Partially $\mathcal{PT}$ symmetric optical potentials with all-real spectra and soliton families in multidimensions

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Multidimensional complex optical potentials with partial parity-time ($\mathcal{PT}$) symmetry are proposed. The usual $\mathcal{PT}$ symmetry requires that the potential is invariant under complex conjugation and simultaneous reflection in all spatial directions. However, we show that if the potential is only partially $\mathcal{PT}$ symmetric, i.e., it is invariant under complex conjugation and reflection in a *single* spatial direction, then it can also possess all-real spectra and continuous families of solitons. These results are established analytically and corroborated numerically.

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In optics, light propagation is often modeled by Schrödinger-type equations [1]. If the medium contains gain and loss, the optical potential of the Schrödinger equation would be complex. A surprising finding in recent years is that, if this complex potential satisfies parity-time ($\mathcal{PT}$) symmetry, then the linear spectrum can still be all-real, thus admitting stationary light transmission [2–7]. Here $\mathcal{PT}$ symmetry means that the potential is invariant under complex conjugation and simultaneous reflection in all spatial directions. In one dimension (1D), $\mathcal{PT}$ symmetry condition is $V^*(x) = V(-x)$; in 2D, this condition is $V^*(x,y) = V(-x,-y)$; and so on. Besides all-real spectra, $\mathcal{PT}$ symmetric potentials have been found to support continuous families of optical solitons [8–12]. But if the complex potential is not $\mathcal{PT}$ symmetric, then the linear spectrum is often nonreal, and soliton families often do not exist [13]. Other findings on $\mathcal{PT}$ systems can be found in [14–29].

In this Letter, we show that in multidimensions, if the complex potential is not $\mathcal{PT}$ symmetric but is partially $\mathcal{PT}$ symmetric, then such potentials can still admit all-real spectra and continuous families of solitons. Here partial $\mathcal{PT}$ symmetry means that the potential is invariant under complex conjugation and reflection in a *single* spatial direction (rather than in all spatial directions simultaneously). For example, in 2D, partially $\mathcal{PT}$ symmetric potentials are such that either $V^*(x,y) = V(-x,y)$ or $V^*(x,y) = V(x,-y)$. Partially $\mathcal{PT}$ symmetric potentials constitute another large class of complex potentials with all-real spectra and soliton families, and they may find interesting applications in optics. For simplicity, we consider the 2D case throughout the Letter, but similar results hold for three and higher dimensions too.

The model for nonlinear propagation of light beams in complex optical potentials is taken as

$$i\Psi_z + \nabla^2\Psi + V(x,y)\Psi + \sigma|\Psi|^2\Psi = 0, \qquad (1)$$

where z is the propagation direction, (x,y) is the transverse plane, $\nabla^2 = \partial_{xx} + \partial_{yy}$, and $\sigma = \pm 1$ is the sign of nonlinearity. The complex potential $V(x,y)$ is assumed to possess the partial $\mathcal{PT}$ symmetry

$$V^*(x,y) = V(-x,y). \qquad (2)$$

The real part of this potential is symmetric in x, and its imaginary part anti-symmetric in x. No symmetry is assumed in the y direction.

First, we show that the spectrum of this partially $\mathcal{PT}$ symmetric potential can be all-real. Eigenvalues of this potential are defined by the Schrödinger equation

$$(\nabla^2 + V)\psi = \lambda\psi, \qquad (3)$$

where λ is the eigenvalue and ψ the eigenfunction.

We start by considering separable potentials where

$$V(x,y) = V_1(x) + V_2(y).$$

For these potentials, the partial $\mathcal{PT}$ symmetry condition (2) implies that

$$V_1^*(x) = V_1(-x), \qquad V_2^*(y) = V_2(y).$$

Thus the function $V_1(x)$ is $\mathcal{PT}$ symmetric and $V_2(y)$ strictly real. Eigenvalues of this separable potential are

$$\lambda = \Lambda_1 + \Lambda_2$$

and the corresponding eigenfunctions are $\psi(x,y) = \Psi_1(x)\Psi_2(y)$, where

$$[\partial_{xx} + V_1(x)]\Psi_1(x) = \Lambda_1\Psi_1(x),$$
$$[\partial_{yy} + V_2(y)]\Psi_2(y) = \Lambda_2\Psi_2(y).$$

Since $V_1(x)$ is $\mathcal{PT}$ symmetric, its eigenvalues Λ_1 can be all real. Since $V_2(y)$ is strictly real, its eigenvalues Λ_2 are all real as well. Thus eigenvalues λ of the separable potential $V(x,y)$ can be all real.

Next we consider separable potentials perturbed by localized potentials,

$$V(x,y) = V_0(x,y) + \epsilon V_p(x,y), \qquad (4)$$

where V_0 is separable, V_p localized, ϵ a small real parameter, and both V_0, V_p satisfy the partial $\mathcal{PT}$ symmetry condition (2). Since V_p is localized, continuous eigenvalues of the perturbed potential V are the same as those of the separable potential V_0 and are thus all-real. We now show that discrete eigenvalues of V are also real.

Suppose λ_0 is a simple discrete real eigenvalue of the separable potential V_0. Since V_0 is partially $\mathcal{PT}$ symmetric, the eigenfunction ψ_0 of λ_0 is partially $\mathcal{PT}$ symmetric as well, i.e., $\psi_0^*(x,y) = \psi_0(-x,y)$. Under perturbation ϵV_p, the perturbed eigenvalue and eigenfunction can be expanded into the following perturbation series:

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \ldots, \psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots.$$

Substituting these expansions and the perturbed potential (4) into Eq. (3), at $O(\epsilon)$ we get

$$L\psi_1 = (\lambda_1 - V_p)\psi_0, \tag{5}$$

where $L \equiv \nabla^2 + V_0 - \lambda_0$. Since λ_0 is a simple eigenvalue, the kernel of the adjoint operator L^* then contains a single eigenfunction ψ_0^*. Then in order for Eq. (5) to be solvable, the solvability condition is that its right-hand side be orthogonal to ψ_0^*, which yields

$$\lambda_1 = \frac{\langle \psi_0^*, V_p\psi_0 \rangle}{\langle \psi_0^*, \psi_0 \rangle}, \tag{6}$$

where the inner product is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x,y)g(x,y)\mathrm{d}x\mathrm{d}y.$$

Since λ_0 is simple, it is easy to show that $\langle \psi_0^*, \psi_0 \rangle \neq 0$.

A key consequence of partial $\mathcal{PT}$ symmetry is that, if functions f and g are both partially $\mathcal{PT}$ symmetric, then their inner product $\langle f, g \rangle$ is real, because

$$\langle f, g \rangle^* = \langle f^*, g^* \rangle = \langle f(-x,y), g(-x,y) \rangle = \langle f, g \rangle.$$

Since ψ_0 and V_p are partially $\mathcal{PT}$ symmetric, the inner products in Eq. (6) then are real; thus λ_1 is real.

Pursuing this perturbation calculation to higher orders, we can show that λ_n is real for all $n \geq 1$; thus the eigenvalue λ remains real under perturbations ϵV_p.

For general partially $\mathcal{PT}$ symmetric potentials, we use numerical methods to establish that their spectra can be all real. To illustrate, we take the complex potential $V(x,y)$ to be localized at four spots:

$$\begin{aligned} V(x,y) = {} & 3(e^{-(x-x_0)^2-(y-y_0)^2} + e^{-(x+x_0)^2-(y-y_0)^2}) \\ & +2(e^{-(x-x_0)^2-(y+y_0)^2} + e^{-(x+x_0)^2-(y+y_0)^2}) \\ & + i\beta[2(e^{-(x-x_0)^2-(y-y_0)^2} - e^{-(x+x_0)^2-(y-y_0)^2}) \\ & + (e^{-(x-x_0)^2-(y+y_0)^2} - e^{-(x+x_0)^2-(y+y_0)^2})], \end{aligned} \tag{7}$$

where x_0, y_0 control the separation distances between these four spots, and β is a real constant. For definiteness, we set $x_0 = y_0 = 1.5$. This potential is not $\mathcal{PT}$ symmetric but is partially $\mathcal{PT}$ symmetric with symmetry Eq. (2). For $\beta = 0.1$, this potential is displayed in Fig. 1 (top row). It is seen that $\mathrm{Re}(V)$ is symmetric in x, $\mathrm{Im}(V)$ anti-symmetric in x, and both $\mathrm{Re}(V)$, $\mathrm{Im}(V)$ are asymmetric in y. The spectrum of this potential is plotted in Fig. 1(c). It is seen that this spectrum contains three discrete eigenvalues and the continuous spectrum, which are all-real. Thus we have numerically established that partially $\mathcal{PT}$ symmetric potentials can have all-real spectra. For these real eigenvalues, their eigenfunctions respect the partial $\mathcal{PT}$ symmetry of the potential.

For potential (7) with varying β, we have found that its spectrum is all-real as long as $|\beta|$ is below a threshold value of 0.214. Above this threshold, a phase transition occurs, where complex eigenvalues appear in the spectrum. This phase transition is illustrated in Fig. 1(d), where the spectrum at $\beta = 0.3$ is shown. Phase transition is a well-known phenomenon of $\mathcal{PT}$ symmetric potentials [2,5,6,8]. We see that it arises in partially $\mathcal{PT}$ symmetric potentials too.

In addition to the four-spot potential (7), we have tested other partially $\mathcal{PT}$ symmetric (but not $\mathcal{PT}$ symmetric) potentials as well, such as the one-spot potential

$$V(x,y) = (1 + i\gamma xy)e^{-x^2-y^2},$$

where γ is a real constant. We have found that the spectrum of this potential is also all-real. In addition, no phase transition is detected here.

Next we examine whether these partially $\mathcal{PT}$ symmetric potentials support continuous families of solitons. These solitons are special solutions of Eq. (1) in the form of

$$\Psi(x,y,t) = \psi(x,y)e^{i\mu z}, \tag{8}$$

where μ is a real propagation constant, and $\psi(x,y)$ satisfies the equation

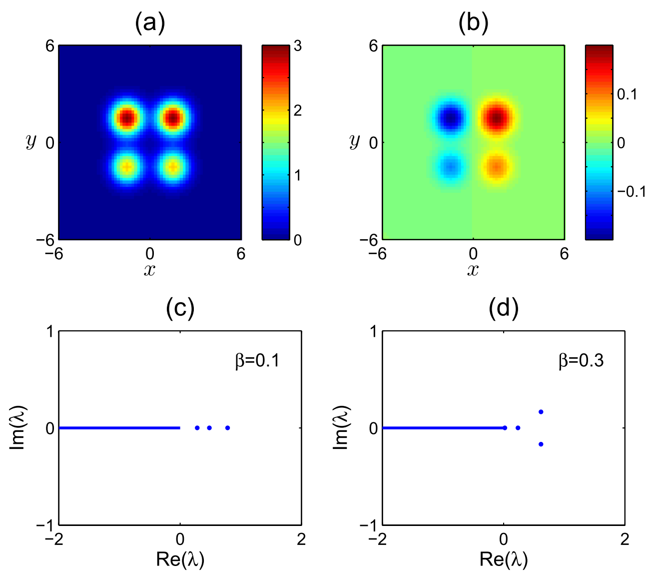

Fig. 1. (a), (b) Real and imaginary parts of the partially $\mathcal{PT}$ symmetric potential (7) for $\beta = 0.1$ and (c), (d) spectrum of this potential for $\beta = 0.1$ and 0.3, respectively.

$$\nabla^2\psi + V(x,y)\psi + \sigma|\psi|^2\psi = \mu\psi \quad (9)$$

and vanishes when (x,y) goes to infinity. In 1D, non-$\mathcal{PT}$ symmetric potentials cannot admit soliton families [13]. However, in higher dimensions, we will show analytically and numerically that partially $\mathcal{PT}$ symmetric potentials do support continuous families of solitons.

First, we show analytically that, from each real discrete eigenvalue of the partially $\mathcal{PT}$ symmetric potential, a continuous family of solitons bifurcates out under each of the focusing and defocusing nonlinearities. Suppose μ_0 is a discrete simple real eigenvalue of the potential and ψ_0 is its eigenfunction, i.e., $L\psi_0 = 0$, where $L \equiv \nabla^2 + V - \mu_0$. Then we seek solitons with the following perturbation expansion

$$\psi(x,y;\mu) = \epsilon^{1/2}[c_0\psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots],$$

where $\epsilon \equiv |\mu - \mu_0| \ll 1$ and c_0 is a certain nonzero constant. Substituting this expansion into Eq. (9), the $O(\epsilon^{1/2})$ equation is automatically satisfied. At $O(\epsilon^{3/2})$, we get the equation for ψ_1 as

$$L\psi_1 = c_0(\rho\psi_0 - \sigma|c_0|^2|\psi_0|^2\psi_0),$$

where $\rho = \mathrm{sgn}(\mu - \mu_0)$. The solvability condition of this ψ_1 equation is that its right-hand side be orthogonal to the adjoint homogeneous solution ψ_0^*. This condition yields an equation for c_0 as

$$|c_0|^2 = \frac{\rho\langle\psi_0^*, \psi_0\rangle}{\sigma\langle\psi_0^*, |\psi_0|^2\psi_0\rangle}. \quad (10)$$

For the real eigenvalue μ_0, its eigenfunction ψ_0 possesses partial $\mathcal{PT}$ symmetry. Thus the two inner products in the above equation are both real. Then for a certain sign of ρ, i.e., when μ is on a certain side of μ_0, the right side of Eq. (10) is positive; hence this equation is solvable for the constant c_0. Since the soliton in Eq. (9) is phase invariant, we can take c_0 to be positive without any loss of generality.

Pursuing this perturbation calculation to higher orders, we can find that this perturbation solution can be constructed to all orders for any small ϵ; thus a continuous family of solitons bifurcates out from the linear eigenmode (μ_0, ψ_0). In this construction process, partial $\mathcal{PT}$ symmetry of the potential is critical. For instance, in the absence of this partial $\mathcal{PT}$ symmetry (and $\mathcal{PT}$ symmetry), it is generally impossible to guarantee the reality of inner products in Eq. (10), which makes this equation unsolvable for c_0.

Next we corroborate these analytical results numerically. The partial $\mathcal{PT}$ potential (7) with $\beta = 0.1$ contains three discrete real eigenvalues [see Fig. 1(c)]. From each of these three eigenmodes, we have found numerically that a soliton family bifurcates out, just as the theory predicted. To illustrate, we take the focusing nonlinearity ($\sigma = 1$). Then power curves of soliton families bifurcated from the first and second eigenmodes of the potential are displayed in Fig. 2. Here the power P is defined as $\iint |\psi|^2 dxdy$. Interestingly, these two power curves are connected through a fold bifurcation, meaning that solitons bifurcated from these two eigenmodes belong to the same solution family, and the power of this solution family has an upper bound.

Profiles of solitons on this power curve are also displayed in Fig. 2. Here the amplitude fields of solitons at points "b,c" of the power curve (with $\mu = 1.3$) are plotted on the right column of the figure. It is seen that the soliton at point "b" has higher amplitude, obviously because it is on the upper power branch. The phase fields of these two solitons are similar; thus only the phase field at point "b" is shown. Note that these solitons share the same partial $\mathcal{PT}$ symmetry of the complex potential (7).

Lastly, we examine linear stability of this soliton family. For this purpose, we perturb these solitons by normal modes

$$\Psi(x,y,z) = e^{i\mu z}[\psi(x,y) + f(x,y)e^{\lambda z} + g^*(x,y)e^{\lambda^* z}],$$

where $f,\ g \ll 1$, and λ is the growth rate of the disturbance. Linearization of Eq. (1) for these perturbations yields a linear-stability eigenvalue problem

$$i\begin{bmatrix} M_1 & M_2 \\ -M_2^* & -M_1^* \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} = \lambda\begin{bmatrix} f \\ g \end{bmatrix}, \quad (11)$$

where $M_1 = \nabla^2 + V - \mu + 2\sigma|\psi|^2$, and $M_2 = \sigma\psi^2$.

We solve this eigenvalue problem (11) by the Fourier collocation method. For the four solitons on the power curve of Fig. 2, their eigenvalue spectra are computed and displayed in Fig. 3. It is seen that the soliton at point "a" contains a quartet of complex eigenvalues, and the soliton at point "c" contains a pair of real eigenvalues, thus these two solitons are both linearly unstable. However, solitons at points "b,d" only contain purely imaginary eigenvalues and are thus linearly stable.

Repeating this spectrum computation for other solitons on the power curve of Fig. 2, their linear stability

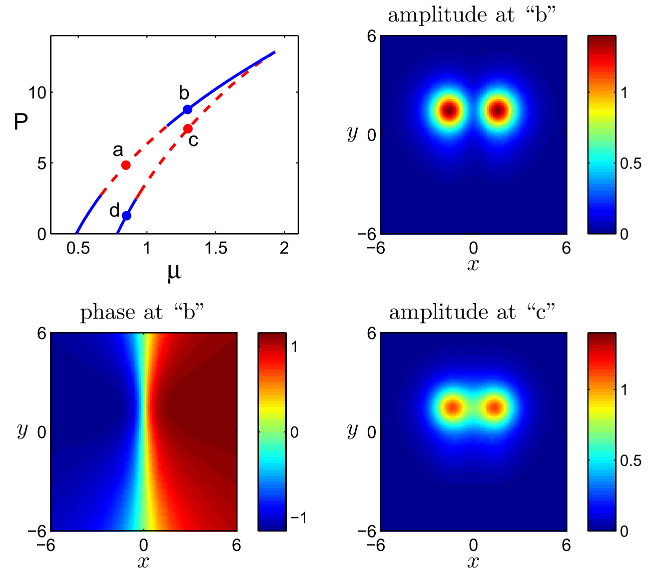

Fig. 2. Upper left: power diagram of the soliton family in potential (7) with $\beta = 0.1$ and $\sigma = 1$ (solid blue are stable and dashed red unstable); upper and lower right: amplitude fields of solitons ($|\psi|$) at points "b,c" of the power curve; lower left: phase field of the soliton at "b".

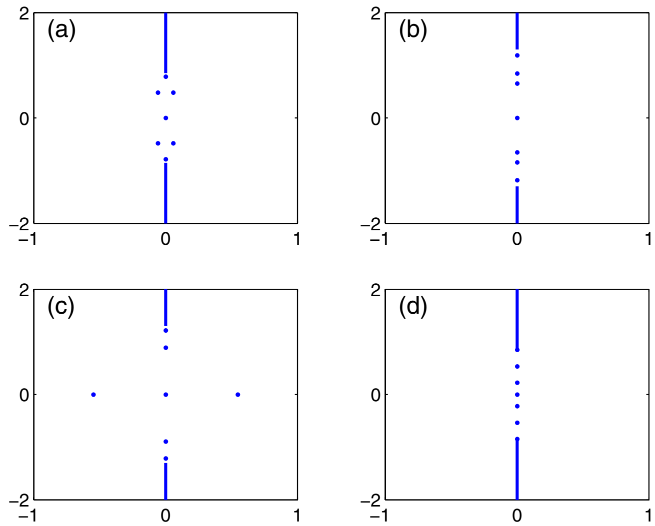

Fig. 3. Linear-stability spectra of the four solitons marked by letters "a,b,c,d" on the power curve of Fig. 2.

is then determined, and the results are indicated on that power curve, with solid blue representing stable solitons and dashed red for unstable ones. Notice that most of the lower power branch is unstable, while most of the upper power branch is stable. This is surprising, since in conservative potentials, solitons on the upper power branch are generally less stable.

In summary, we have proposed a class of multidimensional complex optical potentials that are not $\mathcal{PT}$ symmetric but rather partially $\mathcal{PT}$ symmetric, i.e., they are invariant under complex conjugation and reflection in a single spatial direction. We have shown that these partially $\mathcal{PT}$ symmetric potentials can possess all-real spectra and support continuous families of solitons, similar to $\mathcal{PT}$ symmetric potentials. We have also shown that these soliton families can exhibit multiple power branches, with the upper power branches more stable than the lower ones. These results expand the concept of $\mathcal{PT}$ symmetry in multidimensions, and they may find interesting optical applications.

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