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Resonance- and phase-induced window sequences in vector-soliton collisions

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Abstract

In this Letter, we report two new discoveries in vector-soliton collisions of the nonintegrable coupled nonlinear Schrödinger equations. The first discovery is that, at low cross-phase-modulational coefficients, a sequence of reflection windows similar to that in the ϕ^4 model appears in the collision of two orthogonally polarized and equal-amplitude solitons. This window sequence is caused by a resonance between the translational motion and width oscillations of vector solitons. Analytically, we explain these collision behaviors by a variational model which qualitatively reproduces this window sequence. The second discovery is that, when initial solitons are not orthogonally polarized, their collision can generate a different sequence of reflection windows. This window sequence is induced by the collision's dependence on solitons' phase differences. Our analytical formula for the locations of these phase-induced windows agrees well with our numerics. To our knowledge, this is the first report of *phase*-induced window sequences in solitary-wave collisions in the literature. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Collision of vector solitons in the coupled nonlinear Schrödinger (NLS) equations arises in many important physical situations. Examples include pulse propagation in birefringent fibers and wavelength-division-multiplexed systems [1,2], operation of optical logic gates [3], and beam steering in Kerr or photo-refractive crystals [4–6]. Theoretically, vector-soliton collisions have been studied intensively in the past ten years. It has been shown that soliton transmission, reflection, trapping and spiraling can all occur [6,7]. More surprisingly, a fractal structure has been found where transmission, reflection and trapping occur in an intertwined, fractal manner [8]. In this Letter, we report yet two new discoveries on vector-soliton collisions. The first one is that, at low cross-phase-modulational (XPM) coefficients, a sequence of reflection windows similar to that in the ϕ^4 model appears in the collision of two orthogonally polarized and equal-amplitude solitons. We will show that this window sequence is caused by a resonance between the translational motion and width oscillations of vector solitons. Its reappearance in our model suggests its "universality" in solitary-wave collisions. Recalling the fractal structure at moderate XPM coefficients in the same system [8], we see that this window sequence in the coupled NLS equations can bifurcate into more compli-

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cated resonance structures by tuning the XPM coefficient. The second discovery is that, when initial solitons are not orthogonally polarized, their collision can generate a different sequence of reflection windows. We will show that this window sequence is induced by the collision's dependence on the two solitons' phase differences. The existence of this phase-induced window sequence in solitary-wave collisions is very novel, and does not appear to have been reported before in the literature.

2. Collision of orthogonally polarized solitons at small XPM coefficients

We cast the coupled NLS equations in the birefringent fiber context as [1,2]

$$iA_z + A_{tt} + \left(|A|^2 + \beta|B|^2\right)A = 0, \tag{1}$$

$$iB_z + B_{tt} + \left(|B|^2 + \beta|A|^2\right)B = 0, \tag{2}$$

where A and B are envelopes of the electrical fields in the two orthogonal polarizations of the fiber, t is the retarded time, z is the propagation distance, and β is the XPM coefficient. All variables have been nondimensionalized. When $\beta = 0$, the system is decoupled into two NLS equations; when $\beta = 1$, the system is called the Manakov model. Both cases are integrable [7,9]. The system is nonintegrable for other β values.

In this section, we study the collision of two orthogonally polarized and equal-amplitude solitons at small but nonzero XPM coefficient β. For this purpose, we take $\beta = 0.2$, and the initial condition as

$$A(t,0) = \sqrt{2}\,\mathrm{sech}\left(t + \frac{1}{2}\Delta_0\right)e^{(1/4)iV_0 t}, \tag{3}$$

$$B(t,0) = \sqrt{2}\,\mathrm{sech}\left(t - \frac{1}{2}\Delta_0\right)e^{-(1/4)iV_0 t}. \tag{4}$$

Here V_0 (> 0) is the collision velocity, and Δ_0 ($\gg 1$) is the initial pulse separation. Note that in this case, the collision outcome is independent of the initial separation Δ_0 as long as Δ_0 is large enough. In our numerical simulations, we take $\Delta_0 = 20$. We simulated Eqs. (1) and (2) by a split-step method. We also checked its results independently by a Fourier/Runge–Kutta 4 method.

Our numerical simulations reveal that, at high collision velocities, solitons (3) and (4) pass through each other, and small shadows are created. At low collision velocities, these solitons trap each other and form a bound state. At certain windows of moderate collision velocities, the solitons are reflected back. In each collision, some radiation is emitted as well. We define the exit velocity V of a transmissional or reflectional collision as the relative separation velocity of the exit solitons. For transmissional collision, V is positive. For reflectional collision, V is negative. In a trapping collision, we simply assign V as zero. Numerically, we have determined the exit velocities (by tracking the positions of $|A|$ and $|B|$'s maximum values) at various collision velocities. The results are displayed in Fig. 1(a). A very interesting phenomenon in this figure is the appearance of a large number of reflection windows converging to a critical velocity $V_c = 0.93563$. Here the critical velocity is the smallest collision velocity above which all collisions are transmissional. In these reflection windows, solitons (3) and (4) always pass each other only twice: the first pass occurs when they come together, and the second pass occurs when they escape from each other. But the distance between these two passes varies from one window to another. In addition, solitons between passes show a significant amount of width and amplitude oscillations. To illustrate, we pick the bottom point $V_0 = 0.9214$ of the second reflection window in Fig. 1(a), and plot the $|A|$ contour of the collision in Fig. 1(b) (the $|B|$ contour is just a horizontal reflection of the $|A|$ contour about the $t = 0$ axis). Collisions in other reflection windows are similar, except that the numbers of width oscillations between passes increases consecutively from one window to the next. If we define the collision distance Z_n as the distance between two passes when the collision velocity is at the bottom of the nth window, then we find that Z_n is almost a perfect linear function of the window index n as

$$\omega Z_n = 2n\pi + \delta, \tag{5}$$

where the least-square linear fit gives $\omega = 0.7976$ and $\delta = 0.9764$. Here ω is the width oscillation frequency of solitons between passes. We also find that the locations of these reflection windows are very well approximated by the formula

$$\left(V_c^2 - V_n^2\right)^{-1/2} = \mu n + \theta, \tag{6}$$

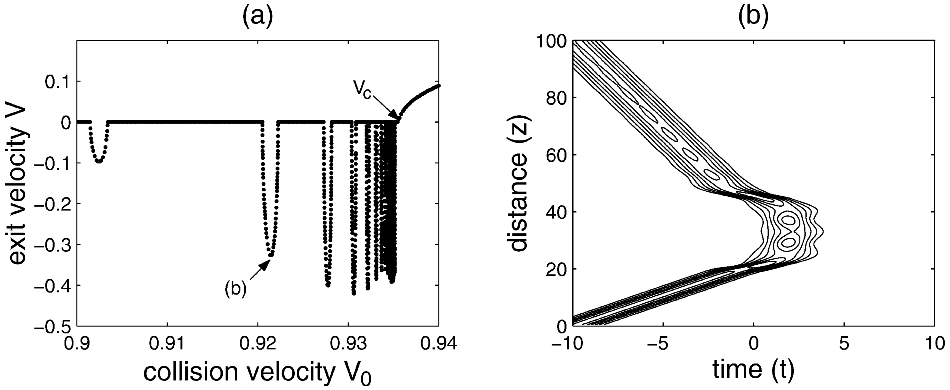

Fig. 1. Collision results for $\beta = 0.2$: (a) exit velocity graph; (b) $|A|$ contour at $V_0 = 0.9214$ (bottom of the second window in (a)).

where V_n is the center position of the nth window, V_c is the critical velocity mentioned above, and the least-square linear fit gives $\mu = 2.0588$ and $\theta = 0.0375$.

The exit velocity graph 1(a) and relations (5) and (6) are very similar to those of kink-antikink collisions in the ϕ^4-type models [10]. Even though vector solitons are localized entities, while kinks and antikinks are not, these similarities are remarkable. They suggest that reflection windows of ϕ^4 models are somewhat "universal" in solitary-wave collisions. In the ϕ^4-type models, the window sequence was attributed to a resonance between kink/antikink's translational motion and internal modes or impurity modes [10]. In the present case, this window sequence is induced by a resonance between the translational motion and width oscillations of vector solitons, as Fig. 1(b) clearly suggests. Specifically, when the collision distance is multiples of the width oscillation period $(2\pi/\omega)$ plus an offset parameter, energy stored in width oscillations of solitons during the first pass is released back into the translational motion of solitons during the second pass, and reflectional collision is then obtained. Note that width oscillations are caused by radiation modes, not internal modes [11]. This is a minor difference between resonance mechanisms in the ϕ^4 and our systems.

To analytically explain the above collision results, we employ a variational model with ansatz

$$A(t,z) = \sqrt{2}a\,\mathrm{sech}\left(\frac{t+\Delta/2}{w}\right) \times \exp\left\{i\left[\frac{v}{4}\left(t+\frac{\Delta}{2}\right) + \frac{b}{2w}\left(t+\frac{\Delta}{2}\right)^2 + \sigma\right]\right\}, \tag{7}$$

$$B(t,z) = \sqrt{2}a\,\mathrm{sech}\left(\frac{t-\Delta/2}{w}\right) \times \exp\left\{i\left[-\frac{v}{4}\left(t-\frac{\Delta}{2}\right) + \frac{b}{2w}\left(t-\frac{\Delta}{2}\right)^2 + \sigma\right]\right\}. \tag{8}$$

This ansatz was suggested by Fig. 1(b), and it incorporates soliton position (Δ), velocity (v), amplitude (a), width (w), chirp (b) and phase (σ) variations throughout collisions. We note that this same variational approach has been used by Ueda and Kath [12] in the study of internal oscillations of vector solitons. Here we apply it to the soliton-collision problem. Substituting the above ansatz into the Lagrangian form of Eqs. (1) and (2), and variations with respect to each of the soliton parameters taken, the following system of ordinary differential equations (ODEs) will be derived [12]:

$$\frac{d^2\Delta}{dz^2} = \frac{16K\beta}{w^2}F'(\alpha), \tag{9}$$

$$\frac{d^2 w}{dz^2} = \frac{16}{\pi^2 w^2}\left\{\frac{1}{w} - K - 3\beta K\left[\alpha F(\alpha)\right]'\right\}. \tag{10}$$

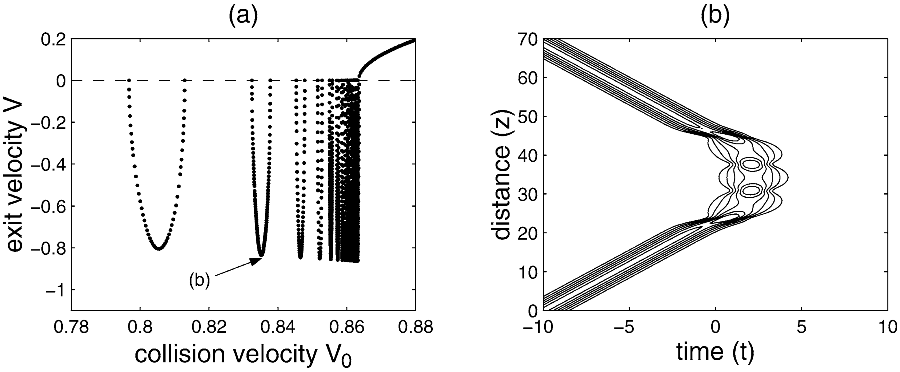

Fig. 2. Collision results of the variational model for $\beta = 0.2$: (a) exit velocity graph; (b) $|A|$-ansatz contour at $V_0 = 0.8352$ (bottom of the second window in (a)).

Here $v = \Delta'(z)$, $b = w'(z)/2$, $F(\alpha) = (\alpha \cosh\alpha - \sinh\alpha)/\sinh^3\alpha$, $\alpha = \Delta/w$, and $K = a^2 w = \text{const}$. Corresponding to the initial solitons (3) and (4), the initial conditions for the above ODEs are

$$\Delta(0) = 20, \qquad \Delta'(0) = V_0,$$
$$w(0) = 1, \qquad w'(0) = 0. \tag{11}$$

In addition, $K = 1$ and $\beta = 0.2$. The exit velocity V of the ODE model is the $v(t)$ value after the collision has completed and the solitons (7) and (8) have separated far apart again $[\Delta(t) \gg 1]$.

Our numerical simulations of the ODE model (9) and (10) show that there is a critical velocity $V_c = 0.86338$, above which all collisions are transmissional. Below V_c, a sequence of two-pass reflection windows appears, just like in the original partial differential equations (PDEs) (see Fig. 1(a)). This window sequence is shown in Fig. 2(a). At the bottom $V_0 = 0.8352$ of the second reflection window in Fig. 2(a), the contour of the $|A|$ ansatz (7) is shown in Fig. 2(b). Obviously, this collision dynamics is qualitatively identical to that in Fig. 1(b). In other windows of Fig. 2(a), collision contours are similar to Fig. 2(a), except that numbers of width oscillations change consecutively from one window to the next. Collision distances at the bottoms of those reflection windows in the ODE model also depend linearly on window index n in form (5) with a little different coefficients $\omega_v = 0.9993$ and $\delta_v = 3.9494$. Window locations in the ODE model are also well approximated by Eq. (6) while $\mu_v = 1.3234$ and $\theta_v = 0.5496$ now. The above qualitative agreements indicate that our variational model has correctly captured the main collision features of the PDE system. Since the key ingredient of our variational ansatz is the inclusion of position variation and width oscillations of solitons, the variational results thus reinforce our previous conclusion that the mechanism for this window sequence is a resonance between the translational motion and width oscillations of solitons.

3. Collision of nonorthogonal vector solitons

In this section, we study the collision of two nonorthogonal vector solitons. For this purpose, we take the initial condition as

$$A(t,0) = r_1\left(t + \frac{\Delta_0}{2}\right)e^{(1/4)iV_0 t} + r_2\left(t - \frac{\Delta_0}{2}\right)e^{-(1/4)iV_0 t}, \tag{1}$$

$$B(t,0) = r_2\left(t + \frac{\Delta_0}{2}\right)e^{(1/4)iV_0 t} + r_1\left(t - \frac{\Delta_0}{2}\right)e^{-(1/4)iV_0 t}, \tag{2}$$

where Δ_0 ($\gg 1$) is the initial vector-soliton separation, V_0 is the collision velocity, functions $[r_1(t), r_2(t)]$ are nondegenerate vector solitons with propagation constants (ω_1^2, ω_2^2), and they satisfy the ODE system

$$r_{1tt} - \omega_1^2 r_1 + \left(r_1^2 + \beta r_2^2\right) r_1 = 0, \tag{3}$$

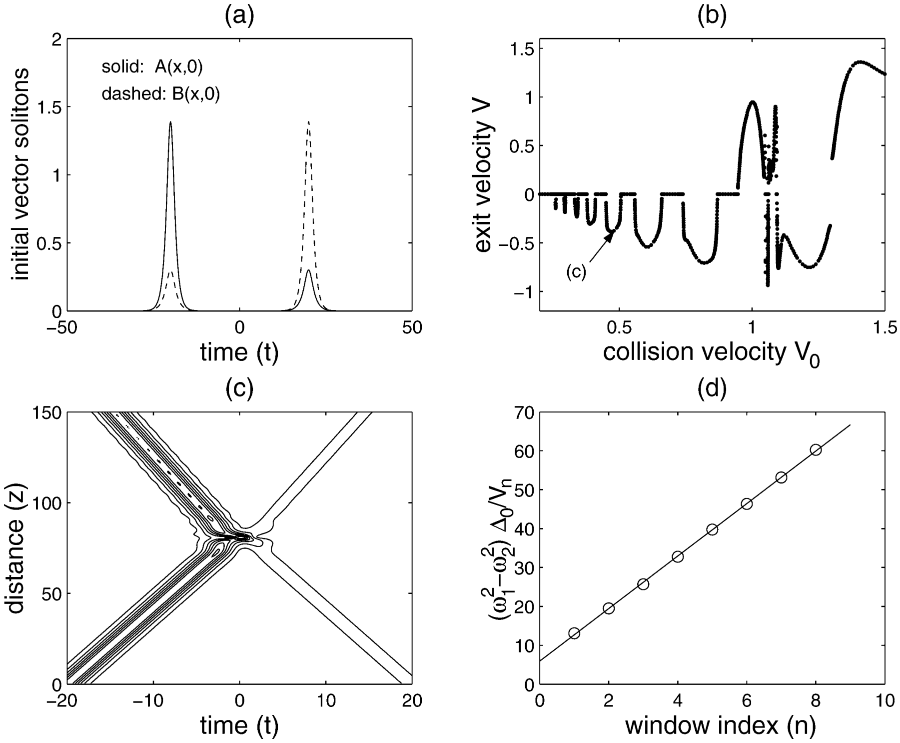

Fig. 3. Collision results for weakly nonorthogonal vector solitons at $\beta = 2/3$. Parameters in the initial conditions (1) and (2) are $\omega_1 = 1$, $\omega_2 = 0.78$, and $\Delta_0 = 40$. (a) The initial conditions $A(t,0)$ (solid) and $B(t,0)$ (dashed); (b) the exit velocity graph; (c) $|A|$ contour at $V_0 = 0.4781$, center of the fourth reflection window in (b) (from the right); (d) $(\omega_2^2 - \omega_1^2)\Delta_0/V_n$ graph (circles), where V_n is the center position of the nth reflection window in (b). The solid line is the least-square linear fit.

$$r_{2tt} - \omega_2^2 r_2 + \left(r_2^2 + \beta r_1^2\right) r_2 = 0. \tag{4}$$

For a detailed discussion of vector-soliton solutions in this ODE system, see [13]. It is noted that collisions of nonorthogonal vector solitons depend on initial phase differences between the two solitons. In our initial conditions (1) and (2), we have set these phase differences as zero for simplicity. This phase dependence results in the collision's dependence on initial position separation Δ_0, no matter how large Δ_0 is.

We have carried out collision simulations of two vector solitons (1) and (2) with $\beta = 2/3$, initial separation $\Delta_0 = 40$, and propagation constants $(\omega_1, \omega_2) = (1, 0.78)$ at various collision velocities. It is noted that $\beta = 2/3$ is the XPM coefficient in linearly birefringent fibers [1]. For these system parameters, the initial vector solitons (1) and (2) are displayed in Fig. 3(a). As we can see, each vector soliton here has a wave-shadow structure, and the two vector solitons are weakly nonorthogonal. In this case, the exit velocity versus collision velocity graph is displayed in Fig. 3(b). This graph is quite different from that in [8] for the collision of two initially orthogonal vector solitons at the same XPM coefficient β. This indicates that even a weak nonorthogonality has a significant effect on the collision structure and dynamics.

The novel feature of Fig. 3(b) is the appearance of a sequence of reflection windows on the left-hand side of the graph. This window sequence is fundamentally different from that in Fig. 1(a) on the geometrical feature, dynamics and mechanism. Geometrically, this window sequence converges toward the left. Dynamically, collisions in all these windows are simple reflections. This is shown in Fig. 3(c) where the $|A|$ contour at the middle point $V_0 = 0.4781$ of the fourth reflection window (from the right) is plotted. This simple reflection is quite different from the two-pass collision in Fig. 1(b). Collisions in other windows of this sequence are nearly the same as Fig. 3(c).

What mechanism creates this sequence of reflection windows? Apparently, the mechanism here is not the resonance as in Fig. 1(a). The similarity of collisions in these windows offers the following answer. In a moving vector soliton with position ξ_0, velocity v, propagation constants (ω_1^2, ω_2^2), and phases (γ_1, γ_2), the A and B components' phases at the soliton center are $(\omega_1^2 + (1/4)v^2)z + (1/2)v\xi_0 + \gamma_1$ and $(\omega_2^2 + (1/4)v^2)z + (1/2)v\xi_0 + \gamma_2$, respectively [13]. Now when the two vector solitons in the initial conditions (1) and (2) move toward each other, their phase differences at soliton centers are $(\omega_2^2 - \omega_1^2)z$ and $-(\omega_2^2 - \omega_1^2)z$ in the A and B components. We know that collisions of nonorthogonal vector solitons depend on their phase differences at the time of collision. With initial separation Δ_0 and collision velocity V_0, collision occurs at distance $z \approx \Delta_0/V_0$. Then phase differences of soliton centers at collision are $(\omega_2^2 - \omega_1^2)\Delta_0/V_0$ and $-(\omega_2^2 - \omega_1^2)\Delta_0/V_0$ in the A and B components. Our key observation is that, at two collision velocities, if the above phase differences at collision differ by a multiple of 2π, then the collision outcomes should be roughly the same. Mathematically, it means that at collision velocities V_n $(n = 1, 2, \ldots)$ where

$$\frac{(\omega_2^2 - \omega_1^2)\Delta_0}{V_n} = \tau n + \phi, \qquad (5)$$

slope $\tau = 2\pi$ and ϕ is a constant, collision dynamics should be similar. To check this formula, we have determined the center points V_n of those reflection windows. When quantities $(\omega_2^2 - \omega_1^2)\Delta_0/V_n$ are plotted in Fig. 3(d) versus window index n (circles), we see that the dependence is indeed linear! A least-square linear fit (solid line in Fig. 3(d)) shows that the actual slope is $\tau = 6.7525$, which is rather close to the theoretical value 2π. These good agreements confirm that this sequence of reflection windows is indeed caused by the phase dependence of the collision as described above. Furthermore, linear relation (5) does hold very well. We emphasize that the phase-induced window sequence in Fig. 3(b) is a new phenomenon associated only with nonorthogonal vector-soliton collisions. To our knowledge, such window sequences have not been reported before in the literature. We also note that these reflection windows are very wide, thus their experimental verification is highly feasible. The suggested experimental parameters for such verification can be found in [1,8].

An important further question is how the window sequences discovered in this Letter change as XPM coefficient β and polarizations of initial vector solitons vary. This question is computation-intensive, and it falls beyond the scope of the present Letter.

4. Conclusion

In summary, we have reported two new discoveries on vector-soliton collisions in the nonintegrable coupled NLS equations. The first one is a window sequence similar to ϕ^4's at low XPM coefficient β. This window sequence is caused by a resonance between the translational motion and width oscillations of solitons. Our variational model analytically explained these collision results very well. The re-appearance of this window sequence in the present system suggests its “universality” in solitary-wave collisions. The second discovery is a novel phase-induced window sequence in collisions of nonorthogonal vector solitons. Our analytical formula for the locations of these windows agrees well with the numerics. The experimental verification of these results in optical fibers is highly feasible.

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