Integrable properties of the general coupled nonlinear Schrödinger equations

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In this paper, a general integrable coupled nonlinear Schrödinger system is investigated. In this system, the coefficients of the self-phase modulation, cross-phase modulation, and four-wave mixing terms are more general while still maintaining integrability. The *N*-soliton solutions in this system are obtained by the Riemann–Hilbert method. The collision dynamics between two solitons is also analyzed. It is shown that this collision exhibits some new phenomena (such as soliton reflection) which have not been seen before in integrable systems. In addition, the recursion operator and conservation laws for this system are also derived. © *2010 American Institute of Physics.* [doi:10.1063/1.3290736]

I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation has been widely recognized as a ubiquitous mathematical model for describing the evolution of a slowly varying wave packet in a general nonlinear wave system; thus, it plays an important role in a wide range of physical subjects such as nonlinear optics,[1,2] water waves,[3] and plasma physics. The discovery of integrability of this equation by Zakharov and Shabat in 1971 (Ref. 4) has made a big impact on the studies of all these areas. In certain physical situations, two or more wave packets of different carrier frequencies appear simultaneously, and their interactions are then governed by the coupled NLS equations. Examples include nonlinear light propagation in a birefringent optical fiber or a wavelength-division-multiplexed system,[1,2,5] the evolution of two surface wave packets in deep water,[6] the interaction of Bloch-wave packets in a periodic system,[7] spinor Bose–Einstein condensates,[8,9] and so on. On the integrability of these coupled systems, Manakov[10] showed first that if the coupling is only through cross-phase modulation (XPM), and the XPM coefficient is equal to the self-phase modulation (SPM) coefficient, then this system (now called the Manakov system) is integrable. Multisoliton solutions in the Manakov system have also been extensively investigated by the inverse scattering method and the Hirota method,[10–13] and an interesting phenomenon of polarization rotation after collision has been found. Later studies revealed that when the XPM coefficient is opposite of the SPM coefficient, the system is still integrable.[14–16] The two- and three-soliton solutions in this model were obtained by the Hirota method in Ref. 17, and a phenomenon of energy redistribution between solitons after collision was reported. More general forms of integrable coupled NLS equations were also mentioned in Refs. 14 and 16, but multisoliton solutions

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in such systems have not been examined yet. Given the importance of the general coupled NLS equations for various physical problems, these equations deserve careful and detailed investigations.

In this paper, we consider the general coupled NLS system (GCNLSS) of the form

$$ip_t + p_{xx} + 2(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*)p = 0,$$

$$iq_t + q_{xx} + 2(a|p|^2 + c|q|^2 + bpq^* + b^*qp^*)q = 0, \tag{1}$$

where a and c are real constants, b is a complex constant, and "$*$" denotes complex conjugation. Physically, a and c terms describe the SPM and XPM effects, and b and b^* terms describe the four-wave mixing effects. When $a=c$ and $b=0$, this system reduces to the Manakov system.[10] When $a=-c$ and $b=0$, this system reduces to the case considered in Refs. 15 and 17. In this paper, we allow constants a, b, and c to be arbitrary, so Eq. (1) is much more general and may be used to describe a wider variety of physical processes. Equation (1) is still integrable for arbitrary a, b, and c values. Its Lax pair is[18]

$$\phi_x = \begin{pmatrix} i\lambda & 0 & p \\ 0 & i\lambda & q \\ r_1 & r_2 & -i\lambda \end{pmatrix}\phi, \tag{2a}$$

$$\phi_t = \begin{pmatrix} -2i\lambda^2 - ipr_1 & -ipr_2 & ip_x - 2p\lambda \\ -iqr_1 & -2i\lambda^2 - iqr_2 & iq_x - 2\lambda q \\ -ir_{1x} - 2\lambda r_1 & -ir_{2x} - 2\lambda r_2 & ipr_1 + iqr_2 + 2i\lambda^2 \end{pmatrix}\phi, \tag{2b}$$

where $r_1=-ap^*-bq^*$, $r_2=-b^*p^*-cq^*$, λ is a spectral parameter, and $\phi(x,t,\lambda)$ is a vector or a matrix function. It is noted that Eq. (1) is more general than the integrable two-component NLS equations discussed in Ref. 16 [see Eq. (3) there].

In this paper, the general integrable coupled NLS system (1) will be carefully analyzed. First, we will derive its N-soliton solutions by the Riemann–Hilbert method. Using these exact solutions, we will examine the collision dynamics between two solitons and relate the after-collision soliton parameters with the precollision ones. From these analytical formulas and selective solution plots, we will show that the soliton collision in this general system exhibits some new phenomena (such as soliton reflection) which have not been seen in integrable systems. In addition, we will derive the recursion operator for this system. As a result, the whole hierarchy associated with this system will be obtained. Lastly, the infinite conservation laws of this system are also derived. These results significantly enhance our understanding of the integrable properties of this GCNLSS.

II. THE RIEMANN–HILBERT FORMULATION

In this section, we present the scattering and inverse scattering methods for Eq. (1) using the Riemann–Hilbert formulation.[16,19,20] These results will lay the ground for us to derive the N-soliton solutions in Sec. III.

Let us consider Eq. (1) for the localized solutions, i.e., we assume that potentials p and q decay to zero sufficiently fast as $x,t\to\pm\infty$. In the Riemann–Hilbert formulation, we treat ϕ in Eq. (2) as a fundamental matrix of the two linear equations. From (2), we note that when $x,t\to\pm\infty$, one has $\phi=e^{-i\lambda\Lambda x+2i\lambda^2\Lambda t}$, where $\Lambda=\mathrm{diag}(-1,-1,1)$. This motivates us to introduce the variable transformation

$$\phi = Je^{-i\lambda\Lambda x+2i\lambda^2\Lambda t}, \tag{3}$$

where J is (x,t)-independent at infinity. Inserting (3) into (2), we get

$$J_x = -i\lambda[\Lambda,J] + QJ, \tag{4a}$$

$$J_t = 2i\lambda^2[\Lambda, J] + VJ, \tag{4b}$$

with

$$Q = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ r_1 & r_2 & 0 \end{pmatrix}, \quad V = -2\lambda Q + \begin{pmatrix} -ipr_1 & -ipr_2 & ip_x \\ -iqr_1 & -iqr_2 & iq_x \\ -ir_{1x} & -ir_{2x} & ipr_1 + iqr_2 \end{pmatrix}.$$

Here $[\Lambda, J] = \Lambda J - J\Lambda$ is the commutator, $\mathrm{tr}(Q) = \mathrm{tr}(V) = 0$, and

$$Q^\dagger = -BQB^{-1}, \tag{5}$$

where † represents the Hermitian of a matrix, and

$$B = \begin{pmatrix} a & b^* & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the scattering problem, we first introduce matrix Jost solutions $J_\pm(x,\lambda)$ of (4a) with the asymptotic condition

$$J_\pm \to I \quad \text{when } x \to \pm\infty, \tag{6}$$

where I is a 3×3 unit matrix. Here, the subscript in $J_\pm$ refers to which end of the x-axis the boundary conditions are set. Then, due to $\mathrm{tr}(Q) = 0$ and Abel's formula, we have $\det(J_\pm) = 1$ for all x. Next we denote $E = e^{-i\lambda\Lambda x}$. Since $\Psi \equiv J_+E$ and $\Phi \equiv J_-E$ are both solutions of (2a), they must be linearly related, i.e.,

$$J_-E = J_+ES(\lambda), \quad \lambda \in \mathbb{R}, \tag{7}$$

where

$$S(\lambda) = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

is the scattering matrix and $\mathbb{R}$ is the set of real numbers. Notice that $\det(S(\lambda)) = 1$ since $\det(J_\pm) = 1$. In addition, (Φ, Ψ) satisfies the spectral equation (2a), i.e.,

$$\phi_x + i\lambda\Lambda\phi = Q\phi. \tag{8}$$

If we treat the $Q\phi$ term in the above equation as an inhomogeneous term and notice that the solution to the homogeneous equation on its left hand side is E, then using the method of variation in parameters as well as the boundary condition (6), we can turn (8) into Volterra integral equations for (Φ, Ψ). These equations can be cast in terms of $J_\pm$ as

$$J_-(\lambda,x) = I + \int_{-\infty}^{x} e^{i\lambda\Lambda(y-x)}Q(y)J_-(\lambda,y)e^{i\lambda\Lambda(x-y)}\mathrm{d}y, \tag{9a}$$

$$J_+(\lambda,x) = I - \int_{x}^{\infty} e^{i\lambda\Lambda(y-x)}Q(y)J_+(\lambda,y)e^{i\lambda\Lambda(x-y)}\mathrm{d}y. \tag{9b}$$

Thus, $J_\pm$ allows analytical continuations off the real axis $\lambda \in \mathbb{R}$ as long as the integrals on their right hand sides converge. We can easily see that the integral equation for the third column of J_- contains only the exponential factor $e^{i\lambda(x-y)}$ which decays when λ is in the upper half plane $\mathbb{C}_+$, and the integral equation for the first two columns of J_+ contains only the exponential factor $e^{i\lambda(y-x)}$

which also decays when λ is in the upper half plane $\mathbb{C}_+$. Thus, these three columns can be analytically continued to the upper half plane $\lambda \in \mathbb{C}_+$. Similarly, we find that the first two columns of J_- and the third column of J_+ can be analytically continued to the lower half plane $\lambda \in \mathbb{C}_-$. If we express (Φ, Ψ) as a collection of columns

$$\Phi = [\phi_1, \phi_2, \phi_3], \quad \Psi = [\psi_1, \psi_2, \psi_3],$$

then the Jost solutions

$$P^+ = [\psi_1, \psi_2, \phi_3] e^{i\lambda\Lambda x} = J_+ H_2 + J_- H_1 \tag{10}$$

are analytic in $\lambda \in \mathbb{C}_+$, and the Jost solutions $[\phi_1, \phi_2, \psi_3] e^{i\lambda\Lambda x}$ are analytic in $\lambda \in \mathbb{C}_-$. Here, $H_1 = \mathrm{diag}(0,0,1)$, $H_2 = \mathrm{diag}(1,1,0)$. In addition, from the Volterra integral equation (9), we find that

$$P^+(x,\lambda) \to I \quad \text{as} \quad \lambda \in \mathbb{C}_+ \to \infty,$$

$$[\phi_1, \phi_2, \psi_3] e^{i\lambda\Lambda x} \to I \quad \text{as} \quad \lambda \in \mathbb{C}_- \to \infty.$$

Next we construct the analytic counterpart of P^+ in the $\mathbb{C}_-$. Note that the adjoint equation of (4a) reads as

$$K_x = -i\lambda[\Lambda, K] - KQ. \tag{11}$$

The inverse matrices $J_\pm^{-1}$ solve this adjoint equation. If we express Φ^{-1} and Ψ^{-1} as a collection of rows,

$$\Phi^{-1} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}, \quad \Psi^{-1} = \begin{bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \end{bmatrix},$$

then by similar techniques as used above, we can show that adjoint Jost solutions

$$P^- = e^{-i\lambda\Lambda x} \begin{bmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\phi}_3 \end{bmatrix} = H_2 J_+^{-1} + H_1 J_-^{-1} \tag{12}$$

are analytic for $\lambda \in \mathbb{C}_-$. In the same way, we see that

$$P^-(x,\lambda) \to I \quad \text{as} \quad \lambda \in \mathbb{C}_- \to \infty.$$

Now we have constructed two matrix functions P^+ and P^-, which are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. On the real line, they are related by

$$P^-(x,\lambda) P^+(x,\lambda) = G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{13}$$

where

$$G(x,\lambda) = E(H_1 + H_2 S)(H_1 + S^{-1} H_2) E^{-1} = E \begin{pmatrix} 1 & 0 & s_{13} \\ 0 & 1 & s_{23} \\ s_{21}s_{32} - s_{31}s_{22} & s_{31}s_{12} - s_{11}s_{32} & 1 \end{pmatrix} E^{-1}. \tag{14}$$

Equation (13) is just a matrix Riemann–Hilbert problem. The asymptotics

$$P^\pm(x,\lambda) \to I \quad \text{as} \quad \lambda \to \infty \tag{15}$$

provides the canonical normalization condition for this Riemann–Hilbert problem.

The solution to the Riemann–Hilbert problem (13) will not be unique unless the zeros of $\det(P^+)$ and $\det(P^-)$ in the upper and lower half of the λ plane are also specified, and the kernel structures of $P^{\pm}$ at these zeros are provided. From the definitions of $P^{\pm}$ as well as the scattering relations between J^+ and J^-, we see that

$$\det P^+(x,\lambda) = s_{33}(\lambda), \quad \det P^-(x,\lambda) = \hat{s}_{33}(\lambda), \tag{16}$$

where $\hat{s}_{33}=(S^{-1})_{33}=s_{11}s_{22}-s_{12}s_{21}$. Suppose that s_{33} has zeros $\{\lambda_k \in \mathbb{C}_+, 1\le k\le N\}$, and $\hat{s}_{33}$ has zeros $\{\hat{\lambda}_k \in \mathbb{C}_-, 1\le k\le N\}$. For simplicity, we assume that all zeros $\{(\lambda_k,\hat{\lambda}_k), k=1,\ldots,N\}$ are simple zeros of $(s_{33},\hat{s}_{33})$, which is the generic case. In this case, each of $\ker P^+(\lambda_k)$ and $\ker P^-(\hat{\lambda}_k)$ contains only a single column vector v_k and a row vector $\hat{v}_k$, respectively,

$$P^+(\lambda_k)v_k = 0, \quad \hat{v}_k P^-(\hat{\lambda}_k) = 0, \quad 1 \le k \le N. \tag{17}$$

If the Riemann–Hilbert problem (13) with the normalization condition (15) and zero structure (17) can be solved, then one can readily reconstruct the potential Q as follows. Notice that P^+ is the solution of the spectral problem (4a). Thus, if we expand P^+ at large λ as

$$P^+(x,\lambda) = I + \frac{1}{\lambda}P_1^+(x) + O(\lambda^{-2}), \quad \lambda \to \infty, \tag{18}$$

and inserting this expansion into (4a), then by comparing $O(1)$ terms, we find that

$$Q = i[\Lambda, P_1^+] = \begin{pmatrix} 0 & 0 & -2iP_{13} \\ 0 & 0 & -2iP_{23} \\ 2iP_{31} & 2iP_{32} & 0 \end{pmatrix}. \tag{19}$$

Thus, the potentials p and q can be reconstructed as

$$p = -2iP_{13}, \quad q = -2iP_{23}, \tag{20}$$

where $P_1^+=(P_{ij})$.

The symmetry property (5) of the potential Q gives rise to symmetry properties in the scattering matrix as well as in the Jost functions. Taking the Hermitian of the spectral equation (4a), we have

$$(J_{\pm}^{\dagger})_x = -i\lambda[\Lambda, J_{\pm}^{\dagger}] + J_{\pm}^{\dagger}Q^{\dagger}. \tag{21}$$

Right-multiplying this equation by B and using (5) yields

$$(J_{\pm}^{\dagger}B)_x = -i\lambda[\Lambda, J_{\pm}^{\dagger}B] - J_{\pm}^{\dagger}BQ. \tag{22}$$

This equation shows that $J_{\pm}^{\dagger}B$ is also a fundamental matrix of the adjoint equation (11). Recalling that $J_{\pm}^{-1}$ satisfies (11) as well, we see that $J_{\pm}^{\dagger}B$ must be linearly related to $J_{\pm}^{-1}$, i.e., $J_{\pm}^{\dagger}B=CJ_{\pm}^{-1}$, where C is x-independent. Using the large-x boundary conditions of $J_{\pm}$, we find that $C=B$. So we find that $J_{\pm}$ satisfies the involution property

$$J_{\pm}^{\dagger} = BJ_{\pm}^{-1}B^{-1}.$$

From this property as well as definitions (10) and (12) for $P^{\pm}$, we can see that the analytic solution $P^{\pm}$ satisfies the involution property

$$(P^+)^{\dagger}(\lambda^*) = BP^-(\lambda)B^{-1}. \tag{23}$$

Then, in view of the relation $J_-E=J_+ES$, we see that S satisfies the involution property

$$S^{\dagger}(\lambda^{*}) = BS^{-1}(\lambda)B^{-1}. \tag{24}$$

Due to this involution property, we have the symmetry relation

$$\hat{\lambda}_k = \lambda_k^{*} \tag{25}$$

for the zeros of $s_{33}(\lambda)$ and $\hat{s}_{33}(\lambda)$. To obtain the symmetry properties for the eigenvectors v_k and $\hat{v}_k$, we take the Hermitian of the first equation in (17). Upon the use of the involution properties (23) and (25), we get

$$v_k^{\dagger} B P^{-}(\hat{\lambda}_k) = 0. \tag{26}$$

Then, comparing it with the second equation in (17), we see that eigenvectors $(v_k, \hat{v}_k)$ satisfy the involution property

$$\hat{v}_k = v_k^{\dagger} B. \tag{27}$$

To obtain soliton solutions, we set $G=I$ in (13). The solutions to this special Riemann–Hilbert problem have been derived from Refs. 19, 21, and 22 and the result is

$$P^{+}(\lambda) = I + \sum_{j,k=1}^{N} \frac{v_j (M^{-1})_{jk} \hat{v}_k}{\lambda - \hat{\lambda}_k}, \tag{28}$$

where the matrix M is given by

$$M_{jk} = \left(\frac{\hat{v}_j v_k}{\lambda_j^{*} - \lambda_k} \right).$$

The zeros λ_k and $\hat{\lambda}_k$ are time independent. To find the spatial and temporal evolutions for vectors $v_k(x,t)$, we take the x-derivative to the equation $P^{+}v_k=0$. By using (4a), one gets

$$P^{+}(\lambda_k, x)\left(\frac{\mathrm{d}v_k}{\mathrm{d}x} + i\lambda_k \Lambda v_k \right) = 0, \tag{29}$$

thus

$$\frac{\mathrm{d}v_k}{\mathrm{d}x} = -i\lambda_k \Lambda v_k. \tag{30}$$

The time dependence of v_k can be found in a similar way. Combining these results, we get

$$v_k(x,t) = e^{-i\lambda_k \Lambda x + 2i\lambda_k^2 \Lambda t} v_{k0}, \tag{31a}$$

$$\hat{v}_k(x,t) = \hat{v}_{k0} e^{i\lambda_k^{*} \Lambda x - 2i\lambda_k^{*2} \Lambda t} B, \tag{31b}$$

where $(v_{k0}, \hat{v}_{k0})$ are now constants.

III. THE *N*-SOLITON SOLUTIONS AND THEIR DYNAMICS

The N-soliton solutions in Eq. (1) are obtained from the analytical function P^{+} in (28) together with the potential reconstruction formula (20), and these solutions are

$$p(x,t) = -2iP_{13} = -2i\left(\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k \right)_{13}, \tag{32}$$

$$q(x,t) = -2iP_{23} = -2i\left(\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k\right)_{23}. \tag{33}$$

The negative signs in p and q can be scaled out. Thus, the general N-soliton solution in system (1) can be written out explicitly as

$$\begin{bmatrix} p(x,t) \\ q(x,t) \end{bmatrix} = 2i \sum_{j,k=1}^{N} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} e^{\theta_k^* - \theta_j} (M^{-1})_{jk}, \tag{34}$$

where

$$M_{jk} = \frac{1}{\lambda_j^* - \lambda_k} [(a\alpha_j^* \alpha_k + b\beta_j^* \alpha_k + b^* \alpha_j^* \beta_k + c\beta_j^* \beta_k) e^{-(\theta_k + \theta_j^*)} + e^{\theta_k + \theta_j^*}], \tag{35}$$

and $\theta_k = -i\lambda_k x + 2i\lambda_k^2 t$. Here, we have chosen $v_{k0} = [\alpha_k, \beta_k, 1]^T$ without loss of generality. In what follows, we will investigate the dynamics of the one- and two-soliton solutions in this system in more detail.

A. Single-soliton solutions

To get the single-soliton solutions, we set $N=1$ in formula (34). If $a\alpha_1^* \alpha_1 + b\beta_1^* \alpha_1 + b^* \alpha_1^* \beta_1 + c\beta_1^* \beta_1 < 0$, then this single soliton will have a singularity. In this paper, we require this quantity to be always positive to avoid singular solutions. Then, denoting

$$\lambda_1 = \lambda_{11} + i\lambda_{12}, \quad a\alpha_1^* \alpha_1 + b\beta_1^* \alpha_1 + b^* \alpha_1^* \beta_1 + c\beta_1^* \beta_1 = e^{-2\eta_1} > 0,$$

this single-soliton solution can be written as

$$\begin{bmatrix} p \\ q \end{bmatrix} = \mathbf{r}_1 e^{\eta_1} (2\lambda_{12}) e^{\theta_1^* - \theta_1} \operatorname{sech}(\theta_1^* + \theta_1 + \eta_1), \tag{36}$$

where $\mathbf{r}_1 = [\alpha_1, \beta_1]^T$. Let us also introduce the polarization vector

$$\mathbf{u}_1 \equiv \mathbf{r}_1 e^{\eta_1} = \frac{\mathbf{r}_1}{\sqrt{\langle \mathbf{r}_1, \mathbf{r}_1 \rangle}}, \tag{37}$$

where the inner product $\langle \cdot, \cdot \rangle$ between two column vectors of length 2 here is defined as

$$\langle \mathbf{f}_1, \mathbf{f}_2 \rangle \equiv \mathbf{f}_1^\dagger B_1 \mathbf{f}_2, \tag{38}$$

with

$$B_1 = \begin{bmatrix} a & b^* \\ b & c \end{bmatrix}.$$

According to this inner product definition,

$$\langle \mathbf{r}_1, \mathbf{r}_1 \rangle = a\alpha_1^* \alpha_1 + b\beta_1^* \alpha_1 + b^* \alpha_1^* \beta_1 + c\beta_1^* \beta_1, \tag{39}$$

$$\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 1, \tag{40}$$

and the soliton (36) will be nonsingular when $\langle \mathbf{r}_1, \mathbf{r}_1 \rangle > 0$. It is noted that for the Manakov system, $B_1 = I$, thus $\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = \mathbf{u}_1^\dagger \mathbf{u}_1$. In this case, due to Eq. (40) above, the polarization vector $\mathbf{u}_1$ is a unit vector (i.e., $\mathbf{u}_1^\dagger \mathbf{u}_1 = 1$); thus, its two components are less than or equal to 1 in magnitude. However, for other (a,b,c) values, $\langle \mathbf{r}_1, \mathbf{r}_1 \rangle$ can be very small. In that case, the polarization vector $\mathbf{u}_1$ would no longer be a unit vector, and it may have very large component values. This is a new feature of

the GCNLSS (1), and it will have important consequences for soliton collisions in this system (see below).

Under the above notations, we see that

$$\theta_1 + \theta_1^* = 2\lambda_{12}(x - 4\lambda_{11}t), \quad \theta_1^* - \theta_1 = 2i\lambda_{11}x - 4i(\lambda_{11}^2 - \lambda_{12}^2)t.$$

Thus, the single-soliton solution (36) can be rewritten as

$$(p,q)^T(x,t) = \mathbf{u}_1 \cdot 2\lambda_{12}\,\mathrm{sech}[2\lambda_{12}(x - 4\lambda_{11}t) + \eta_1]\exp\{2i\lambda_{11}x - 4i(\lambda_{11}^2 - \lambda_{12}^2)t\}. \tag{41}$$

Its amplitude function has the shape of a hyperbolic secant with peak amplitude of $2\lambda_{12}\mathbf{u}_1$, and its velocity is $4\lambda_{11}$. Notice that this soliton's peak amplitude depends not only on the imaginary part λ_{12} of the eigenvalue λ_1, but also on the polarization vector $\mathbf{u}_1$. While the eigenvalue λ_1 does not change when this soliton collides with another soliton, the polarization vector $\mathbf{u}_1$ does (see below). This means that the soliton's amplitude can change after collision. The power of this soliton is

$$P \equiv \int_{-\infty}^{\infty} (|p|^2 + |q|^2)\mathrm{d}x = 4\lambda_{12}\mathbf{u}_1^\dagger\mathbf{u}_1. \tag{42}$$

In Sec. II, we always let the zeros λ_k be in the upper half plane and its conjugate in the lower half plane, so the imaginary part λ_{12} of λ_1 is always positive. In the Manakov system, the polarization vector $\mathbf{u}_1$ is a unit vector. In that case, the soliton's power depends only on λ_{12}, thus does not change after soliton collisions. However, for more general (a,b,c) values, the polarization vector $\mathbf{u}_1$ is not a unit vector; thus, the power of the soliton depends on $\mathbf{u}_1$ and can change after soliton collisions. This means that during soliton collisions in the general system (1), the power can transfer from one soliton to the other. More details on soliton collisions will be examined next.

B. Collisions of two solitons

The two-soliton solution in Eq. (1) corresponds to $N=2$ in the general N-soliton solution (34). This solution can also be written out explicitly. Below we examine this solution with different velocity parameters $\lambda_{11} \neq \lambda_{21}$, i.e., we consider the collision between two solitons in Eq. (1). Suppose that $\lambda_{11} < \lambda_{21}$, i.e., at $t=-\infty$, soliton-1 is at the right side of soliton-2 and moves slower. After collision, they will scatter with their polarizations, positions, and phases all changed. These changes can be explained analytically by means of asymptotic analysis. Indeed, when $t \to \pm\infty$, one can readily find that the solution (34) approaches two single solitons as

$$(p,q)^T \to \mathbf{r}_1^- e^{\eta_1^-}(2\lambda_{12})e^{\theta_1^* - \theta_1}\,\mathrm{sech}(\theta_1^* + \theta_1 + \eta_1^-) + \mathbf{r}_2^- e^{\eta_2^-}(2\lambda_{22})e^{\theta_2^* - \theta_2}\,\mathrm{sech}(\theta_2^* + \theta_2 + \eta_2^-), \quad t \to -\infty \tag{43}$$

and

$$(p,q)^T \to \mathbf{r}_1^+ e^{\eta_1^+}(2\lambda_{12})e^{\theta_1^* - \theta_1}\,\mathrm{sech}(\theta_1^* + \theta_1 + \eta_1^+) + \mathbf{r}_2^+ e^{\eta_2^+}(2\lambda_{22})e^{\theta_2^* - \theta_2}\,\mathrm{sech}(\theta_2^* + \theta_2 + \eta_2^+), \quad t \to +\infty, \tag{44}$$

where

$$e^{-2\eta_k^-} = \langle \mathbf{r}_k^-, \mathbf{r}_k^- \rangle, \quad e^{-2\eta_k^+} = \langle \mathbf{r}_k^+, \mathbf{r}_k^+ \rangle, \quad k = 1,2. \tag{45}$$

Here, the intermediate variables $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ in the solution (34) are related to the pre- and after-collision soliton vectors $\mathbf{r}_1^-$ and $\mathbf{r}_2^+$ as

$$\mathbf{r}_1^- = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \cdot \frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2}, \quad \mathbf{r}_2^+ = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \cdot \frac{\lambda_2 - \lambda_1^*}{\lambda_2 - \lambda_1}, \tag{46}$$

and these soliton vectors $(\mathbf{r}_1^-, \mathbf{r}_2^-, \mathbf{r}_1^+, \mathbf{r}_2^+)$ are inter-related as

$$\mathbf{r}_1^+ = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2^*}\left\{\mathbf{r}_1^- - \mathbf{r}_2^+ \frac{\langle \mathbf{r}_2^+, \mathbf{r}_1^- \rangle}{\langle \mathbf{r}_2^+, \mathbf{r}_2^+ \rangle} \cdot \frac{\lambda_2 - \lambda_2^*}{\lambda_1 - \lambda_2^*}\right\}, \tag{47}$$

$$\mathbf{r}_2^- = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1^*}\left\{\mathbf{r}_2^+ - \mathbf{r}_1^- \frac{\langle \mathbf{r}_1^-, \mathbf{r}_2^+ \rangle}{\langle \mathbf{r}_1^-, \mathbf{r}_1^- \rangle} \cdot \frac{\lambda_1 - \lambda_1^*}{\lambda_2 - \lambda_1^*}\right\}. \tag{48}$$

Defining the polarization vectors of these single solitons by

$$\mathbf{u}_k^- \equiv \mathbf{r}_k^- e^{\eta_k^-} = \frac{\mathbf{r}_k^-}{\sqrt{\langle \mathbf{r}_k^-, \mathbf{r}_k^- \rangle}}, \quad \mathbf{u}_k^+ \equiv \mathbf{r}_k^+ e^{\eta_k^+} = \frac{\mathbf{r}_k^+}{\sqrt{\langle \mathbf{r}_k^+, \mathbf{r}_k^+ \rangle}}, \tag{49}$$

then when $t \to -\infty$,

$$(p,q)^T \to \mathbf{u}_1^-(2\lambda_{12}) e^{\theta_1^* - \theta_1} \operatorname{sech}(\theta_1^* + \theta_1 + \eta_1^-) + \mathbf{u}_2^-(2\lambda_{22}) e^{\theta_2^* - \theta_2} \operatorname{sech}(\theta_2^* + \theta_2 + \eta_2^-), \tag{50}$$

and when $t \to +\infty$,

$$(p,q)^T \to \mathbf{u}_1^+(2\lambda_{12}) e^{\theta_1^* - \theta_1} \operatorname{sech}(\theta_1^* + \theta_1 + \eta_1^+) + \mathbf{u}_2^+(2\lambda_{22}) e^{\theta_2^* - \theta_2} \operatorname{sech}(\theta_2^* + \theta_2 + \eta_2^+). \tag{51}$$

Inserting relations (47) and (48) into the polarization-vector expression (49), we find that the polarization vectors of these single-soliton pre- and after collision are related by

$$\mathbf{u}_1^+ = e^{\phi} \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2^*}\left\{\mathbf{u}_1^- - \mathbf{u}_2^+ \langle \mathbf{u}_2^+, \mathbf{u}_1^- \rangle \cdot \frac{\lambda_2 - \lambda_2^*}{\lambda_1 - \lambda_2^*}\right\}, \tag{52}$$

$$\mathbf{u}_2^- = e^{\phi} \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1^*}\left\{\mathbf{u}_2^+ - \mathbf{u}_1^- \langle \mathbf{u}_1^-, \mathbf{u}_2^+ \rangle \cdot \frac{\lambda_1 - \lambda_1^*}{\lambda_2 - \lambda_1^*}\right\}, \tag{53}$$

where

$$e^{-2\phi} = \left|\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2^*}\right|^2 \left\{1 + \frac{(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*)}{|\lambda_1 - \lambda_2^*|^2} |\langle \mathbf{u}_1^-, \mathbf{u}_2^+ \rangle|^2\right\}. \tag{54}$$

In addition, the position-shift relations are

$$\eta_1^+ - \eta_1^- = \phi, \quad \eta_2^+ - \eta_2^- = -\phi. \tag{55}$$

These polarization and position-shift relations are not convenient to use, however, as the right hand sides of these relations involve the after-collision soliton polarizations $\mathbf{u}_2^+$. To overcome this problem, we will solve $\mathbf{u}_1^+$ and $\mathbf{u}_2^+$ in terms of $\mathbf{u}_1^-$ and $\mathbf{u}_2^-$ as Manakov and Tsuchida did for the Manakov equations.[10,13] First, we rewrite Eq. (53) as

$$\mathbf{u}_2^- = e^{\phi} \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1^*}\left\{I - \mathbf{u}_1^-(\mathbf{u}_1^-)^A \cdot \frac{\lambda_1 - \lambda_1^*}{\lambda_2 - \lambda_1^*}\right\}\mathbf{u}_2^+, \tag{56}$$

where $(\mathbf{u}_1^-)^A \equiv (\mathbf{u}_1^-)^\dagger B_1$, and I is the two-by-two identity matrix. Notice that

$$(\mathbf{u}_1^-)^A \mathbf{u}_1^- = \langle \mathbf{u}_1^-, \mathbf{u}_1^- \rangle = 1, \tag{57}$$

then the following identity can be easily verified:

$$\left\{I - \mathbf{u}_1^-(\mathbf{u}_1^-)^A \cdot \frac{\lambda_1 - \lambda_1^*}{\lambda_2 - \lambda_1^*}\right\}\left\{I + \mathbf{u}_1^-(\mathbf{u}_1^-)^A \cdot \frac{\lambda_1 - \lambda_1^*}{\lambda_2 - \lambda_1}\right\} = I. \tag{58}$$

Using this identity, we can invert Eq. (56) to obtain $\mathbf{u}_2^+$ in terms of $\mathbf{u}_1^-$ and $\mathbf{u}_2^-$. Then, we insert this $\mathbf{u}_2^+$ formula into (52) and (54). After straightforward simplifications, the formulas for $\mathbf{u}_1^+$ and $e^{-2\phi}$

in terms of $\mathbf{u}_1^-$ and $\mathbf{u}_2^-$ will be obtained. Summarizing these results, we find that the polarization vectors of the two single solitons after collision are related to those before collision as

$$\mathbf{u}_1^+ = e^{-\phi}\frac{\lambda_1^* - \lambda_2}{\lambda_1^* - \lambda_2^*}\left\{\mathbf{u}_1^- - \mathbf{u}_2^-\langle \mathbf{u}_2^-, \mathbf{u}_1^-\rangle \cdot \frac{\lambda_2 - \lambda_2^*}{\lambda_1^* - \lambda_2^*}\right\}, \tag{59}$$

$$\mathbf{u}_2^+ = e^{-\phi}\frac{\lambda_2 - \lambda_1^*}{\lambda_2 - \lambda_1}\left\{\mathbf{u}_2^- + \mathbf{u}_1^-\langle \mathbf{u}_1^-, \mathbf{u}_2^-\rangle \cdot \frac{\lambda_1 - \lambda_1^*}{\lambda_2 - \lambda_1}\right\}, \tag{60}$$

where

$$e^{-2\phi} = \left|\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2^*}\right|^2\left\{1 - \frac{(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*)}{|\lambda_1 - \lambda_2|^2}|\langle \mathbf{u}_1^-, \mathbf{u}_2^-\rangle|^2\right\}^{-1}. \tag{61}$$

When $B_1 = I$, Eq. (1) reduces to the Manakov system. In this case, the inner product (38) becomes the usual one in the complex-vector space, and formulas (59)–(61) reduce to those obtained by Manakov and Tsuchida in Refs. 10 and 13. Note that when comparing the above formulas with those in Refs. 10 and 13, the eigenvalue λ_k in our notations corresponds to $-\zeta_k^*$ in Refs. 10 and 13 because our spectral parameter λ in (2a) is opposite of ζ in Refs. 10 and 13.

It is interesting to observe that the polarization formulas (59)–(61) for the GCNLSS (1) are very similar to those in the Manakov system.[10,13] The only difference is that one needs to define the inner products in these formulas differently according to the system parameters (a,b,c) [see Eq. (38)]. This fact shows that the whole family of GCNLSS (1) indeed shares many common features and can be treated collectively together.

Relations (59)–(61) show that in the soliton collisions in Eq. (1), polarization vectors of individual solitons will change in the generic case. Since a soliton's amplitude and power depend on its polarization vector (see above), the amplitudes and powers of the two solitons will generically change after collision. The only two exceptions occur when $\langle \mathbf{u}_1^-, \mathbf{u}_2^-\rangle = 0$ and when $\mathbf{u}_2^-$ is proportional to $\mathbf{u}_1^-$ [in the latter case, Eq. (1) degenerates to a single NLS equation]. In these two special cases, the polarization vectors of the two solitons do not change (except a phase shift), thus both the amplitudes and powers of these solitons remain the same after collision. Similar phenomenon has also been reported in the Manakov system before.[10] However, the special case of $\langle \mathbf{u}_1^-, \mathbf{u}_2^-\rangle = 0$ is not always possible in the GCNLSS (1) if one does not allow singularities in individual solitons. Let us take $a = c = 0$ and $b = 1$ as an example. Denoting the precollision soliton vectors $(\mathbf{r}_1^-, \mathbf{r}_2^-)$ in Eq. (43) as $\mathbf{r}_k^- = (g_k, h_k)^T$, then the condition $\langle \mathbf{u}_1^-, \mathbf{u}_2^-\rangle = 0$ is equivalent to $\langle \mathbf{r}_1^-, \mathbf{r}_2^-\rangle = 0$, i.e.,

$$g_1^* h_2 + h_1^* g_2 = 0. \tag{62}$$

If we multiply this equation by $g_1 g_2^*$ and take its real part, we get

$$|g_1|^2\,\mathrm{Re}(g_2^* h_2) + |g_2|^2\,\mathrm{Re}(g_1 h_1^*) = 0. \tag{63}$$

On the other hand, in order for the two precollision solitons to be nonsingular, we must have $\langle \mathbf{r}_k^-, \mathbf{r}_k^-\rangle > 0$ $(k = 1, 2)$, i.e.,

$$\mathrm{Re}(g_1 h_1^*) > 0, \quad \mathrm{Re}(g_2 h_2^*) > 0. \tag{64}$$

Obviously, the two equations (63) and (64) contradict with each other, meaning that for nonsingular solitons, the polarization-preserving collision of $\langle \mathbf{u}_1^-, \mathbf{u}_2^-\rangle = 0$ cannot happen when $a = c = 0$ and $b = 1$. In this system, except the degenerate case of $\mathbf{u}_2^-$ being proportional to $\mathbf{u}_1^-$, the two solitons always change their polarizations after collision.

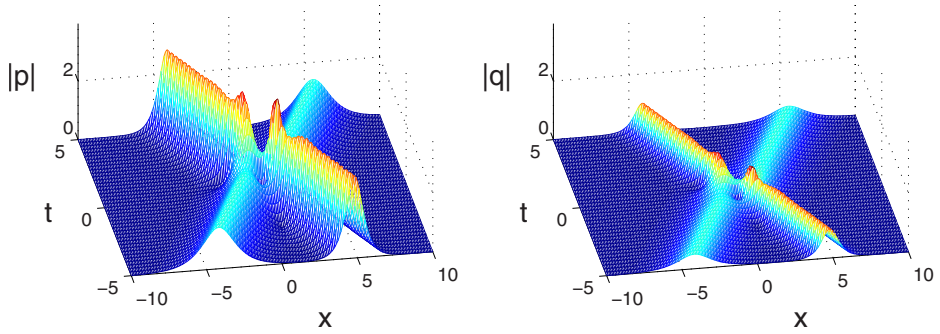

FIG. 1. (Color online) Soliton collision in Eq. (65) with parameter values (66) in the solution (34). This collision is equivalent to that in the NLS equation.

C. Examples of soliton collisions

Now we use some examples to demonstrate various collision scenarios in Eq. (1). Since collisions in the Manakov system ($a=c=1$, $b=0$) and a focusing-defocusing coupled system ($a=-c=1$, $b=0$) have been examined in previous works,[10–13,17] we will focus on a different system with $a=c=0$ and $b=1$ below, i.e.,

$$ip_t + p_{xx} + 2(pq^* + qp^*)p = 0,$$

$$iq_t + q_{xx} + 2(pq^* + qp^*)q = 0. \tag{65}$$

In this system, only four-wave mixing terms are present, but not SPM and XPM terms. In all our examples, we will specify λ_1, λ_2, α_1, β_1, α_2, and β_2 in the solution (34). The polarization vectors $\mathbf{u}_1^-$, $\mathbf{u}_2^-$, $\mathbf{u}_1^+$, and $\mathbf{u}_2^+$ can be readily obtained from the various formulas presented above.

1. Soliton transmission

In our first example, we take

$$\lambda_1 = -0.2 + i, \quad \lambda_2 = 0.2 + 0.5i, \quad \alpha_1 = 2 + 2i, \quad \beta_1 = 1, \quad \alpha_2 = -\alpha_1, \quad \beta_2 = -\beta_1 \tag{66}$$

in the solution (34), and this collision is displayed in Fig. 1. In this case, the two solitons pass through each other, and their polarizations do not change. The reason for this is that $\mathbf{u}_2^-$ can be found to be proportional to $\mathbf{u}_1^-$, thus this collision degenerates to that of two solitons in the NLS equation.

2. Soliton reflection (I)

In our second example, we take

$$\lambda_1 = -0.2 + i, \quad \lambda_2 = 0.2 + i, \quad \alpha_1 = 2 + 2i, \quad \beta_1 = 1, \quad \alpha_2 = -\alpha_1, \quad \beta_2 = -1 + 0.24i, \tag{67}$$

and this collision is displayed in Fig. 2. At $t=-\infty$, the left soliton has much more power than the right one. After collision when this left soliton passes to the right, its power has diminished dramatically. At the same time, when the initial right soliton passes to the left after collision, its power has increased dramatically. Thus a lot of power has transferred from one soliton to the other during this collision (a similar phenomenon was also reported for $a=-c$ and $b=0$ in Ref. 17). Certainly, the polarization vectors of the two solitons have changed a lot after collision as well, which directly induced this power redistribution. When comparing this collision with the soliton transmission in Fig. 1, we can see that these two collisions are very different. Here, the collision visually looks like the initial two solitons are bounced back by the collision; thus, we can call this collision soliton reflection. In this reflection, the two solitons do not come together and coalesce. Rather, they stay apart from each other during the whole reflection process. To the best of our

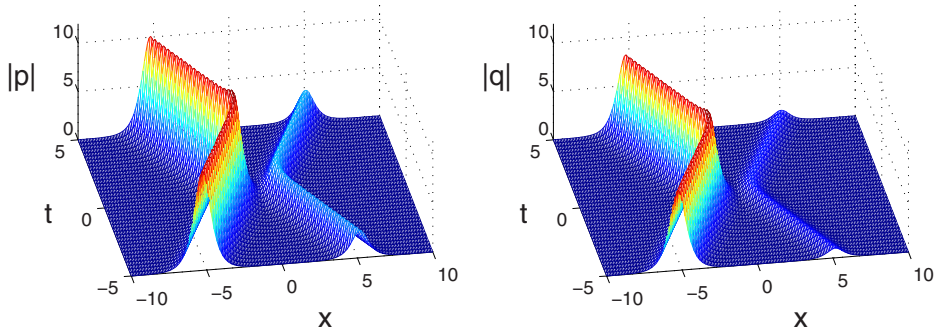

FIG. 2. (Color online) Soliton reflection in Eq. (65) with parameter values (67) in the solution (34).

knowledge, this soliton reflection has not been reported before in integrable systems. It is noted that in certain nonintegrable systems, reflections of solitary waves after collisions have been reported.[23] However, those reflections are due to entirely different reasons (by some kind of resonance mechanism), and they only appear in nonintegrable systems. In addition, in those reflections, the two solitary waves first pass through each other, then turn around and pass each other again, and then escape. That differs from the reflection here in Fig. 2 where the two solitons visually never pass each other.

3. Soliton reflection (II)

In our third example, we take

$$\lambda_1 = -0.2 + i, \quad \lambda_2 = 0.2 + i, \quad \alpha_1 = 2 + 2i, \quad \beta_1 = 1, \quad \alpha_2 = -\alpha_1, \quad \beta_2 = -1 - 0.3i. \tag{68}$$

These parameters are the same as those in Fig. 2, except that β_2 is slightly different. This collision is displayed in Fig. 3. This collision is a little similar to that in Fig. 2 and is another example of soliton reflections. Its main difference from Fig. 2 is that the q component here has much less power than the p component (in Fig. 2, the powers in the p and q components were comparable). However, even in this case, soliton reflection can still occur.

4. Soliton reflection (III)

In our last example, we take

$$\lambda_1 = -0.2 + i, \quad \lambda_2 = 0.2 + 0.5i, \quad \alpha_1 = 2 + 2i, \quad \beta_1 = 1, \quad \alpha_2 = -\alpha_1, \quad \beta_2 = -0.55. \tag{69}$$

These parameters are the same as those in Fig. 1, except that β_2 differs a little. This collision is displayed in Fig. 4. In this collision, when the initial left soliton (which is higher and broader) passes to the right after collision, its width does not change, but its intensity has dropped signifi-

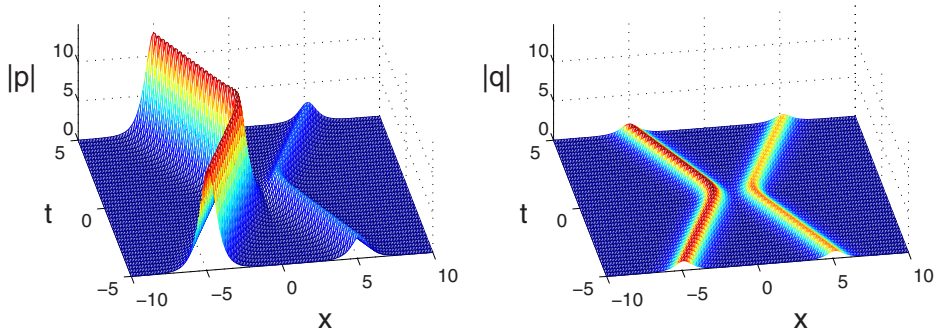

FIG. 3. (Color online) Soliton reflection in Eq. (65) with parameter values (68) in the solution (34).

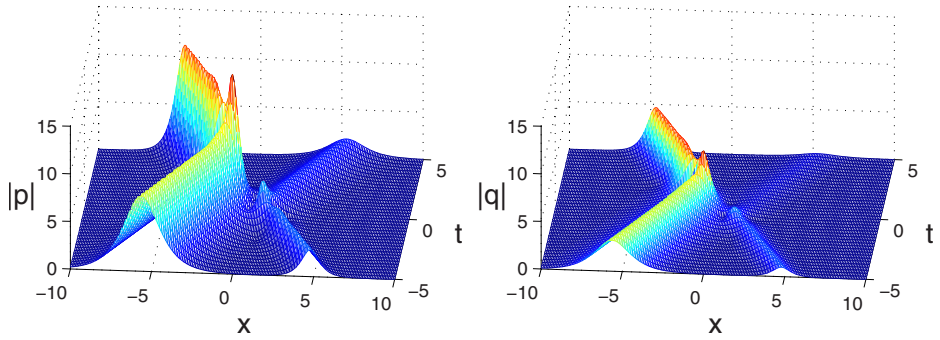

FIG. 4. (Color online) Soliton reflection in Eq. (65) with parameter values (69) in the solution (34).

cantly. Meanwhile, when the initial right soliton (which is lower and narrower) passes to the left after collision, its width does not change, but its intensity has gained significantly. Thus, a lot of energy transfer has taken place between these two solitons of different widths during the collision as well. Visually, this collision looks like the initial left soliton (with much more power) is bounced back by the collision with the other soliton (with much less power). Meanwhile, it has become narrower and taller. This collision is also quite distinctive and not often seen in integrable systems.

IV. THE RECURSION OPERATOR AND CONSERVATION LAWS

In this section, we derive the recursion operator and the infinite integrable hierarchy associated with the spectral problem (2a). In addition, we will derive the infinite conservation laws for this integrable hierarchy. The spectral problem (2a) is a third order system with two potential functions p and q. Since our derivation of the recursion operator and conservation laws can be trivially extended to any higher-order system of the type (2a), we will work with this general higher-order spectral problem instead. Of course, the recursion operator and infinite conservation laws for Eq. (1) can be pulled out of these general results very trivially.

We consider the following generalized Zakharov–Shabat spectral problem and the related time evolution equation:

$$\phi_x = M\phi, \quad M = \begin{pmatrix} i\lambda I_n & \mathbf{q}^T \\ \mathbf{r} & -i\lambda \end{pmatrix}, \tag{70a}$$

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B^T \\ C & D \end{pmatrix}, \tag{70b}$$

where $\phi=(\phi_1,\phi_2,\ldots,\phi_{n+1})^T$, $\mathbf{q}$ and $\mathbf{r}$ are row vectors $\mathbf{q}=(q_1,q_2,\ldots,q_n)$, $\mathbf{r}=(r_1,r_2,\ldots,r_n)$, λ is a spectral parameter, I_n is the nth order identity matrix, A is an nth order matrix, B and C are both n-component row vectors, and D a scalar variable.

We first derive the recursion operator and the infinite hierarchy for the above integrable system. The method we will use is a modification of that was originally developed in Ref. 24 for the Zakharov–Shabat spectral problem[4] (see also Refs. 25–27). The zero curvature equation

$$M_t - N_x + [M,N] = 0 \tag{71}$$

yields

$$0 = -A_x + \mathbf{q}^T C - B^T \mathbf{r}, \tag{72a}$$

$$0 = \mathbf{q}_t - B_x + 2i\lambda B + D\mathbf{q} - \mathbf{q}A^T, \tag{72b}$$

$$0 = \mathbf{r}_t - C_x - 2i\lambda C + \mathbf{r}A - D\mathbf{r}, \tag{72c}$$

$$0 = -D_x + \mathbf{r}B^T - C\mathbf{q}^T. \tag{72d}$$

From (72a) and (72d), we get

$$A = \partial^{-1}(\mathbf{q}^T C - B^T \mathbf{r}) + A_0, \tag{73a}$$

$$D = \partial^{-1}(\mathbf{r}B^T - C\mathbf{q}^T) + D_0 \tag{73b}$$

with constant matrix A_0 and scalar D_0, and then rewrite (72b) and (72c) into

$$\begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix}_t = 2i\lambda \begin{pmatrix} -B \\ C \end{pmatrix} + L \begin{pmatrix} -B \\ C \end{pmatrix} + \begin{pmatrix} D_0 I_n - A_0^T & 0 \\ 0 & D_0 I_n - A_0 \end{pmatrix} \begin{pmatrix} -\mathbf{q} \\ \mathbf{r} \end{pmatrix}, \tag{74}$$

where

$$L = \begin{pmatrix} -\partial + \mathbf{q}\partial^{-1} \cdot \mathbf{r}^T + \mathbf{q}\partial^{-1}\mathbf{r}^T \cdot & \mathbf{q}\partial^{-1} \cdot \mathbf{q}^T + \mathbf{q}\partial^{-1}(\cdot)^T \mathbf{q} \\ -\mathbf{r}\partial^{-1}(\cdot)^T \mathbf{r} + \mathbf{r}\partial^{-1} \cdot \mathbf{r}^T & \partial - \mathbf{r}\partial^{-1}\mathbf{q}^T \cdot - \mathbf{r}\partial^{-1} \cdot \mathbf{q}^T \end{pmatrix}, \tag{75}$$

$\partial = \partial/\partial x$, $\partial^{-1}\partial = \partial\partial^{-1} = 1$. Here, we have made use of "dot" so that L looks a little bit more clearly. When L acts on a variable, these dots are the places where the variable enters the operation. We note that a special case of this recursion operator, for the vector NLS equations of the Manakov type, has been given in Ref. 28.

To derive evolution equations, we expand

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{j=1}^{m} \begin{pmatrix} -b_j \\ c_j \end{pmatrix} (-2i\lambda)^{m-j} \tag{76}$$

and take $A_0 = -D_0 I_n = i(-2i\lambda)^{n-1}\lambda I_n$. It then follows from (74) that

$$\begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix}_t = L \begin{pmatrix} -b_m \\ c_m \end{pmatrix},$$

$$\begin{pmatrix} -b_{m-j} \\ c_{m-j} \end{pmatrix} = L \begin{pmatrix} -b_{m-j-1} \\ c_{m-j-1} \end{pmatrix} \quad (j = 0, 1, \ldots, m-1,),$$

$$\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -\mathbf{q} \\ \mathbf{r} \end{pmatrix}.$$

Thus, we obtain the hierarchy

$$i \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix}_t = L^m \begin{pmatrix} -\mathbf{q} \\ \mathbf{r} \end{pmatrix} \quad (m = 1, 2, \ldots), \tag{77}$$

where we have substituted $-it$ for t.

The first two nontrivial equations in the hierarchy (77) are $(m=2,3)$

$$i \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix}_t = \begin{pmatrix} -\mathbf{q}_{xx} + 2\mathbf{q}(\mathbf{q}\mathbf{r}^T) \\ \mathbf{r}_{xx} - 2\mathbf{r}(\mathbf{q}\mathbf{r}^T) \end{pmatrix}, \tag{78}$$

$$i\begin{pmatrix}\mathbf{q}\\ \mathbf{r}\end{pmatrix}_t = \begin{pmatrix}\mathbf{q}_{xxx} - 3\mathbf{q}(\mathbf{r}\mathbf{q}_x^T) - 3\mathbf{q}_x(\mathbf{r}\mathbf{q}^T)\\ \mathbf{r}_{xxx} - 3\mathbf{r}(\mathbf{q}\mathbf{r}_x^T) - 3\mathbf{r}_x(\mathbf{q}\mathbf{r}^T)\end{pmatrix}. \tag{79}$$

In their Lax pairs,

$$N_2 = -i\begin{pmatrix}2\lambda^2 I_n + \mathbf{q}^T\mathbf{r} & -2\lambda i\mathbf{q} - \mathbf{q}_x\\ -2\lambda i\mathbf{r} + \mathbf{r}_x & -2\lambda^2 - \mathbf{q}\mathbf{r}^T\end{pmatrix}, \tag{80}$$

$$N_3 = -i\begin{pmatrix}-4\lambda^3 iI_n - 2\lambda i\mathbf{q}^T\mathbf{r} + \mathbf{q}^T\mathbf{r}_x - \mathbf{q}_x^T\mathbf{r} & -4\lambda^2\mathbf{q} + 2\lambda i\mathbf{q}_x + \mathbf{q}_{xx} - 2\mathbf{q}(\mathbf{q}\mathbf{r}^T)\\ -4\lambda^2\mathbf{r} - 2\lambda i\mathbf{r}_x + \mathbf{r}_{xx} - 2\mathbf{r}(\mathbf{q}\mathbf{r}^T) & 4\lambda^3 i + 2\lambda i\mathbf{q}\mathbf{r}^T - \mathbf{q}\mathbf{r}_x^T + \mathbf{q}_x\mathbf{r}^T\end{pmatrix}. \tag{81}$$

We note that Eq. (78) admits the reduction

$$\mathbf{r}^T = W\mathbf{q}^{*T}, \tag{82}$$

where W is a nonsingular $n\times n$ Hermite matrix, i.e.,

$$W = W^\dagger, \quad \det(W) \neq 0. \tag{83}$$

In the special case of $n=2$, if we denote

$$W = -\begin{pmatrix}a & b\\ b^* & c\end{pmatrix}, \quad a,c \in \mathbb{R}, \quad ac - |b|^2 \neq 0, \tag{84}$$

then the reduction in (78) is

$$iq_{j,t} + q_{j,xx} + 2q_j(a|q_1|^2 + c|q_2|^2 + bq_1q_2^* + b^*q_1^*q_2) = 0 \quad (j=1,2), \tag{85}$$

which is just the GCNLSS (1).

Equation (79) also admits the reduction (82). If we take W as the identity matrix, and $\mathbf{q} = (q_1, q_1^*)$, then Eq. (79) becomes the Sasa–Satsuma equation for the variable q_1.[29] Further equations can be derived by taking other reductions and higher equations in the hierarchy (77).

Finally, we derive the infinite conserved quantities for the hierarchy (77). Our method will be an extension of that originally used in Ref. 30 for the Ablowitz–Kaup–Newell–Segur hierarchy.[24] We rewrite (70a) as

$$\phi_{j,x} = -i\lambda\phi_j + q_j\phi_{n+1} \quad (j=1,2,\ldots,n), \tag{86a}$$

$$\phi_{n+1,x} = \sum_{j=1}^{n} r_j\phi_j + i\lambda\phi_{n+1}. \tag{86b}$$

By introducing

$$\omega_j = \frac{\phi_j}{\phi_{n+1}} \quad (j=1,2,\ldots,n), \tag{87}$$

from (86), we get the coupled Riccati equations

$$\omega_{j,x} = -2i\lambda\omega_j + q_j - \omega_j\sum_{i=1}^{n} r_i\omega_i \quad (j=1,2,\ldots,n). \tag{88}$$

To solve these Riccati equations, we expand $\{\omega_j\}$ into power series in $(2i\lambda)^{-1}$,

$$\omega_j = \sum_{k=1}^{\infty} \omega_{j,k}(2i\lambda)^{-k} \quad (j = 1,2,\ldots,n). \tag{89a}$$

When this expansion is inserted into the Riccati equation (88), by comparing terms of the same order in $(2i\lambda)^{-1}$, we get

$$\omega_{j,1} = q_j, \quad \omega_{j,2} = -q_{j,x}, \tag{89b}$$

$$\omega_{j,k+1} = -\omega_{j,k,x} - \sum_{i=1}^{n} r_i \sum_{s=1}^{k-1} \omega_{i,s}\omega_{j,k-s} \quad (k = 1,2,\ldots) \tag{89c}$$

for $j=1,2,\ldots,n$. To get conservation laws, we take out the last components of (70a) and (70b) and then divide them by ϕ_{n+1}. This yields

$$(\ln \phi_{n+1})_x = \sum_{j=1}^{n} r_j\omega_j + i\lambda,$$

$$(\ln \phi_{n+1})_t = \sum_{j=1}^{n} C_j\omega_j + D,$$

where C_j is the jth component of the vector C in the evolution equation (70b). Next, noting that $\lambda_t=0$ and making use of the commutating relationship $(\ln \phi_{n+1})_{xt}=(\ln \phi_{n+1})_{tx}$, we get

$$\left(\sum_{j=1}^{n} r_j\omega_j\right)_t = \left(\sum_{j=1}^{n} C_j\omega_j + D\right)_x. \tag{90}$$

Then, inserting the expansion (89) of ω_j as well as the expressions of C and D into the above equation, an infinite number of local conservation laws would be obtained. The conserved densities in these conservation laws are

$$\rho_k = \sum_{j=1}^{n} r_j\omega_{j,k} \quad (k = 1,2,\ldots). \tag{91}$$

V. SUMMARY

In this paper, we have analyzed the general integrable coupled NLS system. In this system, the coefficients of the SPM, XPM, and four-wave mixing terms are more general while still maintaining integrability. This general system contains the Manakov equations as a special case. Using the Riemann–Hilbert method, we obtained the general N-soliton formula for the entire system. The collision dynamics between two solitons has also been analyzed. It was shown that this collision exhibits some new phenomena (such as soliton reflection), which are rarely seen in integrable systems. In addition, the unified formulas relating the soliton polarizations before and after collisions have been derived. These formulas generalize those obtained before for the Manakov system. We have further derived the recursion operator and the whole hierarchy associated with this system. It was found that the Sasa–Satsuma equation belongs to this hierarchy. Finally, the infinite number of conservation laws for this system was also obtained. Since the coupled NLS equations arise in a wide variety of physical subjects such as nonlinear optics, water waves, and Bose–Einstein condensates, the results in this paper should prove helpful to the studies of those physical problems.

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