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Classes of non-parity-time-symmetric optical potentials with exceptional-point-free phase transitions

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Paraxial linear propagation of light in an optical waveguide with material gain and loss is governed by a Schrödinger equation with a complex potential. In this Letter, new classes of non-parity-time ($\mathcal{PT}$)-symmetric complex potentials featuring conjugate-pair eigenvalue symmetry are constructed by operator symmetry methods. Due to this eigenvalue symmetry, it is shown that the spectrum of these complex potentials is often all-real. Under parameter tuning in these potentials, a phase transition can also occur, where pairs of complex eigenvalues appear in the spectrum. A distinctive feature of the phase transition here is that the complex eigenvalues may bifurcate out from an interior continuous eigenvalue inside the continuous spectrum; hence, a phase transition takes place without going through an exceptional point. In one spatial dimension, this class of non-$\mathcal{PT}$-symmetric complex potentials is of the form $V(x) = h'(x) - h^2(x)$, where $h(x)$ is an arbitrary $\mathcal{PT}$-symmetric complex function. These potentials in two spatial dimensions are also derived. Diffraction patterns in these complex potentials are further examined, and unidirectional propagation behaviors are demonstrated.

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Parity-time ($\mathcal{PT}$)-symmetric optics has been heavily studied in the past 10 years (see Refs. [1,2] for reviews). $\mathcal{PT}$ symmetry was first introduced as a non-Hermitian generalization of quantum mechanics, where a complex, but $\mathcal{PT}$-symmetric potential, was shown to still possess all-real spectrum [3]. Since the Schrödinger equation in quantum mechanics is equivalent to the beam propagation equation in optics under paraxial approximation, $\mathcal{PT}$ symmetry then spread to optics, where this symmetry can be realized by an even refractive index profile, together with an odd gain–loss landscape [4]. In this optical setting, $\mathcal{PT}$ symmetry was studied experimentally for the first time [5,6]. $\mathcal{PT}$-symmetric potentials do not always possess all-real spectra though. If the gain–loss profile (corresponding to the imaginary part of the complex potential) is too strong, a phase transition can occur, where complex eigenvalues enter the spectrum [3,7]. This phase transition has been demonstrated in a number of optical experiments and utilized for many emerging applications [5,6,8–12]. The interplay between $\mathcal{PT}$ symmetry and nonlinearity has been extensively explored as well [1,2].

Optical $\mathcal{PT}$ symmetry requires the refractive index to be even and the gain–loss profile to be odd in space, which is restrictive and limits its applicability. The generalization of $\mathcal{PT}$-symmetric potentials to allow a more flexible gain–loss profile, while still maintaining all-real spectrum, is thus an important question. Using supersymmetry methods, families of non-$\mathcal{PT}$-symmetric potentials with real spectra have been constructed [13,14], but the gain–loss profiles in such potentials are still very special. Using other methods, classes of non-$\mathcal{PT}$-symmetric potentials with arbitrary gain–loss profiles and all-real spectra were reported in Refs. [15–17], which is a big step forward. The simplest class of such potentials is of the form $V(x) = g^2(x) + ig'(x)$, where $g(x)$ is an arbitrary real function, and the prime represents the derivative [15–17]. Other such potentials with more involved functional forms can be found in Ref. [17].

In this Letter, we report new classes of non-$\mathcal{PT}$-symmetric complex potentials which feature all-real spectra and phase transitions. In one dimension, this class of potentials is of the form $V(x) = h'(x) - h^2(x)$, where $h(x)$ is an arbitrary $\mathcal{PT}$-symmetric complex function. Even though these potentials are very different from those reported in Refs. [15–17], we show that their eigenvalues still possess conjugate-pair symmetry. Thus, their spectra can still be all-real, and a phase transition can also occur by tuning parameters in these potentials. A distinctive phenomenon here is that, when these potentials are localized, their spectra do not admit any discrete real eigenvalues. Thus, when a phase transition occurs, this transition does not go through an exceptional point. Instead, discrete complex eigenvalues bifurcate out from continuous real eigenvalues in the interior of the continuous spectrum. This scenario of a phase transition is different from all those reported before in non-Hermitian systems, where a phase transition was either induced by collisions of discrete real eigenvalues through an exceptional point [1,3,5–8,10–12,17], or from an exotic singular scenario [18]. For periodic potentials at a phase transition, we observe unidirectional beam propagation behaviors with new features beyond those reported earlier for $\mathcal{PT}$-symmetric potentials.

Linear paraxial propagation of light in an optical waveguide with material gain and loss is governed by the Schrödinger equation

$$i\Psi_z + \Psi_{xx} + V(x)\Psi = 0, \tag{1}$$

where z is the distance of propagation, x is the transverse coordinate, and $V(x)$ is a complex potential whose real part is the index of refraction and whose imaginary part represents gain and loss in the waveguide. Looking for eigenmodes of the form $\Psi = e^{-i\mu z}\psi(x)$, we arrive at the eigenvalue problem

$$L\psi = -\mu\psi, \tag{2}$$

where $L = \partial_{xx} + V(x)$ is a Schrödinger operator and μ is an eigenvalue.

To derive new non-$\mathcal{PT}$-symmetric complex potentials with all-real spectra, we follow the operator symmetry strategy which we have used before [17]. This strategy is based on the following observation: if there exists an operator η such that L and its complex conjugate L^* are related by a similarity relation

$$\eta L = L^*\eta, \tag{3}$$

then the eigenvalues of L come in conjugate pairs if the kernel of η is empty. This conjugate-pair eigenvalue symmetry guarantees that either the spectrum of L is all-real, or a phase transition occurs when pairs of complex eigenvalues bifurcate out. This relation resembles the condition for pseudo-Hermiticity [19], but we do not require η to be invertible here.

The key difference between our current approach and the one in Ref. [17] is that, instead of choosing η as pure differential operators, we now take η to be a combination of the parity operator $\mathcal{P}$ and differential operators. In one spatial dimension, the parity operator is defined as $\mathcal{P}f(x) \equiv f(-x)$. In the simplest case, we take η to be a combination of the parity operator and a first-order differential operator, i.e.,

$$\eta = \mathcal{P}[\partial_x + h(x)], \tag{4}$$

where $h(x)$ is a complex function to be determined. Substituting this η into the similarity condition (3), we get the following two equations:

$$V(x) - V^*(-x) = 2h'(x), \tag{5}$$

$$[V(x) - V^*(-x)]h(x) = h''(x) - V'(x). \tag{6}$$

From the first equation, we see that $[h^*(-x)]_x = h'(x)$; thus,

$$h^*(-x) = h(x) + c_1, \tag{7}$$

where c_1 is a constant. Substituting Eq. (5) into Eq. (6) and integrating once, we get

$$V(x) = h'(x) - h^2(x) + c_2, \tag{8}$$

where c_2 is another constant. Lastly, inserting Eqs. (7) and (8) into Eq. (5), we obtain

$$c_1^2 + 2c_1h(x) + c_2 - c_2^* = 0. \tag{9}$$

In order for the potential $V(x)$ in Eq. (8) not to be a constant, the function $h(x)$ should not be identically zero. Thus, Eq. (9) dictates that $c_1 = 0$ and c_2 is real. The former condition means that the complex function $h(x)$ is $\mathcal{PT}$-symmetric in view of Eq. (7). Regarding the latter condition, since a real constant in a potential can be easily removed by a simple shift of the eigenvalue, we can set $c_2 = 0$ without loss of generality. In the end, we find that for new complex potentials of the form

$$V(x) = h'(x) - h^2(x), \tag{10}$$

where $h(x)$ is a $\mathcal{PT}$-symmetric complex function, i.e., $h^*(x) = h(-x)$, the Schrödinger operator L satisfies the similarity condition (3) with η given in Eq. (4). Because of this, eigenvalues of L exhibit complex-conjugate symmetry; hence, the spectrum of L can be all-real, and a phase transition could occur, similar to $\mathcal{PT}$-symmetric potentials.

In these new potentials, $h(x)$ is an arbitrary $\mathcal{PT}$-symmetric function. This implies that the imaginary part of the potential $V(x)$ (corresponding to the gain and loss profile) can also be arbitrary. To see this, we write $h(x) = h_1(x) + ih_2(x)$, where h_1 is the real part of h which is even and h_2 is the imaginary part of h which is odd. Then $\mathrm{Im}(V) = h_2'(x) - 2h_1(x)h_2(x)$. Notice that $h_2'(x)$ is even, and $h_1(x)h_2(x)$ is odd. For any arbitrary gain–loss profile, $G(x) = \mathrm{Im}(V)$, we can always write it as $G(x) = G_1(x) + G_2(x)$, where $G_1(x) \equiv [G(x) + G(-x)]/2$ is even, and $G_2(x) \equiv [G(x) - G(-x)]/2$ is odd. Then under the choice of $h_2'(x) = G_1(x)$ and $h_1(x) = -G_2(x)/2h_2(x)$, where $h_2(x) = \int G_1(x)\mathrm{d}x$ is selected to be an odd function, the resulting gain–loss profile $\mathrm{Im}(V) = h_2'(x) - 2h_1(x)h_2(x)$ would be equal to $G(x)$. This means that the new class of potentials (10) can accommodate any arbitrary gain–loss profile. (The refractive index would need to be engineered accordingly though.)

In the above construction, if we choose η to be a combination of the parity operator $\mathcal{P}$ and higher-order differential operators, additional families of new non-$\mathcal{PT}$-symmetric complex potentials with all-real spectra could be derived.

To illustrate the all-real spectra and a phase transition of this class of non-$\mathcal{PT}$-symmetric potentials, we take

$$h(x) = b_0 + b_1 \cos x + ib_2 \sin x, \tag{11}$$

which is a periodic function with real constants b_0, b_1, and b_2. This function $h(x)$ is $\mathcal{PT}$-symmetric as required. We also fix $b_0 = b_1 = 1$ and allow b_2 to vary. When $b_2 = 0.98$, the resulting complex periodic potential $V(x)$ is depicted in the upper left panel of Fig. 1. It is easy to see that this potential is non-$\mathcal{PT}$-symmetric (even under any x-coordinate shift). The diffraction relation of this periodic potential is shown in the upper right panel of Fig. 1. These diffraction curves are all-real, meaning that the whole spectrum of L is all-real. However, when $b_2 = 1.02$, the lowest two Bloch bands collide, and complex eigenvalues appear near the two edges of the Brillouin zone; see the lower panels of Fig. 1. In this case, the spectrum of L becomes partially complex. The phase transition occurs at $b_2 = 1$. These behaviors are qualitatively similar to that in $\mathcal{PT}$-symmetric periodic potentials [4].

The dynamics of beam propagation in these new complex potentials is of high interest. To examine this, we take the above periodic potential at a phase transition ($b_2 = 1$) and launch a broad beam into it at opposite angles. Specifically, our initial condition is taken as

$$\Psi(x, 0) = e^{-x^2/100 + i\beta x}, \tag{12}$$

where the real constant β is proportional to the initial launch angle. When $\beta = \pm 2$ (opposite angles), the evolutions of these two beams are obtained by computing Eq. (1) and displayed in Fig. 2. We find that when the beam is launched toward the left, it does not really travel in that direction. Instead, it

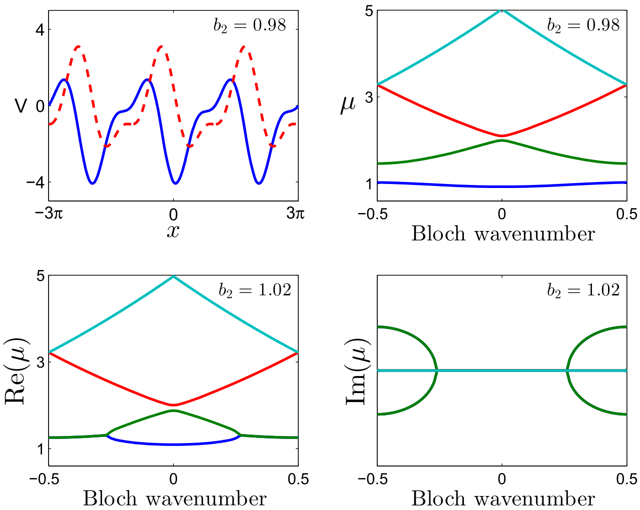

Fig. 1. Diffraction relations of non-$\mathcal{PT}$-symmetric periodic potentials (10) with $h(x)$ given in Eq. (11) and $b_0 = b_1 = 1$. Upper left: real (solid blue) and imaginary (dashed red) parts of the complex potential at $b_2 = 0.98$. Upper right: diffraction curves of the potential in the upper left panel over the Brillouin zone $-0.5 \le k \le 0.5$. Lower panels: real and imaginary parts of the diffraction curves over the Brillouin zone at $b_2 = 1.02$.

spreads in both directions (see the left panel). On the other hand, if the beam is launched to the right, it indeed moves away toward that direction. Thus, this non-$\mathcal{PT}$-symmetric periodic potential exhibits highly nonreciprocal unidirectional behavior. While non-reciprocity and unidirectional propagation have been reported in $\mathcal{PT}$-symmetric photonic lattices before [5,6,9,20,21], this behavior in the underlying non-$\mathcal{PT}$-symmetric lattices is worthy of report.

The previous example was a periodic potential induced by a periodic function $h(x)$. When $h(x)$ is chosen as a localized function, a localized non-$\mathcal{PT}$-symmetric potential would result. For localized potentials, the continuous spectrum is the semi-infinite interval $0 \le \mu < \infty$; and discrete real eigenvalues, if any, are located in the interval $-\infty < \mu < 0$. An important phenomenon here is that localized potentials (10) do not admit any discrete real eigenvalues, which contrasts with other non-$\mathcal{PT}$-symmetric potentials reported in Refs. [16,17]. We will use the contradiction argument to prove this fact. Suppose $\mu = -k^2 < 0$, with $k > 0$, is a discrete real eigenvalue in a localized potential (10). Since L is a second-order differential operator, its discrete eigenvalue μ can only have geometric

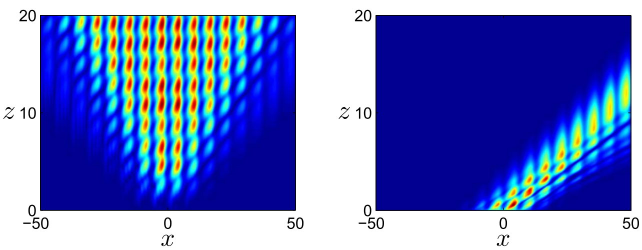

Fig. 2. Propagation of broad beams (12) launched at opposite angles into the non-$\mathcal{PT}$-symmetric lattice (10) and (11) at a phase transition ($b_2 = 1$). Left, $\beta = -2$; right, $\beta = 2$.

multiplicity one, meaning that the corresponding eigenfunction ψ is unique. Applying the operator η to the eigenvalue equation $L\psi = k^2\psi$ and recalling the symmetry relation (3), we get $L^*(\eta\psi) = k^2(\eta\psi)$. Taking the complex conjugate to this equation, we get $L(\eta\psi)^* = k^2(\eta\psi)^*$. This means that $(\eta\psi)^*$ is also an eigenfunction of L at eigenvalue μ. Since the eigenfunction for eigenvalue μ is unique, we see that $(\eta\psi)^*$ and ψ must be linearly dependent on each other, i.e., $(\eta\psi)^* = \alpha\psi$, where α is a complex constant. In view of the expression of η in Eq. (4), this relation can be rewritten as

$$[\partial_x + h(x)]\psi(x) = \alpha^*\psi^*(-x). \tag{13}$$

Now we examine this relation as $x \to \pm\infty$. Since the potential $V(x)$ is localized, the large-x asymptotics of $\psi(x)$ is

$$\psi(x) \to a_\pm e^{-k|x|}, \qquad x \to \pm\infty,$$

where $a_\pm$ are complex constants which cannot be both zero. Since $h(x)$ is also localized, as $x \to \pm\infty$, the contribution of the $h(x)$ term in Eq. (13) is subdominant and will be ignored. Then, inserting the above ψ-asymptotics into (13), we get two parameter conditions:

$$-ka_+ = \alpha^* a_-^*, \qquad ka_- = \alpha^* a_+^*.$$

Dividing these two equations and rearranging terms, we get

$$|a_+|^2 + |a_-|^2 = 0,$$

which is impossible since $a_\pm$ cannot be both zero. Thus, localized potentials (10) do not admit discrete real eigenvalues.

The fact of localized potentials (10) not admitting discrete real eigenvalues is a distinctive property, and it has important implications. Since there are no discrete real eigenvalues, a phase transition in these potentials clearly cannot be induced from collisions of such eigenvalues through an exceptional point. Instead, complex eigenvalues will have to bifurcate out from the continuous spectrum. Below, we will use an example to show that this is exactly the case. In this example, we take

$$h(x) = d_1 \text{sech}x + i d_2 \text{sech}x \tanh x, \tag{14}$$

which is $\mathcal{PT}$-symmetric for real constants d_1 and d_2. We also fix $d_1 = 1$. Then for two different d_2 values of 1 and 2, the resulting non-$\mathcal{PT}$-symmetric localized potentials and their spectra are plotted in Fig. 3. We see that neither spectrum contains discrete real eigenvalues, which corroborates our analytical result above. When $d_2 = 1$, the spectrum is all-real (see the upper right panel) but, when $d_2 = 2$, a conjugate pair of discrete eigenvalues $\mu \approx 0.7067 \pm 0.4961i$ appears (see the lower right panel). The phase transition occurs at $d_2 \approx 1.385$. A closer examination reveals that the two complex eigenvalues bifurcate out from $\mu_0 \approx 0.8062$, which is in the interior of the continuous spectrum. We also noticed that the discrete (localized) eigenfunctions of the two complex eigenvalues bifurcate out from two different continuous (nonlocal) eigenfunctions of the real eigenvalue μ_0, rather than from a single coalesced eigenfunction. This reveals two facts: (1) these discrete eigenmodes bifurcate out from continuous eigenmodes, rather than embedded isolated eigenmodes, in the interior of the continuous spectrum; (2) this phase transition does not go through an exceptional point. The second fact is particularly significant because, to the best of our knowledge, all phase transitions reported before in both finite- and infinite-dimensional non-Hermitian systems occurred due to either a collision of real eigenvalues forming an exceptional point, where different eigenvectors or eigenfunctions coalesce [1,3,7,17] or through an exotic singular scenario, where complex

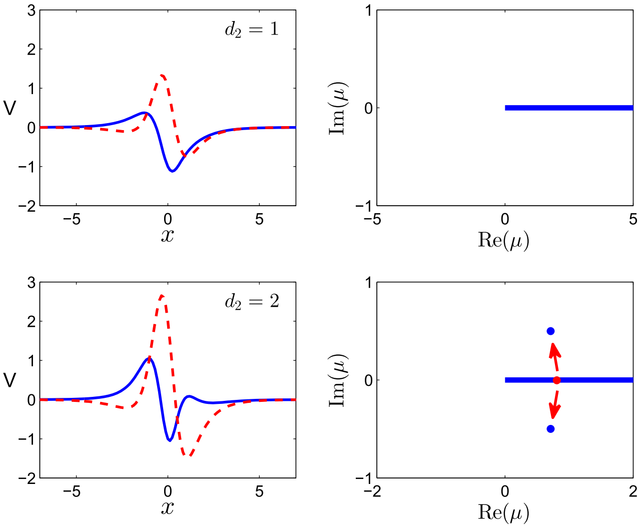

Fig. 3. Spectra of localized potentials (10) with $h(x)$ given in Eq. (14), and $d_1 = 1$. (The d_2 values are shown inside the panels.) Left column: real (solid blue) and imaginary (dashed red) parts of the complex potentials. Right column: spectra of potentials in the left column (the red arrows in the lower panel indicate that the two complex eigenvalues in the spectrum bifurcate out from the red dot in the interior of the continuous spectrum when a phase transition happens).

eigenvalues bifurcate out from infinity of the real axis [18]. This is the first instance where a phase transition occurs without an exceptional point or a singular point. We also point out that, from a mathematical point of view, we have not seen discrete eigenvalues bifurcating out of a continuous eigenvalue in the interior of the continuous spectrum before, and this is the first such example. Recently, an analytical explanation of this bifurcation was given in [22].

Lastly, we show that our construction of non-$\mathcal{PT}$-symmetric complex potentials above can be extended to higher dimensions. Let us consider paraxial light propagation in a three-dimensional waveguide, which gives rise to a two-dimensional (2D) Schrödinger operator

$$L = \partial_{xx} + \partial_{yy} + V(x, y). \tag{15}$$

To construct new complex potentials $V(x, y)$ with all-real spectra, we still impose the similarity condition (3), and choose the operator η to be a combination of the parity operator and differential operators. In 2D, the parity operator $\mathcal{P}$ can take different forms, either a full parity operator $\mathcal{P}f(x, y) = f(-x, -y)$, or a partial parity operator $\mathcal{P}f(x, y) = f(-x, y)$ or $\mathcal{P}f(x, y) = f(x, -y)$ [23]. For simplicity, we choose η to be a combination of one of those parity operators and a first-order differential operator in x, i.e.,

$$\eta = \mathcal{P}[\partial_x + a(x)]. \tag{16}$$

Inserting this η into the similarity condition (3) and after some algebra, we obtain the resulting 2D complex potential as

$$V(x, y) = a'(x) - a^2(x) + \phi(y), \tag{17}$$

where functions $a(x)$ and $\phi(y)$ satisfy the parity conditions

$$\mathcal{P}a(x) = a^*(x), \qquad \mathcal{P}\phi(y) = \phi^*(y). \tag{18}$$

This separable 2D complex potential satisfies the similarity condition (3); thus, its complex eigenvalues come in conjugate pairs, which implies that its spectrum can be all-real. Other choices of the operator η could lead to additional classes of 2D non-$\mathcal{PT}$-symmetric complex potentials with all-real spectra.

In summary, we have derived new classes of non-$\mathcal{PT}$-symmetric optical potentials featuring conjugate-pair eigenvalue symmetry in its spectrum by operator symmetry methods. Due to this eigenvalue symmetry, it is shown that the spectrum of these complex potentials is often all-real. Under parameter tuning in these potentials, a phase transition can also occur, where pairs of complex eigenvalues appear in the spectrum. A remarkable finding is that when these potentials are localized, they do not admit any discrete real eigenvalues. Then a phase transition cannot go through an exceptional point. Instead, it is only induced by complex eigenvalues bifurcating from an interior continuous eigenvalue inside the continuous spectrum, which is very novel. Since these new potentials allow an arbitrary gain–loss profile, they may find applications such as non-$\mathcal{PT}$-symmetric lasers with more flexible laser cavities.

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