial x^2}-r^2-r_x\int_{-\infty}^{x}dy\,r, \tag{4.8}$$

then the mKdV hierarchy (4.1) simply becomes

$$r_t+C(4L_m^+)r_x=0. \tag{4.9}$$

This equation is the scaler form of the mKdV hierarchy (4.1). Notice that it is very similar to the KdV hierarchy (2.1). When $C(k)=-k$, Eq. (4.9) gives the mKdV equation

$$r_t+r_{xxx}+6r^2r_x=0. \tag{4.10}$$

Other mKdV hierarchy equations can be obtained by choosing different phase velocity functions $C(k)$.

The rest of this section runs parallel to Secs. II and III. We first note that the mKdV hierarchy (4.9) admits a family of soliton solutions whose amplitude is a free parameter. With a scaling of variables, we can normalize the amplitude to be 1. The normalized soliton is

$$r(x,t)=\operatorname{sech}\{x-C(-1)t\}. \tag{4.11}$$

In moving coordinates,

$$\overline{x}=x-C(-1)t, \quad \overline{t}=t, \tag{4.12}$$

and with the bars dropped, the mKdV hierarchy (4.9) becomes

$$r_t+[C(4L_m^+)-C(-1)]r_x=0, \tag{4.13}$$

and

$$r_0(x)=\operatorname{sech}x \tag{4.14}$$

is its soliton solution.

Next, we linearize the mKdV hierarchy (4.13) around its soliton (4.14). We set

$$r(x,t)=r_0(x)+\tilde{r}(x,t), \tag{4.15}$$

where $\tilde{r}\ll 1$, and substitute it into Eq. (4.13). With the higher order terms in $\tilde{r}$ neglected, Eq. (4.13) becomes the linearized equation

$$\tilde{r}_t+L_{mh}\tilde{r}=0, \tag{4.16}$$

where L_{mh} is the linearization operator. Here the subscript "mh" refers to "mKdV hierarchy." Similar to the KdV and NLS hierarchy cases, we have the following factorization theorem for L_{mh} and its adjoint operator L_{mh}^A.

Theorem 5: *For any mKdV hierarchy equation (4.13) where $C(z)$ is an entire function, the linearization operator L_{mh} and its adjoint operator L_{mh}^A expanded around the soliton (4.14) have the following factorizations:*

$$L_{mh}=\overline{M}(4L_{m0}^{+})L_{\mathrm{mkdv}}, \tag{4.17}$$

$$L_{mh}^{A}=\overline{M}(4L_{m0})L_{\mathrm{mkdv}}^{A}. \tag{4.18}$$

Here operator L_{m0}^{+} is defined as L_{m}^{+} with $r(x,t)$ replaced by $r_0(x)$, i.e.,

$$L_{m0}^{+}=-\frac{1}{4}\frac{\partial^2}{\partial x^2}-r_0^2-r_{0x}\int_{-\infty}^{x}dy\, r_0, \tag{4.19}$$

L_{m0} is the adjoint operator of L_{m0}^{+}, which is

$$L_{m0}=-\frac{1}{4}\frac{\partial^2}{\partial x^2}-r_0^2-r_0\int_{x}^{\infty}dy\, r_{0y}, \tag{4.20}$$

L_{mkdv} is the linearization operator of the mKdV equation,

$$L_{\mathrm{mkdv}}=\frac{\partial^3}{\partial x^3}+(6r_0^2-1)\frac{\partial}{\partial x}+(6r_0^2)_x, \tag{4.21}$$

its adjoint operator L_{mkdv}^{A} is

$$L_{\mathrm{mkdv}}^{A}=-\frac{\partial^3}{\partial x^3}-(6r_0^2-1)\frac{\partial}{\partial x}, \tag{4.22}$$

and function $\overline{M}(z)$ is defined as

$$\overline{M}(z)\equiv\frac{C(-1)-C(z)}{1+z}. \tag{4.23}$$

The proof of this theorem is very similar to those of Theorems 1 and 3 for the KdV and NLS hierarchies, and is thus omitted here.

Also similar to the KdV and NLS hierarchies, we can verify that operators L_{m0}^{+} and L_{mkdv} are commutable, and L_{m0} and L_{mkdv}^{A} are commutable, i.e.,

$$L_{m0}^{+}L_{\mathrm{mkdv}}=L_{\mathrm{mkdv}}L_{m0}^{+}, \quad L_{m0}L_{\mathrm{mkdv}}^{A}=L_{\mathrm{mkdv}}^{A}L_{m0}. \tag{4.24}$$

This fact implies that L_{mkdv} (L_{mkdv}^{A}) and L_{m0}^{+} (L_{m0}) share the same set of eigenfunctions. Then the factorization formulas (4.17) and (4.18) indicate that these same sets of eigenstates are then shared by linearization operators of all mKdV hierarchy equations. The complete sets of eigenstates for mKdV linearization operators L_{mkdv} and L_{mkdv}^{A} have not been reported before in the literature. But we can obtain them from eigenstates of L_{m0}^{+} and L_{m0}, as will be done below.

For general r and q potentials in the Zakharov–Shabat eigenvalue problem (3.70) and (3.71), Eq. (3.79) holds. When $q=-r$, as is the case for the mKdV hierarchy (4.1), it is easy to see that

$$\overline{\phi}_1(x,\zeta)=\phi_2(x,-\zeta), \quad \overline{\phi}_2(x,\zeta)=-\phi_1(x,-\zeta). \tag{4.25}$$

Thus we find from (3.79) that the two relations

$$4L_s^{+2}\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix}=4\zeta^2\begin{bmatrix}\phi_2^2\\ -\phi_1^2\end{bmatrix} \tag{4.26}$$

and

$$4L_s^{+2}\begin{bmatrix}\phi_1^2\\ -\phi_2^2\end{bmatrix}=4\zeta^2\begin{bmatrix}\phi_1^2\\ -\phi_2^2\end{bmatrix}\tag{4.27}$$

hold simultaneously. Subtracting the second equation from the first one, we get

$$4L_s^{+2}\begin{bmatrix}\phi_2^2-\phi_1^2\\ \phi_2^2-\phi_1^2\end{bmatrix}=4\zeta^2\begin{bmatrix}\phi_2^2-\phi_1^2\\ \phi_2^2-\phi_1^2\end{bmatrix}.\tag{4.28}$$

Recalling Eqs. (4.2) and (4.7), the above equation leads to

$$L_m^+(\phi_2^2-\phi_1^2)=\zeta^2(\phi_2^2-\phi_1^2),\tag{4.29}$$

i.e., $\phi_2^2-\phi_1^2$ are eigenfunctions of operator L_m^+. Similarly, we can show that

$$L_m(\psi_1^2+\psi_2^2)=\zeta^2(\psi_1^2+\psi_2^2)\tag{4.30}$$

for general r and q potentials. So $\psi_1^2+\psi_2^2$ are eigenfunctions of operator L_m.

For soliton potentials $r_0=-q_0=\operatorname{sech} x$, the Zakharov–Shabat eigenstates $(\psi_1,\psi_2)^T$ and $(\phi_1,\phi_2)^T$ have been given in Eqs. (3.76) and (3.78). We define two sets of functions

$$\Psi(x,k)\equiv(\phi_2^2-\phi_1^2)|_{\zeta=k/2}=\frac{1}{(k+i)^2}[(\tanh x+ik)^2-\operatorname{sech}^2 x]e^{-ikx},\quad -\infty<k<\infty,\tag{4.31}$$

$$\Psi_1(x)=\operatorname{sech} x\tanh x,\quad \Psi_2(x)=\operatorname{sech} x(1-x\tanh x),\tag{4.32}$$

and

$$\Phi(x,k)\equiv(\psi_1^2+\psi_2^2)|_{\zeta=k/2}=-\frac{1}{(k+i)^2}[(\tanh x-ik)^2+\operatorname{sech}^2 x]e^{ikx},\quad -\infty<k<\infty,\tag{4.33}$$

$$\Phi_1(x)=\operatorname{sech} x,\quad \Phi_2(x)=x\operatorname{sech} x.\tag{4.34}$$

Then from Eqs. (4.29) and (4.30) we see that $\Psi(x,k)$ $[\Phi(x,k)]$ are continuous eigenfunctions of operators L_{m0}^+ (L_{m0}) with

$$L_{m0}^+\Psi(x,k)=\frac{k^2}{4}\Psi(x,k),\tag{4.35}$$

$$L_{m0}\Phi(x,k)=\frac{k^2}{4}\Phi(x,k),\tag{4.36}$$

and $\Psi_j(x)$ $[\Phi_j(x)]$, $j=1,2$, are discrete eigenmodes or generalized eigenmodes of L_{m0}^+ (L_{m0}) with

$$L_{m0}^+\Psi_1=-\tfrac{1}{4}\Psi_1,\quad L_{m0}^+\Psi_2=-\tfrac{1}{4}\Psi_2+\tfrac{1}{2}\Psi_1,\tag{4.37}$$

and

$$L_{m0}\Phi_1=-\tfrac{1}{4}\Phi_1,\quad L_{m0}\Phi_2=-\tfrac{1}{4}\Phi_2+\tfrac{1}{2}\Phi_1.\tag{4.38}$$

Commutability of operators L_{mkdv} (L_{mkdv}^A) and L_{m0}^+ (L_{m0}) indicates that the above eigenmodes of L_{m0}^+ (L_{m0}) are also eigenstates of L_{mkdv} (L_{mkdv}^A). Indeed, the set

$$\{\Psi(x,k),\quad \text{real};\ \Psi_1(x),\Psi_2(x)\}\tag{4.39}$$

are all the linearly independent eigenstates and generalized eigenstates of operator L_{mkdv}, where

$$L_{\mathrm{mkdv}}\Psi(x,k)=ik(k^2+1)\Psi(x,k), \tag{4.40}$$

$$L_{\mathrm{mkdv}}\Psi_1=0, \quad L_{\mathrm{mkdv}}\Psi_2=-2\Psi_1; \tag{4.41}$$

and the set

$$\{\Phi(x,k), \ \ k \ \ \text{real}; \ \ \Phi_1(x),\Phi_2(x)\} \tag{4.42}$$

are all the linearly independent eigenstates and generalized eigenstates of operator L^A_{mkdv}, where

$$L^A_{\mathrm{mkdv}}\Phi(x,k)=ik(k^2+1)\Phi(x,k), \tag{4.43}$$

$$L^A_{\mathrm{mkdv}}\Phi_1=0, \quad L^A_{\mathrm{mkdv}}\Phi_2=-2\Phi_1. \tag{4.44}$$

To calculate the inner products between eigenstates (4.39) and adjoint eigenstates (4.42), we first recall Eqs. (3.81) and (3.86) and inner-product relations such as (3.46) in Sec. III. Explicitly, such inner-product relations tell us that

$$\int_{-\infty}^{\infty}[\phi_2^2(x,k/2)\psi_1^2(x,k'/2)-\phi_1^2(x,k/2)\psi_2^2(x,k'/2)]dx=-2\pi a^2\delta(k-k'), \tag{4.45}$$

and

$$\int_{-\infty}^{\infty}[\phi_2^2(x,k/2)\psi_2^2(x,k'/2)-\phi_1^2(x,k/2)\psi_1^2(x,k'/2)]dx=0, \tag{4.46}$$

where a is given in Eq. (3.50). Adding these two equations together, and recalling Eqs. (4.31) and (4.33), we find the inner product between $\Psi(x,k)$ and $\Phi(x,k')$ as

$$\langle\Psi(x,k),\Phi(x,k')\rangle=-2\pi a^2\delta(k-k'). \tag{4.47}$$

Similarly, we get

$$\langle\Psi_1,\Phi_2\rangle=\langle\Psi_2,\Phi_1\rangle=1, \tag{4.48}$$

and all other inner products are zero.

Next, we show that each of the two sets (4.39) and (4.42) is complete. In Sec. III, we have known that the L_{NLS} eigenstates

$$\left\{\begin{bmatrix}\phi_2^2\\-\phi_1^2\end{bmatrix}_{\zeta=k/2},\begin{bmatrix}\bar{\phi}_2^2\\-\bar{\phi}_1^2\end{bmatrix}_{\zeta=-k/2},k \ \ \text{real}; \ \ \operatorname{sech}x\begin{bmatrix}1\\-1\end{bmatrix},\operatorname{sech}x\tanh x\begin{bmatrix}1\\1\end{bmatrix},\right.$$

$$\left.\operatorname{sech}x(x\tanh x-1)\begin{bmatrix}1\\1\end{bmatrix},x\operatorname{sech}x\begin{bmatrix}1\\-1\end{bmatrix}\right\} \tag{4.49}$$

is complete. Here $(\phi_1,\phi_2)^T$ and $(\bar{\phi}_1,\bar{\phi}_2)^T$ are given in Eq. (3.78). Recalling Eq. (4.25), we see that the set (4.49) with $(\bar{\phi}_2^2,-\bar{\phi}_1^2)^T_{\zeta=-k/2}$ replaced by $(\phi_1^2,-\phi_2^2)^T_{\zeta=k/2}$ is also complete. Thus, for any function $f(x)$ in L_2 functional space, we can expand the vector function $[f(x),f(x)]^T$ into this complete set and get

$$\begin{bmatrix} f(x) \\ f(x) \end{bmatrix} = \int_{-\infty}^{\infty} \left\{ c_1(k) \begin{bmatrix} \phi_2^2 \\ -\phi_1^2 \end{bmatrix}_{\zeta=k/2} + c_2(k) \begin{bmatrix} \phi_1^2 \\ -\phi_2^2 \end{bmatrix}_{\zeta=k/2} \right\} dk + \alpha_1 \operatorname{sech} x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$+ \alpha_2 \operatorname{sech} x \tanh x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_3 \operatorname{sech} x (x \tanh x - 1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_4 x \operatorname{sech} x \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{4.50}$$

where $c_1(k), c_2(k), \alpha_j, 1 \leq j \leq 4$ are constants. Adding the two components of Eq. (4.50) together, we get

$$f(x) = \frac{1}{2} \int_{-\infty}^{\infty} [c_1(k) - c_2(k)](\phi_2^2 - \phi_1^2)_{\zeta=k/2}\, dk + \alpha_2 \operatorname{sech} x \tanh x + \alpha_3 \operatorname{sech} x (x \tanh x - 1). \tag{4.51}$$

In view of the definitions (4.31)–(4.34), Eq. (4.51) means that the set (4.39) is complete. Similarly, the completeness of the set (4.42) can also be proved. The closure relation for these two sets is then

$$-\int_{-\infty}^{\infty} \frac{1}{2\pi a^2} \Psi(x,k) \Phi(x',k)\, dk + \Psi_1(x) \Phi_2(x') + \Psi_2(x) \Phi_1(x') = \delta(x-x'). \tag{4.52}$$

With the complete sets of eigenstates (4.31) to (4.34) for L_{mkdv} and L_{mkdv}^A obtained, then from the factorization formulas (4.17) and (4.18) and eigenfunction relations (4.35)–(4.38) and (4.40), (4.41), (4.43), and (4.44), the following theorem naturally follows.

Theorem 6: *The linearization operators L_{mh} of all mKdV hierarchy equations (4.13) expanded around the soliton (4.14) share the same complete set of L_{mkdv} eigenstates (4.31) and (4.32), and the adjoint linearization operators L_{mh}^A of all mKdV hierarchy equations share the same complete set of L_{mkdv}^A eigenstates (4.33) and (4.34). Furthermore,*

$$L_{mh} \Psi(x,k) = -ik\{C(k^2) - C(-1)\} \Psi(x,k), \tag{4.53}$$

$$L_{mh} \Psi_1(x) = 0, \quad L_{mh} \Psi_2(x) = -2\overline{M}(-1) \Psi_1(x), \tag{4.54}$$

$$L_{mh}^A \Phi(x,k) = -ik\{C(k^2) - C(-1)\} \Phi(x,k), \tag{4.55}$$

$$L_{mh}^A \Phi_1(x) = 0, \quad L_{mh}^A \Phi_2(x) = -2\overline{M}(-1) \Phi_1(x). \tag{4.56}$$

Obviously, Theorem 6 is the counterpart of Theorems 2 and 4. A by-product of this theorem is that the linear equation $L_{mh} u = \lambda u$ and its adjoint equation $L_{mh}^A \overline{u} = \overline{\lambda} \overline{u}$ are completely solvable, and all their solutions are given by $\Psi(x,k)$ and $\Phi(x,\overline{k})$, respectively (with k and $\overline{k}$ complex in general).

V. CONCLUDING REMARKS

In this article, we constructed complete sets of eigenfunctions for linearized KdV, NLS and mKdV hierarchy equations expanded around single-soliton solutions. We showed that all equations within the same integrable hierarchy share the *same* complete sets of eigenfunctions. Furthermore, these eigenstates are intimately connected to the squared eigenstates of the associated eigenvalue problem. We then explicitly obtained these eigenstates, derived their inner products, and established their closure relations. Even though our analysis is just for the KdV, NLS and mKdV hierarchies, it is quite obvious that similar results should hold for other integrable hierarchies as well.

The value of this work, as we see it, is that it allows one to develop a direct soliton perturbation theory for all equations in the same integrable hierarchy. As we mentioned in the Introduction, the direct soliton perturbation theory has a simplistic appeal, and its key component is the complete set of eigenfunctions for the linearized integrable equation expanded around solitons. In the literature, this theory was developed only for the sine-Gordon, Benjamin–Ono, KdV and NLS equations[16–22] (it was also developed for the nonintegrable cubic-quintic NLS equation,[23] but that theory was not complete). Based on our results in this article, however, one can now readily develop direct soliton perturbation theories for the mKdV equation and all higher order KdV, NLS and mKdV hierarchy equations. From a physical point of view, at the moment, only the lowest order hierarchy equations such as the KdV and NLS equations found most physical applications.[1,5] However, higher order hierarchy equations do become relevant in certain situations. One example is the fifth-order KdV hierarchy equation, which arises in shallow water waves.[8,9] So perturbation theories for higher order equations are in order. As physical systems become more complex, and their studies get more quantitative, we expect more applications of soliton perturbation theory on higher order hierarchy equations in the coming years. In some cases, such perturbation theories on higher order equations prove to be very beneficial, as they reveal interesting soliton dynamics which is totally absent in the perturbed lowest order hierarchy equations. The comprehensive study of embedded-soliton dynamics in the perturbed fifth-order KdV hierarchy equation is one good example.[32]

We would like to point out that the key results of this article actually were already hinted by the inverse-scattering-based soliton perturbation theory, which was developed in the 1970s.[10–13] In those works, the soliton perturbation theory was developed for a large class of integrable equations associated with the Zakharov–Shabat eigenvalue problem. For all those equations, the expansion basis for the potentials was the same squared eigenstates of the Zakharov–Shabat system. The main contribution of this article is that we have shown those squared eigenstates also solve the linearized equations of an entire hierarchy expanded around soliton solutions, thus they are also the expansion basis in the direct soliton perturbation theory. This connection indicates that the direct soliton perturbation theory and inverse-scattering soliton perturbation theory are intimately related.

Lastly, we note that the eigenfunctions we constructed in this article are only for integrable equations linearized around single-soliton solutions. However, these results can be extended to linearization around any time-dependent solution such as a multi-soliton solution. In this general case, the elegant factorization results of linearization operators for the single-soliton case (see Theorems 1, 3 and 5) are no longer valid. But we can still show that the squared eigenstates of the eigenvalue problem also solve the linearized equation, thus they still form the complete set of eigenfunctions for the linearization operator around any time-dependent solution. This fact then allows one to develop a direct multi-soliton perturbation theory for a large class of integrable equations including the KdV, NLS and mKdV hierarchies. We will report these results in a future article.

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