

# Symmetry breaking of solitons in one-dimensional parity-time-symmetric optical potentials

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Symmetry breaking of solitons in a class of one-dimensional parity-time ( $PT$ ) symmetric complex potentials with cubic nonlinearity is reported. In generic  $PT$ -symmetric potentials, such symmetry breaking is forbidden. However, in a special class of  $PT$ -symmetric potentials  $V(x) = g^2(x) + \alpha g(x) + ig'(x)$ , where  $g(x)$  is a real and even function and  $\alpha$  a real constant, symmetry breaking of solitons can occur. That is, a branch of non- $PT$ -symmetric solitons can bifurcate out from the base branch of  $PT$ -symmetric solitons when the base branch's power reaches a certain threshold. At the bifurcation point, the base branch changes stability, and the bifurcated branch can be stable. © 2014 Optical Society of America

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Light propagation in optical waveguides is often modeled by Schrödinger-type equations under the paraxial approximation [1,2]. If the waveguide contains gain and loss, the optical potential of the Schrödinger equation would be complex [3,4]. A surprising phenomenon is that if this complex potential satisfies parity-time ( $PT$ ) symmetry, i.e., if the refractive index is even and the gain-loss profile odd, then the linear spectrum can still be all-real, thus allowing stationary light transmission [5–8]. A number of other unusual properties have been discovered in  $PT$ -symmetric systems as well [7–32]. One of them is that  $PT$ -symmetric potentials can support continuous families of  $PT$ -symmetric solitons [8,17–22], in contrast with typical dissipative systems where solitons only exist as isolated solutions [33].

However, symmetry breaking of solitons in  $PT$ -symmetric potentials has never been reported. Based on a perturbative analysis, it was shown that this symmetry breaking requires an infinite number of nontrivial conditions to be satisfied simultaneously, which cannot occur in a generic  $PT$ -symmetric potential [34]. From a similar analysis, it was shown that a generic non- $PT$ -symmetric complex potential could not support continuous families of solitons either [35]. Recently, a continuous family of solitons was reported in a special non- $PT$ -symmetric potential  $V(x) = g^2(x) + ig'(x)$ , where  $g(x)$  was a real asymmetric localized function [36]. This is a surprising result, and it suggests that symmetry breaking of solitons might also be possible in special types of  $PT$ -symmetric potentials.

In this Letter, we show that for the special type of  $PT$ -symmetric potentials  $V(x) = g^2(x) + \alpha g(x) + ig'(x)$ , where  $g(x)$  is an arbitrary real and even function, and  $\alpha$  is an arbitrary real constant; symmetry breaking of solitons can indeed occur. Specifically, in this class of complex potentials, a branch of non- $PT$ -symmetric solitons can bifurcate out from the base branch of  $PT$ -symmetric solitons when the base branch's power reaches a certain threshold. But if the  $PT$  potential deviates from this special form, then symmetry breaking generically disappears. Symmetry breaking of solitons in this class of  $PT$ -symmetric potentials is a surprising phenomenon, because infinitely many nontrivial conditions in Ref. [34]

are miraculously satisfied simultaneously, which makes this symmetry breaking possible.

The model for nonlinear propagation of light beams in one-dimensional complex optical potentials is taken as

$$i\Psi_z + \Psi_{xx} + V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \quad (1)$$

where  $z$  is the propagation direction,  $x$  is the transverse direction, and  $\sigma = \pm 1$  is the sign of cubic nonlinearity. The complex potential  $V(x)$  is assumed to be  $PT$ -symmetric, i.e.,

$$V^*(x) = V(-x), \quad (2)$$

where the asterisk “\*” represents complex conjugation.

Solitons in Eq. (1) are sought of the form

$$\Psi(x, z) = \psi(x)e^{i\mu z}, \quad (3)$$

where  $\mu$  is a real propagation constant, and  $\psi(x)$  is a localized function which satisfies the equation

$$\psi_{xx} + V(x)\psi + \sigma|\psi|^2\psi = \mu\psi. \quad (4)$$

For a generic  $PT$ -symmetric potential  $V(x)$ , its linear spectrum, i.e., the set of eigenvalues  $\mu$  for the linear Schrödinger equation

$$\psi_{xx} + V(x)\psi = \mu\psi, \quad (5)$$

can be all-real [5–8]. In addition, such a potential often admits continuous families of  $PT$ -symmetric solitons [8,17–22]. However, it is very hard for a  $PT$  potential to admit families of non- $PT$ -symmetric solitons. Indeed, such solution families have never been reported to the author's best knowledge. The analytical reason is that in order for such non- $PT$  solution families to exist in a  $PT$  potential, infinitely many nontrivial conditions have to be met, which is impossible in generic  $PT$ -symmetric potentials [34].

In this Letter, we consider a special class of  $PT$ -symmetric potentials

$$V(x) = g^2(x) + \alpha g(x) + ig'(x), \quad (6)$$

where  $g(x)$  is an arbitrary real and even function, and  $\alpha$  is an arbitrary real constant. This form of the potential contains that used in Ref. [36] as a special case (when  $\alpha = 0$ ). Even though this potential can be reduced to the form of Ref. [36] through shifts of  $g(x) + \alpha/2 \rightarrow g(x)$  and  $\mu + \alpha^2/4 \rightarrow \mu$ , we find this new form of the potential to be more flexible and easier to work with (see later text).

Note that the Schrödinger eigenvalue problem [Eq. (5)] with potential [Eq. (6)] can be transformed to Zakharov–Shabat-type eigenvalue problems [36,37]. This fact can be used to establish all-real spectra under these potentials.

For this class of potentials [Eq. (6)], we will show that continuous families of non- $PT$ -symmetric solitons can appear through symmetry-breaking bifurcations. In real symmetric potentials, symmetry breaking of solitons is a well-known phenomenon [38]. But in complex  $PT$ -symmetric potentials, such symmetry breaking is very novel. We will numerically demonstrate this symmetry breaking by a number of examples. Comparison with the analytical conditions for symmetry breaking in Ref. [34] will also be made.

We first consider a localized double-hump function of  $g(x)$

$$g(x) = A(e^{-(x+x_0)^2} + e^{-(x-x_0)^2}), \quad (7)$$

with  $A = 2$ ,  $x_0 = 1.2$  and  $\alpha = 1$ . The corresponding  $PT$  potential  $V(x)$  from Eq. (6) is plotted in Fig. 1(a). The linear spectrum of this potential is all-real, and it contains three positive discrete eigenvalues, the largest being 3.6614. From this largest discrete eigenmode, a family of  $PT$ -symmetric solitons bifurcates out. Under focusing nonlinearity ( $\sigma = 1$ ), the power curve of this solution

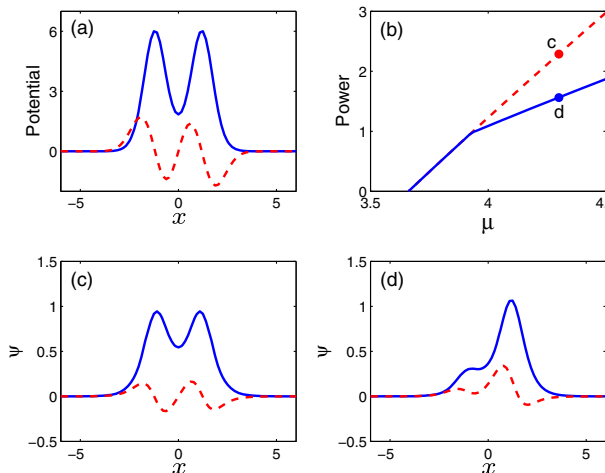


Fig. 1. Symmetry breaking of solitons in the  $PT$ -symmetric potential [Eq. (6)] for a double-hump function [Eq. (7)] with  $A = 2$ ,  $x_0 = 1.2$ ,  $\alpha = 1$ , and  $\sigma = 1$ . (a) Profile of the potential  $V(x)$  [solid blue,  $\text{Re}(V)$ ; dashed red,  $\text{Im}(V)$ ]. (b) Power diagram (solid blue, stable branches; dashed red, unstable branch). (c)  $PT$ -symmetric soliton at point “c” of the power diagram. (d) Non- $PT$ -symmetric soliton at point “d” of the power diagram. In (c) and (d), solid blue is  $\text{Re}(\psi)$ , dashed is red  $\text{Im}(\psi)$ , and  $\mu = 4.3$ .

family is shown in Fig. 1(b), and the soliton profile at the marked point “c” (with  $\mu = 4.3$ ) is displayed in Fig. 1(c). Here the soliton’s power is defined as  $P = \int_{-\infty}^{\infty} |\psi(x; \mu)|^2 dx$ . What is interesting is that, at the propagation constant  $\mu_c \approx 3.9287$  of this base power branch, a family of non- $PT$ -symmetric solitons bifurcates out. The power curve of this non- $PT$ -symmetric family is also shown in Fig. 1(b). At the marked point “d” of the bifurcated power branch, the non- $PT$ -symmetric solution is displayed in Fig. 1(d). It is seen that most of the energy in this soliton resides on the right side of the potential. In order to ascertain these non- $PT$ -symmetric solitons are true solutions to Eq. (4), we have computed them using the Newton-conjugate-gradient method [2] and 32 significant digits (in Matlab with a multiprecision toolbox). These solutions are found to satisfy Eq. (4) to an accuracy of  $1 \times 10^{-30}$ , confirming that they are indeed true solutions.

Because Eq. (4) is  $PT$ -symmetric, if  $\psi(x)$  is a solution, so is  $\psi^*(-x)$ . Thus, for each of the non- $PT$ -symmetric solitons  $\psi(x; \mu)$  in Fig. 1(b), there is a companion soliton  $\psi^*(-x; \mu)$  whose energy resides primarily on the left side of the potential. In other words, the bifurcation in Fig. 1(b) is a pitchfork-type symmetry-breaking bifurcation. This bifurcation resembles that in conservative systems with real symmetric potentials, which is remarkable because the present potential is dissipative (with gain and loss).

Linear stability of these solitons can be determined by perturbing them with normal modes,  $\Psi(x, t) = e^{i\mu z}[\psi(x) + q(x)e^{\lambda z} + r^*(x)e^{\lambda^* z}]$ , where  $q, r \ll \psi$ . Inserting this perturbation into Eq. (1), a linearized eigenvalue problem for  $(q, r)$  can be derived, with  $\lambda$  being the eigenvalue. If eigenvalues with positive real parts exist, then the soliton is linearly unstable. Otherwise it is linearly stable.

We have determined the linear-stability spectra of these solitons by the Fourier collocation method [2]. We found that the base branch of  $PT$ -symmetric solitons is stable before the bifurcation point ( $\mu < \mu_c$ ). After the bifurcation point, this base branch becomes unstable due to the presence of a real positive eigenvalue. However, the bifurcated branch of non- $PT$ -symmetric solitons is stable. These stability results are marked on the power diagram of Fig. 1(b). To corroborate these linear-stability results, we perturb the two solitons in Figs. 1(c) and 1(d) by 1% random-noise perturbations, and their nonlinear evolutions are displayed in Figs. 2(a) and 2(b). It is seen from Fig. 2(a) that the  $PT$ -symmetric soliton in Fig. 1(c) breaks up and becomes non- $PT$ -symmetric. Upon further propagation, the solution bounces back to almost

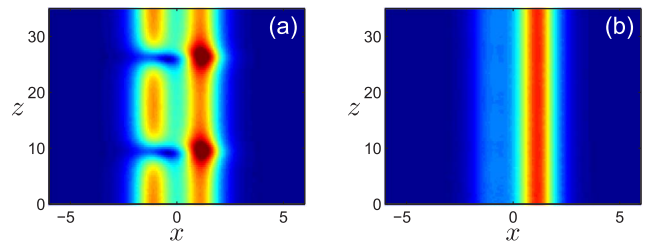


Fig. 2. (a) and (b) Nonlinear evolutions of the two solitons in Figs. 1(c) and 1(d) under 1% random-noise perturbations.

$PT$ -symmetric again, followed by another breakup. In contrast, Fig. 2(b) shows that the non- $PT$ -symmetric soliton in Fig. 1(d) is stable against perturbations.

For the  $PT$ -symmetric potential [Eq. (6)] with  $g(x)$  given by Eq. (7), we have also tested many other values of  $A$ ,  $x_0$  and  $\alpha$ , and observed symmetry-breaking bifurcations as well. Interestingly, we found that symmetry breaking can still occur even if the potential is above the phase-transition point, i.e., when the linear spectrum of the potential is not all-real [5–7]. For instance, with the same  $A$  and  $x_0$  values as above, when  $\alpha = -0.9$ , the linear spectrum of the resulting potential [Eq. (6)] is displayed in Fig. 3(a). This spectrum contains a pair of complex eigenvalues, indicating that this  $PT$ -symmetric potential is above the phase-transition point. This spectrum also contains a discrete real eigenvalue  $\mu \approx 0.5817$ , from which a continuous family of  $PT$ -symmetric solitons bifurcates out. The power curve of these  $PT$ -symmetric solitons (for  $\sigma = 1$ ) is shown in Fig. 3(b). The low-power segment of this solution branch is unstable since the potential is above the phase-transition point. However, when the power is above 1.14, these  $PT$ -symmetric solitons become stable. At the power value of approximately 1.8, symmetry breaking occurs, where a branch of non- $PT$ -symmetric solitons bifurcates out. Meanwhile, the base branch of  $PT$ -symmetric solitons loses stability. The presence of symmetry breaking and existence of stable non- $PT$ -symmetric solitons in a  $PT$ -symmetric potential above phase transition is remarkable.

In addition to the double-hump function [Eq. (7)], symmetry breaking can occur in the  $PT$ -symmetric potential [Eq. (6)] for many other types of real and even functions of  $g(x)$ . To demonstrate, we now choose a periodic function for  $g(x)$

$$g(x) = \sin^2 x, \quad (8)$$

and take  $\alpha = 6$ . The corresponding periodic  $PT$ -symmetric potential [Eq. (6)] is displayed in Fig. 4(a). In this potential under focusing nonlinearity ( $\sigma = 1$ ), a family of  $PT$ -symmetric “dipole” solitons exists in the semi-infinite gap. The power curve of this “dipole” family is plotted in Fig. 4(b), and the profile of the  $PT$ -symmetric soliton at point “c” of this power curve is shown in Fig. 4(c). At  $\mu_c \approx 4.5801$  of this  $PT$ -symmetric power curve, a family of non- $PT$ -symmetric solitons bifurcates out. Its power curve is also shown in Fig. 4(b) (the middle

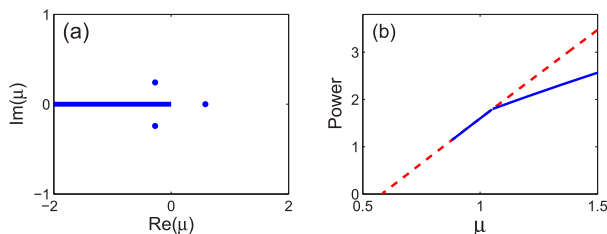


Fig. 3. Symmetry breaking of solitons in the  $PT$ -symmetric potential [Eqs. (6) and (7)] above the phase transition point (with  $A = 2$ ,  $x_0 = 1.2$ ,  $\alpha = -0.9$ , and  $\sigma = 1$ ). (a) Linear spectrum of the potential [Eq. (6)]. (b) Power diagram of symmetry-breaking bifurcations (solid blue, stable solitons; dashed red, unstable solitons).

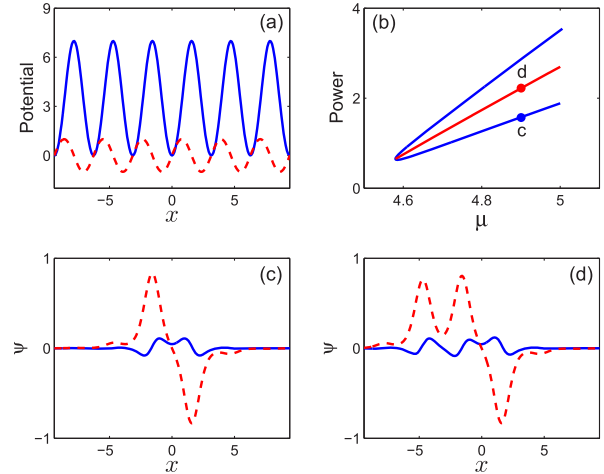


Fig. 4. Symmetry breaking of solitons in a periodic  $PT$ -symmetric potential [Eq. (6)] with  $g(x)$  given by Eq. (8),  $\alpha = 6$  and  $\sigma = 1$ . (a) Profile of the potential  $V(x)$  [solid blue,  $\text{Re}(V)$ ; dashed red,  $\text{Im}(V)$ ]. (b) Power diagram. (c)  $PT$ -symmetric soliton at point “c” of the power diagram. (d) Non- $PT$ -symmetric soliton at point “d” of the power diagram. In (c) and (d), solid blue is  $\text{Re}(\psi)$  and dashed red is  $\text{Im}(\psi)$ .

curve), and the non- $PT$ -symmetric soliton at point “d” of this power branch is displayed in Fig. 4(d).

In a  $PT$ -symmetric potential, symmetry breaking requires infinitely many nontrivial conditions to be satisfied simultaneously [34]. Due to such stringent conditions, symmetry breaking cannot occur in a generic  $PT$ -symmetric potential. But for the special class of potentials [Eq. (6)], those sequences of conditions are miraculously satisfied. For instance, for the potential in Fig. 1, we have numerically checked the first two of those conditions, the first being Eq. (69) in Ref. [34], and the second being mentioned but not presented in Ref. [34]. We verified that those two (nontrivial) conditions are indeed met. Why potentials [Eq. (6)] satisfy all those infinite conditions is an interesting question which merits further investigation. The special nature of potentials [Eq. (6)] is also reflected in the fact that, when the  $PT$  potential  $V(x)$  deviates from those special forms, symmetry breaking generically disappears. To demonstrate, we introduce a real parameter  $\beta$  into those potentials:

$$V(x) = g^2(x) + ag(x) + i\beta g'(x). \quad (9)$$

Then when  $\beta$  moves away from 1, we cannot find symmetry breaking anymore. For instance, from the potentials in Figs. 1–3, when we change  $\beta$  from 1 to 0.9, we find that families of non- $PT$ -symmetric solitons in those figures no longer exist. Similar findings were also reported in [34,39], where for several  $PT$ -symmetric potentials not of the form [Eq. (6)], families of non- $PT$ -symmetric solitons could not be found.

It should be pointed out that even though the potentials [Eq. (6)] are very special among all  $PT$ -symmetric potentials, they still represent a large class of potentials since  $g(x)$  is an arbitrary real even function, and  $\alpha$  is an arbitrary real constant. In addition, even though we chose a cubic nonlinearity in our model [Eq. (1)], we have found that other types of nonlinearities (such as

cubic-quintic nonlinearity) admit symmetry breaking as well inside the class of potentials [Eq. (6)]. Thus, by changing different forms of functions  $g(x)$  and nonlinearities, we may still get a wide variety of symmetry-breaking bifurcations. Whether there are additional types of  $PT$ -symmetric potentials that admit symmetry-breaking bifurcations is an interesting open question. Extension of this symmetry breaking to higher spatial dimensions is another important direction.

In summary, we have reported symmetry breaking bifurcations in a class of  $PT$ -symmetric potentials [Eq. (6)]. From these bifurcations, families of stable non- $PT$ -symmetric solitons can be generated. These results open the door for the study of symmetry breaking in  $PT$ -symmetric potentials under more general circumstances.

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