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Structure of linearization operators of the Korteweg–de Vries hierarchy equations expanded around single-soliton solutions

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Abstract

In this Letter, we analyze the structure of linearization operators of the Korteweg–de Vries (KdV) hierarchy equations expanded around single-soliton solutions. We uncover the remarkable property that these linearization operators can be factored into the integro-differential operator which generates this hierarchy and the linearization operator of the KdV equation. An important consequence of this operator structure is that the linearization operators of all KdV hierarchy equations expanded around single-soliton solutions share the *same* complete set of eigenfunctions. Furthermore, these eigenfunctions are simply related to squared eigenstates of the Schrödinger operator with a soliton potential. Similar results hold for the adjoint linearization operators of this hierarchy. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Integrable equations are nonlinear wave systems which can be solved exactly by the inverse scattering method. Since the pioneering work of Gardner et al. [1] which solved the Korteweg–de Vries (KdV) equation, many other equations such as the nonlinear Schrödinger (NLS), sine–Gordon, modified-KdV, Benjamin–Ono and Manakov equations have been solved (see references in [2]). The original work of Ablowitz et al. [3] further enlarged the family of integrable equations, and introduced the concept of integrable hierarchy for the first time. Each hierarchy is generated by an integro-differential operator and is associated with the same eigenvalue problem. The equations inside a hierarchy are uniquely characterized by their linear dispersion relations. The KdV and NLS hierarchies are two familiar examples.

Integrable equations possess many remarkable properties such as infinite conservation laws and Painlevé property [2,4]. In this Letter we reveal another surprising property on equations inside an integrable hierarchy, i.e., linearization operators of all equations in a hierarchy expanded around a single-soliton solution can be decomposed into the integro-differential operator which generates the hierarchy and the linearization operator of the lowest-order equation in the hierarchy. An immediate consequence of this operator structure is that the linearization operators of all equations in a hierarchy share the *same* complete set of eigenfunctions, and these eigenfunctions are closely related to the squared eigenstates of the associated eigenvalue

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problem. In this Letter, we establish these results only for the KdV hierarchy. Extensions of these results to other hierarchies such as the NLS and modified-KdV hierarchies will be presented elsewhere. In the end of this Letter, we discuss some applications of these results obtained. It is worth noting here that our analysis is independent of the inverse scattering method, even though connections to that method is still visible.

2. Structure of linearization operators in the KdV hierarchy

The KdV hierarchy is of the form [3]

$$q_t + C\left(4L_s^+\right)q_x = 0, \tag{2.1}$$

where $C(k^2)$ is the phase velocity of the linearized equations, and the integro-differential operator L_s^+ is

$$L_s^+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}q_x \int_x^\infty dy. \tag{2.2}$$

Here the subscript s in L_s^+ refers to Schrödinger, as the associated eigenvalue problem for this hierarchy is the Schrödinger equation [1,3]. The adjoint operator of L_s^+ is

$$L_s = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}\int_{-\infty}^x dy\, q_y. \tag{2.3}$$

In this Letter, we require the phase velocity function $C(z)$ to be entire. When $C(z) = -z$, Eq. (2.1) becomes the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0. \tag{2.4}$$

When $C(z) = z^2$, Eq. (2.1) is the fifth-order KdV hierarchy

$$q_t + q_{xxxxx} + 10qq_{xxx} + 20q_xq_{xx} + 30q^2q_x = 0. \tag{2.5}$$

Other members in this hierarchy can be obtained by choosing different functions for the phase velocity $C(z)$.

In the rest of this section, occasions will arise where we want to apply the operator L_s^+ (and L_0^+ to be defined in Eq. (2.11)) on a function $g'(x)$, where $g(x)$ is related to continuous eigenfunctions and is oscillatory at infinity. In such cases, we adopt the following convention for the integral term involved:

$$\int_x^\infty g'(y)\,dy \equiv -g(x). \tag{2.6}$$

This convention echoes the fact that, when we obtain a particular KdV hierarchy equation from (2.1), terms such as $\int_x^\infty q'(y)\,dy$ are always evaluated as $-q(x)$ and so on. This convention applies notably to the commutability relation (2.18) and the factorization formula (2.26) when they operate on continuous eigenfunctions. It applies to the eigenfunction relation (2.35) as well. We emphasize that this convention is only a technical issue. It does not affect our results at all.

We now consider single-soliton solutions of the KdV hierarchy (2.1). One can check that the soliton family

$$q(x,t) = 2\eta^2\mathrm{sech}^2\eta\left\{x - C\left(-4\eta^2\right)t\right\} \tag{2.7}$$

satisfy Eq. (2.1), where η is a free amplitude parameter. By rescaling the variables x and q by η and η^2, respectively, and by denoting $C(\eta^2 z)$ as $C(z)$, we can normalize $\eta = 1$ in the soliton solution (2.7) while keeping the evolution equation (2.1) intact. We also adopt the coordinate system moving with speed $C(-4)$, i.e.,

$$\bar{x} = x - C(-4)t, \qquad \bar{t} = t. \tag{2.8}$$

When the bars are dropped, the KdV hierarchy (2.1) finally becomes

$$q_t + \left[C\left(4L_s^+\right) - C(-4)\right]q_x = 0, \tag{2.9}$$

where

$$q_0(x) = 2\,\mathrm{sech}^2 x \tag{2.10}$$

is its normalized single-soliton solution.

Two operators, L_0^+ and L_0, will be used frequently in the rest of this section. They are defined as L_s^+ and L_s with $q(x,t)$ replaced by $q_0(x)$, i.e.,

$$L_0^+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q_0 + \frac{1}{2}q_{0x}\int_x^\infty dy \tag{2.11}$$

and

$$L_0 = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q_0 + \frac{1}{2}\int_{-\infty}^x dy\, q_{0y}. \tag{2.12}$$

Naturally, L_0 is the adjoint operator of L_0^+, just as L_s is the adjoint operator of L_s^+. Note that

$$L_0^+ q_{0x} = -q_{0x}. \tag{2.13}$$

This relation will be used later in Letter.

Next, we linearize the evolution equation (2.9) around its soliton solution (2.10). We set

$$q(x,t) = q_0(x) + \tilde{q}(x,t), \tag{2.14}$$

where $\tilde{q} \ll 1$. When it is substituted into Eq. (2.9) and higher-order terms discarded, the linearized equation of (2.9) would be in the form

$$\tilde{q}_t + L\tilde{q} = 0, \tag{2.15}$$

where L is the linearization operator. We denote the adjoint operator of L as L^A. For the KdV equation, $C(z) = -z$. In this case, linearization of Eq. (2.9) around soliton (2.10) shows that the linearization operator is

$$L_{\mathrm{kdv}} = \frac{\partial^3}{\partial x^3} + (6q_0 - 4)\frac{\partial}{\partial x} + 6q_{0x}. \tag{2.16}$$

Its adjoint operator is

$$L_{\mathrm{kdv}}^A = -\frac{\partial^3}{\partial x^3} - (6q_0 - 4)\frac{\partial}{\partial x}. \tag{2.17}$$

An important property is that L_0^+ and L_{kdv} are commutable, and L_0 and L_{kdv}^A are commutable, i.e.,

$$L_0^+ L_{\mathrm{kdv}} = L_{\mathrm{kdv}} L_0^+ \tag{2.18}$$

and

$$L_0 L_{\mathrm{kdv}}^A = L_{\mathrm{kdv}}^A L_0. \tag{2.19}$$

These facts can be verified by direct calculations.

The main objective of this Letter is to determine the structure of operators L and L^A for any KdV hierarchy equation (2.9). We first consider the case where the phase velocity $C(z)$ is a power function, i.e., $C(z) = z^n$, where n is any positive integer. In this case, Eq. (2.9) becomes

$$q_t + \left[\left(4L_s^+\right)^n - (-4)^n\right] q_x = 0. \tag{2.20}$$

When Eq. (2.14) is substituted into the operator $4L_s^+$, linearization of $4L_s^+$ is

$$4L_s^+ = 4L_0^+ - 4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy + O\left(\tilde{q}^2\right). \tag{2.21}$$

Thus, linearization of $(4L_s^+)^n$ is

$$\begin{aligned}\left(4L_s^+\right)^n = {} & \left(4L_0^+\right)^n \\ & + \sum_{i=1}^n \left(4L_0^+\right)^{i-1}\left[-4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy\right] \\ & \times \left(4L_0^+\right)^{n-i} + O\left(\tilde{q}^2\right).\end{aligned} \tag{2.22}$$

When this equation is utilized, we find the linearization operator L of the evolution equation (2.20) as

$$\begin{aligned}L\tilde{q} = {} & \left[\left(4L_0^+\right)^n - (-4)^n\right]\frac{\partial \tilde{q}}{\partial x} \\ & + \sum_{i=1}^n \left(4L_0^+\right)^{i-1}\left[-4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy\right] \\ & \times \left(4L_0^+\right)^{n-i} q_{0x}.\end{aligned} \tag{2.23}$$

Recalling Eqs. (2.13) and (2.16), the above equation becomes

$$\begin{aligned}L\tilde{q} &= \sum_{i=1}^n \left(4L_0^+\right)^{i-1}(-4)^{n-i} \\ &\quad \times \left[\left(4L_0^+ + 4\right)\frac{\partial \tilde{q}}{\partial x} - 4q_{0x}\tilde{q} - 2q_0\tilde{q}_x\right] \\ &= \sum_{i=1}^n \left(4L_0^+\right)^{i-1}(-4)^{n-i} \\ &\quad \times \left[-\tilde{q}_{xxx} - (6q_0 - 4)\tilde{q}_x - 6q_{0x}\tilde{q}\right] \\ &= -\sum_{i=1}^n \left(4L_0^+\right)^{i-1}(-4)^{n-i} L_{\mathrm{kdv}}\tilde{q}.\end{aligned} \tag{2.24}$$

Define the function

$$M(z) \equiv \frac{C(-4) - C(z)}{4 + z}. \tag{2.25}$$

Recalling that $C(z) = z^n$, it is easy to see from Eq. (2.24) that the linearization operator L in this case is simply

$$L = M\left(4L_0^+\right) L_{\mathrm{kdv}}. \tag{2.26}$$

As for the adjoint operator L^A, we recall the fact that, for any two operators P and Q, $(PQ)^A = Q^A P^A$, where the superscript A represents the adjoint operator. Since L_0 is the adjoint operator of L_0^+, thus from Eq. (2.24) we have

$$L^A = -\sum_{i=1}^{n} (-4)^{n-i} L^A_{\mathrm{kdv}} (4L_0)^{i-1}. \tag{2.27}$$

But L_0 and L^A_{kdv} are commutable (see Eq. (2.19)). So

$$L^A = -\sum_{i=1}^{n} (4L_0)^{i-1} (-4)^{n-i} L^A_{\mathrm{kdv}}, \tag{2.28}$$

i.e.,

$$L^A = M(4L_0) L^A_{\mathrm{kdv}}. \tag{2.29}$$

The operator structures (2.26) and (2.29) were derived above for power functions of the phase velocity $C(z)$. In the general case, where $C(z)$ is an entire function, we can expand $C(z)$ into a power series. In that way, we can easily prove that expressions (2.26) and (2.29) still hold in the general case.

Eqs. (2.26) and (2.29) are the key results of this Letter. They give the structure of linearization operators of all KdV hierarchy equations expanded around a single-soliton solution. Specifically, these linearization operators L (L^A) can be factored into two simple operators L_0^+ and L_{kdv} (L_0 and L^A_{kdv}). The simplicity of these results is quite remarkable. It is another nice property of the KdV hierarchy in addition to those which have been discovered before [2,4].

An immediate consequence of factorization results (2.26) and (2.29) is that L, L_0^+ and L_{kdv} are mutually commutable, and L^A, L_0 and L^A_{kdv} are mutually commutable. This follows from Eqs. (2.26), (2.29) and the fact that L_0^+ (L_0) and L_{kdv} (L^A_{kdv}) are commutable (see Eqs. (2.18) and (2.19)). Since commutable operators share the same eigenfunctions, we conclude that the linearization operator L (L^A) of any KdV hierarchy equation shares the same eigenfunctions of operator L_0^+ (L_0) or L_{kdv} (L^A_{kdv}). The eigenfunctions of operators L_0^+, L_0, L_{kdv} and L^A_{kdv} have been obtained before [3,5–8]. Below, we simply present the eigenfunctions of operators L_0^+ and L_0, and determine their corresponding eigenvalues under operators L and L^A.

The eigenfunctions of operators L_0^+ and L_0 are given in terms of squared eigenstates of the Schrödinger equation

$$v_{xx} + \left[\zeta^2 + q_0(x)\right] v = 0, \tag{2.30}$$

where $q_0(x)$ is the soliton potential (2.10). Using conventional notations [3], we define the eigenstates $\psi(x,\zeta)$ and $\phi(x,\zeta)$ of Eq. (2.30) as

$$\psi \to e^{i\zeta x}, \quad x \to \infty, \tag{2.31}$$

$$\phi \to e^{-i\zeta x}, \quad x \to -\infty. \tag{2.32}$$

It is easy to check that functions ψ and ϕ above are simply

$$\psi(x,\zeta) = \frac{\zeta + i\tanh x}{\zeta + i} e^{i\zeta x} \tag{2.33}$$

and

$$\phi(x,\zeta) = \frac{\zeta - i\tanh x}{\zeta + i} e^{-i\zeta x}. \tag{2.34}$$

For these eigenstates, we have [3,5]

$$L_0^+ \left(\psi^2\right)_x = \zeta^2 \left(\psi^2\right)_x \tag{2.35}$$

and

$$L_0 \phi^2 = \zeta^2 \phi^2. \tag{2.36}$$

In other words, $(\psi^2)_x$ are eigenfunctions of operator L_0^+, and ϕ^2 are eigenfunctions of L_0. In the above two equations, ζ is an arbitrary complex number. To separate continuous and discrete eigenfunctions, we define two sets

$$\left\{ \frac{\partial \psi^2(x,\zeta)}{\partial x},\ \zeta \text{ real};\ \left.\frac{\partial \psi^2}{\partial x}\right|_{\zeta=i},\ \left.\frac{\partial^2 \psi^2}{\partial x \partial \zeta}\right|_{\zeta=i} \right\} \tag{2.37}$$

and

$$\left\{ \phi^2(x,\zeta),\ \zeta \text{ real};\ \phi^2\big|_{\zeta=i},\ \left.\frac{\partial \phi^2}{\partial \zeta}\right|_{\zeta=i} \right\}. \tag{2.38}$$

Set (2.37) consists of the continuous and discrete eigenfunctions of operator L_0^+. The continuous eigenfunctions $\partial\psi^2(x,\zeta)/\partial x$ satisfy Eq. (2.35). For the discrete eigenfunctions, we have

$$L_0^+ \left.\frac{\partial \psi^2}{\partial x}\right|_{\zeta=i} = -\left.\frac{\partial \psi^2}{\partial x}\right|_{\zeta=i} \tag{2.39}$$

and

$$L_0^+ \left.\frac{\partial^2 \psi^2}{\partial x \partial \zeta}\right|_{\zeta=i} = \left\{ -\frac{\partial^2 \psi^2}{\partial \zeta \partial x} + 2i\left(\psi^2\right)_x \right\}_{\zeta=i}. \tag{2.40}$$

Note that $(\partial^2\psi^2/\partial x \partial\zeta)|_{\zeta=i}$ is a "generalized" eigenfunction of L_0^+. Set (2.38) consists of continuous and discrete eigenfunctions of operator L_0. The continuous eigenfunctions $\phi^2(x,\zeta)$ satisfy Eq. (2.36). The discrete eigenfunctions satisfy

$$L_0\phi^2\big|_{\zeta=i} = -\phi^2\big|_{\zeta=i} \tag{2.41}$$

and

$$L_0\frac{\partial\phi^2}{\partial\zeta}\bigg|_{\zeta=i} = \left(-\frac{\partial\phi^2}{\partial\zeta} + 2i\phi^2\right)\bigg|_{\zeta=i}. \tag{2.42}$$

Both sets (2.37) and (2.38) are complete [5].

We have shown above that L (L^A) and L_0^+ (L_0) share the same eigenfunctions. Indeed, using the decomposition results (2.26) and (2.29), we can check that for operator L,

$$L\frac{\partial\psi^2(x,\zeta)}{\partial x} = 2i\zeta\left\{C\left(4\zeta^2\right) - C(-4)\right\}\frac{\partial\psi^2(x,\zeta)}{\partial x}, \tag{2.43}$$

$$L\frac{\partial\psi^2}{\partial x}\bigg|_{\zeta=i} = 0, \tag{2.44}$$

and

$$L\frac{\partial^2\psi^2}{\partial x\partial\zeta}\bigg|_{\zeta=i} = 16iM(-4)\frac{\partial\psi^2}{\partial x}\bigg|_{\zeta=i}. \tag{2.45}$$

For operator L^A, we have

$$L^A\phi^2(x,\zeta) = 2i\zeta\left\{C\left(4\zeta^2\right) - C(-4)\right\}\phi^2(x,\zeta), \tag{2.46}$$

$$L^A\phi^2\big|_{\zeta=i} = 0, \tag{2.47}$$

and

$$L^A\frac{\partial\phi^2}{\partial\zeta}\bigg|_{\zeta=i} = 16iM(-4)\phi^2\big|_{\zeta=i}. \tag{2.48}$$

Thus, sets (2.37) and (2.38) are eigenfunctions of operators L and L^A as expected. In addition, these eigenfunctions are complete.

3. Discussion

In this Letter, we determined the structure of linearization operators of the KdV hierarchy equations expanded around a single-soliton solution. We showed that these operators can be decomposed into the integro-differential operator which generates the hierarchy and the linearization operator of the KdV equation. Similar decompositions apply to the adjoint linearization operators. As a consequence, we established that linearization operators (adjoint operators) of all KdV hierarchy equations share the same complete set of eigenfunctions, and these eigenfunctions are directly related to squared eigenstates of the Schrödinger equation with a soliton potential. We comment that these results can be extended to other integrable hierarchies. These extensions will be reported elsewhere.

The simple linearization operator structures (2.26) and (2.29) are an important property of the KdV hierarchy. They are another indication of magic associated with integrable hierarchies, or integrable equations in general. Its consequence that all the linearization operators of the KdV hierarchy share the same complete set of eigenfunctions (see Eqs. (2.43)–(2.48)) have important physical and mathematical applications. The first application is the development of a direct soliton perturbation theory for perturbed KdV hierarchy equations. In this theory, complete eigenfunctions of the linearized equation expanded around a single-soliton solution are the key component. Historically, this theory was developed only for the sine–Gordon, NLS, KdV and Benjamin–Ono equations [7–13] (it was also developed for the non-integrable cubic–quintic NLS equation [14], but that theory was not complete). With the result of this Letter, one can now develop a direct soliton perturbation theory for any equation in the KdV hierarchy. We note that the inverse-scattering-based soliton perturbation theory for the KdV hierarchy has been developed for over twenty years [5,15]. But the direct soliton perturbation theory is preferred by many people, since it does not rely on the inverse scattering method and it has a simplistic appeal (see also [16]). In the inverse-scattering soliton perturbation theory for the KdV hierarchy, the expansion basis for the potential is the squared eigenstates of the Schrödinger equation. Because of this, it has long been suspected by people familiar with the inverse scattering method that these squared eigenstates must also solve the linearized equation of any KdV hierarchy equation. In this Letter, we showed for the first time that this is indeed the case. Thus these squared eigenfunctions also form the expansion basis in the direct soliton perturbation theory. These results indicate that, at a deep level, the inverse-scattering soliton perturbation theory and the direct soliton perturbation theory are really equivalent.

Another application of the complete eigenfunctions obtained in this Letter for the KdV hierarchy is in the stability analysis of solitary waves in perturbed KdV hierarchy equations. In this analysis, an important question is whether discrete eigenvalues of the linearization operator expanded around a solitary wave can bifurcate from the continuous spectrum. This calculation could not be done without knowing the complete eigenfunctions of the linearized KdV hierarchy equations expanded around single-soliton solutions. In the literature, this eigenvalue bifurcation analysis was done only for the perturbed NLS, sine–Gordon and Manakov equations [17–19]. With the results of this Letter, it is now possible to perform this analysis for any perturbed KdV hierarchy equation. Extension of our results would make it possible to perform this analysis for other perturbed hierarchy equations such as the NLS hierarchy and modified-KdV hierarchy as well.

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