Fully localized two-dimensional embedded solitons

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We report the prediction of fully localized two-dimensional embedded solitons. These solitons are obtained in a quasi-one-dimensional waveguide array which is periodic along one spatial direction and localized along the orthogonal direction. Under appropriate nonlinearity, these solitons are found to exist inside the Bloch bands (continuous spectrum) of the waveguide and thus are embedded solitons. These embedded solitons are fully localized along both spatial directions. In addition, they are fully stable under perturbations.

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I. INTRODUCTION

Embedded solitons are nonlinear solitary waves whose frequencies (or propagation constants) reside inside the continuous spectrum of the underlying wave system [1]. The existence of embedded solitons is quite counterintuitive, since inside the continuous spectrum, only nonlocal waves with nonvanishing oscillating tails are commonly expected [2]. However, under certain conditions, these oscillating tails are absent, hence truly localized embedded solitons appear inside the continuous spectrum [1,3]. Since embedded solitons exist inside the continuous spectrum and are thus resonant with linear radiation modes, they exhibit some interesting dynamical properties. For instance, isolated embedded solitons are often found to be semistable, i.e., they would persist under energy-enhancing perturbations but perish under energy-reducing perturbations [1,4]. Nonisolated embedded solitons, on the other hand, can be semistable or fully stable, depending on the underlying wave system [5,6]. Embedded solitons have been linked to other physical objects as well. For instance, moving discrete solitary waves in lattices (if they exist) are also embedded solitons [7–9]. So far, almost all embedded solitons reported in the literature are one-dimensional (1D). The soliton trains reported in Ref. [10] exist inside the continuous spectrum and are two dimensional (2D), but these soliton trains are localized only along one spatial direction and nonlocal along the orthogonal direction. It has remained a challenge to find 2D (and higher-dimensional) embedded solitons which are localized in all spatial directions. The reason is that in multidimensions, more stringent conditions need to be satisfied in order for fully localized embedded solitons to exist, thus such solitons are more difficult to find.

From a broader perspective, embedded solitons are intimately related to linear bound states (i.e., localized eigenmodes) embedded inside the continuous spectrum of a wave system. These linear bound states in the continuum were first predicted by von Neumann and Wigner in [11], who showed that the 3D linear Schrödinger equation with certain localized potentials could possess bound states above the potential well (see also Ref. [12]). These predicted bound states were later observed experimentally for electrons in semiconductor heterostructures [13]. In optics, linear bound states inside the continuum have also been predicted in various settings, such as a semi-infinite 1D lattice [14], two parallel dielectric gratings [15], two arrays of thin parallel dielectric cylinders [15], and open 2D quantum dots or optical waveguides [16,17]. Recently, linear 2D bound states in the continuum were demonstrated both theoretically and experimentally for light beams in a quasi-1D waveguide array with two additional waveguides above and below it [18]. The key idea in the construction of linear continuum bound states in Refs. [16–18] is to seek bound states of certain parity which are embedded inside the continuum bands of opposite parity. This idea inspires us to construct fully localized 2D embedded solitons in this article. The above theoretical and experimental investigations on the counterintuitive nonlinear embedded solitons and linear continuum bound states deepened our fundamental understanding of linear and nonlinear wave phenomena, and they could lead to unexpected applications in diverse physical fields.

In this article, we construct fully localized 2D embedded solitons. These solitons are obtained in a quasi-1D waveguide array which is periodic along the horizontal direction and localized along the vertical direction. Under self-defocusing nonlinearity, we find 2D solitons which are symmetric along the vertical direction, and they are embedded in the continuum bands of odd symmetry in the vertical direction. These 2D embedded solitons exist as continuous families, with their propagation constants (or equivalently their powers) as a free parameter. We further show that these embedded solitons are fully stable against perturbations even though they exist inside the continuum bands. In addition, we show how 2D embedded solitons of odd symmetry along the vertical direction can be derived under self-focusing nonlinearity. This construction method for 2D embedded solitons is general, thus these embedded solitons are no longer rare objects but can appear easily in diverse physical situations.

II. TWO-DIMENSIONAL EMBEDDED SOLITONS IN A WAVEGUIDE ARRAY

In this section, we construct 2D embedded solitons. The theoretical model we use is the following 2D nonlinear Schrödinger (NLS) equation with a potential,

$$iU_z + U_{xx} + U_{yy} + n(x,y)U + \sigma|U|^2U = 0. \qquad (1)$$

In spatial optics, this equation models paraxial light transmission in a waveguide under cubic nonlinearity [19]. In this context, U is the complex envelope function of the light's electric field, z is the transmission distance, (x,y) are the transverse coordinates, $n(x,y)$ is the refractive index variation of the waveguide, and $\sigma = \pm 1$ represent self-focusing and self-defocusing nonlinearity respectively (self-focusing nonlinearity is common in most optical materials, and self-defocusing nonlinearity can be realized in certain special materials such as photorefractive crystals [20]). In Bose-Einstein condensates, Eq. (1) models the collective behavior of condensate atoms in a magnetic or optical trap under nonlinear atom-atom interaction (it is called the Gross-Pitaevskii equation in the literature) [21]. In this context, U is the the collective wave function of the condensate, z is time, (x,y) are 2D spatial coordinates, $-n(x,y)$ is the trap potential, and $\sigma = \pm 1$ represent attractive and repulsive atom-atom interaction respectively. Static solitary waves in Eq. (1) are sought in the form

$$U(x,y,z) = u(x,y)e^{-i\mu z}, \tag{2}$$

where μ is the propagation constant and $u(x,y)$ is a real-valued localized function which satisfies the equation

$$u_{xx} + u_{yy} + n(x,y)u + \sigma u^3 = -\mu u. \tag{3}$$

To construct a concrete example of 2D embedded solitons, we take a quasi-1D waveguide array

$$n(x,y) = 6\cos^2 x\, e^{-y^2/4}, \tag{4}$$

which is periodic along the x direction and localized along the y direction. This waveguide is shown in Fig. 1 (left panel). Such a waveguide has been experimentally demonstrated in photorefractive crystals through optical induction [22,23]. We also take $\sigma = -1$ (for self-defocusing nonlinearity). In order to find embedded solitons, we first need to determine the linear continuous spectrum of Eq. (3). For this purpose, we drop the nonlinear term in (3). Since $n(x,y)$ is periodic in x with π period, according to the Bloch theorem, linear eigenmodes of (3) are of the form

$$u(x,y) = e^{ikx}q(x,y), \tag{5}$$

where k is the wave number in the first Brillouin zone $-1 \leqslant k \leqslant 1$ and $q(x,y)$ is an x-periodic function with period π. Inserting (5) into the linear part of (3), we obtain the linear equation for $q(x,y)$ as

$$q_{xx} + 2ikq_x - k^2 q + q_{yy} + n(x,y)q = -\mu q. \tag{6}$$

The continuous spectrum of Eq. (6) consists of the positive axis $\mu \in [0, +\infty)$, where $u(x,y)$ is nonlocal along the y direction, and Bloch bands with $\mu < 0$, where $u(x,y)$ is localized along the y direction. To determine the Bloch bands with $\mu < 0$, we expand $q(x,y)$ into a Fourier series in both x and y and then insert the expansion into Eq. (6) and turn it into a matrix eigenvalue problem, with μ being the eigenvalue and the Fourier coefficients of $q(x,y)$ being the eigenvector [19]. This matrix eigenvalue problem is then solved by conventional algorithms. The resulting diffraction relation $\mu = \mu(k)$ for these Bloch bands is shown in Fig. 1 (right panel). We see that three Bloch bands are obtained. At the edges of the lowest two bands, the corresponding Bloch modes $u(x,y)$ are displayed in Fig. 2. It is important to note that the Bloch modes in the lowest band $\mu \in [-2.9711, -2.7226]$ are symmetric in y, while the Bloch modes in the second band $\mu \in [-1.3030, -0.9260]$ are antisymmetric in y. The existence of different Bloch bands with opposite y parity is important for our construction of 2D embedded solitons.

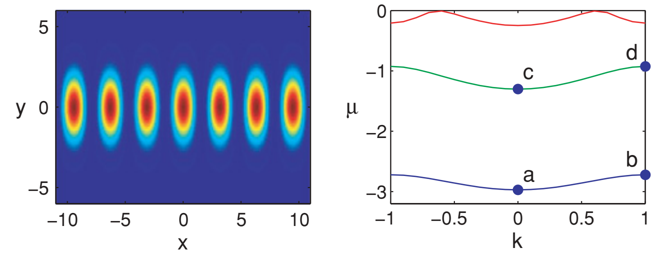

FIG. 1. (Color online) (Left) The quasi-1D waveguide array $n(x,y)$ in our model. (Right) The Bloch bands in this waveguide. The edges of the lowest two Bloch bands are marked by the letters a–d.

When nonlinearity is present, locally confined solitons will bifurcate out from infinitesimal (linear) Bloch modes of band edges [19,24]. Under self-defocusing nonlinearity, these solitons will bifurcate from the upper band edges upward into band gaps. Here we consider the solitons bifurcating from the upper edge of the lowest Bloch band (i.e., point b in Fig. 1). Near edge b, the soliton is a low-amplitude broad packet which decays slowly along the x direction [see Fig. 3(A)]. This soliton is a regular gap soliton since it exists in a band gap. As μ moves further away from the edge b, the soliton becomes more narrow, and its amplitude as well as power becomes higher (see Fig. 3). Here the power P is defined as the integral of u^2 over the xy plane. The most interesting phenomenon about this family of solitons is that, when μ enters into the second Bloch band $[-1.3030, -0.9260]$, the soliton still persists, and it remains fully localized in both x and y directions. To demonstrate, this soliton at $\mu = -1.1$ in the middle of the second Bloch band is displayed in Fig. 3(B). Since this soliton exists inside the continuous spectrum (Bloch bands), it is a fully localized 2D embedded soliton. Likewise, its nearby solitons with μ still inside the second Bloch band are all 2D embedded solitons as well. In other words, this is a continuous family of 2D embedded solitons with its propagation constant μ or power P as a free parameter.

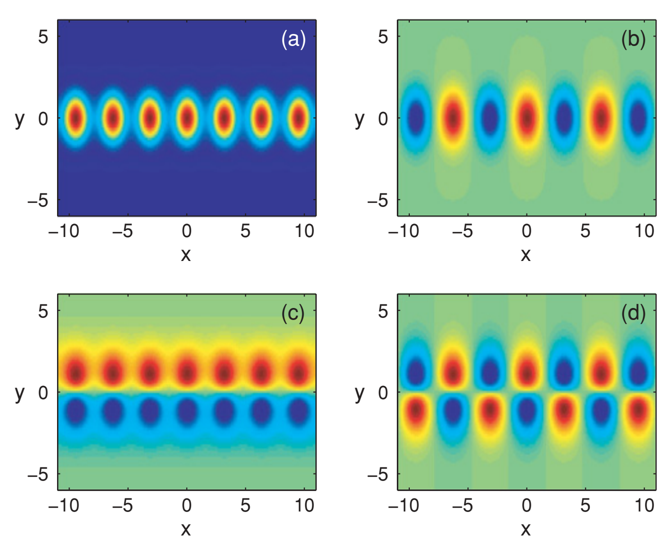

FIG. 2. (Color online) Bloch modes at edges a–d of the lowest two Bloch bands in Fig. 1 respectively.

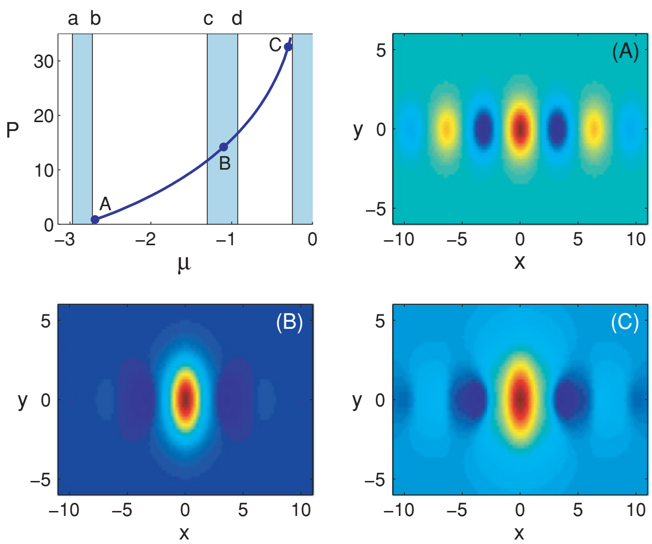

FIG. 3. (Color online) A soliton family under self-defocusing nonlinearity ($\sigma = -1$). (Upper left panel) The power curve; the shaded stripes indicate Bloch bands, and the band edges marked by a–d on top of the panel correspond to the edges of the same marker in Fig. 1. [(A)–(C)] Soliton profiles $u(x,y)$ at locations marked by the same letters on the power curve. The soliton at location B is an embedded soliton which is fully localized in both x and y dimensions.

Why do these 2D embedded solitons exist? Note that these solitons bifurcate out from edge b of the first Bloch band, thus they are symmetric in y (see Fig. 3). Note also that the second Bloch band consists of Bloch modes which are all antisymmetric in y (see Fig. 2). Thus when this y-symmetric soliton branch enters the second Bloch band of y-antisymmetric Bloch modes, even though μ lies in the Bloch band, the soliton does not excite those Bloch modes of opposite y parity, thus it remains fully localized. However, if this soliton moves into the third band $\mu \in [-0.2488, 0]$ (see Fig. 3), since the Bloch modes in this third band are also symmetric in y, this soliton *will* excite these y-symmetric Bloch modes and become delocalized [evidence of this can be seen in Fig. 3(C), where the soliton becomes broad again near the third Bloch band]. Thus one can not find y-symmetric 2D embedded solitons in the third band.

III. STABILITY OF TWO-DIMENSIONAL EMBEDDED SOLITONS

Stability of these 2D embedded solitons in Fig. 3 is an important issue. In previous studies of 1D embedded solitons, since the embedded soliton is in resonance with the continuous spectrum, it could excite the continuum radiation and perish under certain perturbations [1]. Those 1D embedded solitons could also be linearly unstable, leading to their destruction under any generic perturbation [3]. For the 2D embedded solitons in Fig. 3, we have found that they are all linearly stable, i.e., their linear-stability spectra do not contain any eigenvalues with positive real parts. This linear stability for the embedded soliton in Fig. 3(B) is demonstrated in Fig. 4(i). Regarding the question of nonlinear stability, we note that these 2D embedded solitons lie inside the second Bloch band whose Bloch modes

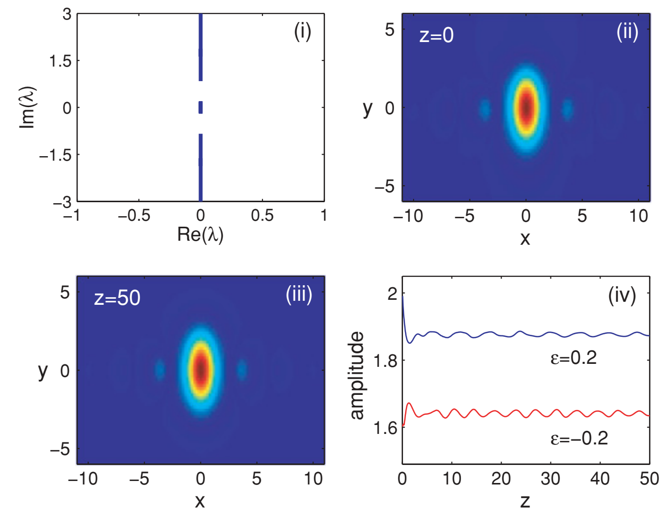

FIG. 4. (Color online) Demonstration of stability for the embedded soliton in Fig. 3(B). (i) The linear-stability spectrum, showing that this soliton is linearly stable; (ii) an initially perturbed embedded soliton (7) with $\epsilon = 0.2$; (iii) evolution of the perturbed soliton in (ii) at $z = 50$; plotted in (ii, iii) are $|U|$ fields; (iv) peak-amplitude evolutions of the perturbed embedded soliton (7) for $\epsilon = 0.2$ and -0.2.

are antisymmetric in y. Thus if the perturbation is y symmetric as the embedded soliton itself, then since the waveguide $n(x,y)$ is also y symmetric, the solution of Eq. (1) will remain y-symmetric for all distances z. Hence the perturbed soliton would not excite y antisymmetric second-band modes, i.e., the soliton would be stable under y symmetric perturbations. A less trivial question is what would happen if the perturbation is not symmetric in y. In this case, the perturbed soliton would excite y-antisymmetric second-band modes since it is resonant with those modes. Then could these y-antisymmetric radiation break up the embedded soliton? Intuitively, we can expect that when the y-antisymmetric component in the perturbation is weak, then these weak antisymmetric components would disperse away through resonance with the second-band modes, and the other dominant y-symmetric component of the solution would adjust its shape into a nearby (y-symmetric) embedded soliton. If so, then these 2D embedded solitons would be nonlinearly fully stable. However, this expectation is under the assumption that energy in the y-symmetric component would not transfer to the y-antisymmetric component during evolution. Since Eq. (1) is nonlinear, this assumption may not hold, because the symmetric and antisymmetric components could couple each other and transfer energy between them. Thus in principle, it is possible for the perturbed soliton to lose a significant amount of radiation to the resonant second-band modes and break up. To clarify this question, we have performed numerical simulations of these embedded solitons under various asymmetric perturbations and found that they are always nonlinearly fully stable. Two typical simulation results are shown in Figs. 4(ii)–4(iv). In these simulations, the embedded soliton $u(x,y)$ is the one in Fig. 3(B), and the perturbed initial state is

$$U(x,y,0) = u(x,y) + \epsilon\,(1 + \sin y)\,e^{-(x^2+y^2)/4}, \qquad (7)$$

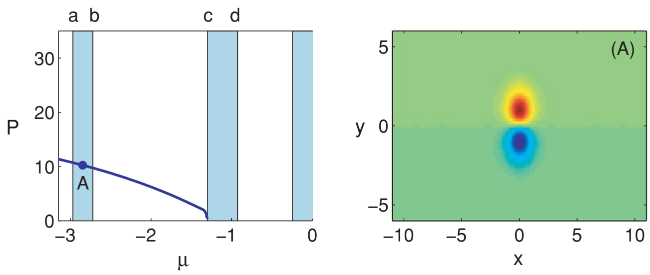

FIG. 5. (Color online) 2D embedded solitons under self-focusing nonlinearity ($\sigma = 1$). (Left panel) The power curve of this soliton family (same notations as Fig. 3); (right panel) an embedded soliton in the first Bloch band (at location A of the power curve).

where ϵ is the strength of perturbations. Note that this perturbation contains both symmetric and antisymmetric components in y. For $\epsilon = 0.2$, this perturbed initial state is shown in Fig. 4(ii). At propagation distance $z = 50$, the solution is shown in Fig. 4(iii). It is seen that this embedded soliton is stable under this perturbation. This stability can be seen more clearly in Fig. 4(iv), where the peak amplitude $|U|_{\max}$ of the solution versus the propagation distance z is displayed. We can see that the peak amplitude approaches a constant value close to the amplitude of the unperturbed soliton at large distances. If we take a different perturbation with $\epsilon = -0.2$, the result is similar, i.e., the peak amplitude of the solution also approaches a constant value close to the amplitude of the unperturbed soliton at large distances [see Fig. 4(iv)]. Thus this embedded soliton is nonlinearly fully stable. This result resembles the full stability of 1D embedded solitons in a generalized third-order NLS equation [6]. It contrasts some other 1D embedded solitons which are semistable and perish under certain types of perturbations [1,5].

It is important to recognize that the method in this article for the construction of 2D embedded solitons is quite general, and it can be used to construct many other fully localized multidimensional embedded solitons in various physical systems. For instance, in the same quasi-1D waveguide array described by Eqs. (1) and (4), if self-focusing nonlinearity is taken (i.e., $\sigma = 1$), then a soliton family which is antisymmetric in y will bifurcate downward from edge c of the second Bloch band (see Figs. 1 and 2). This solution family passes through the first Bloch band whose Bloch modes are symmetric in y. Inside the first Bloch band, these solitons are also fully localized 2D embedded solitons. To demonstrate, the power curve of this soliton family is displayed in Fig. 5 (left panel). At point A inside the first Bloch band, the embedded soliton is shown in Fig. 5(A), which is fully localized in both dimensions. Using similar methods, we can construct fully localized 3D embedded solitons as well. Previously, embedded solitons were generally regarded as rare objects which appear by "accident." Now we see that embedded solitons can arise frequently in diverse physical situations, thus they are an important physical object in nonlinear wave systems.

Now we address why the above results are of interest to physics and mathematics. Intuitively, solitary waves are only expected outside the continuous spectrum. In the past 10 years, the counterintuitive concept of solitons embedded inside the continuous spectrum was proposed and demonstrated in one dimension [1,3]. This concept significantly deepened our general understanding of nonlinear wave phenomena. In addition, it fostered the construction of other physical 1D objects such as moving discrete solitons in lattices since those objects are also embedded solitons under disguise [7–9]. In this article, we demonstrated the existence of embedded solitons in two dimensions and pointed out a way to construct embedded solitons in even higher dimensions (such as three dimensions). This significantly broadened the scope of embedded solitons. It could also stimulate the construction of related objects such as multidimensional moving discrete solitons. From the viewpoint of physical applications, regular (nonembedded) solitons have found applications in numerous situations. Now with the demonstration of families of stable multidimensional embedded solitons in this article, these embedded solitons may find applications analogous to regular solitons in situations where regular solitons do not exist.

IV. SUMMARY

In summary, we have predicted fully localized 2D embedded solitons. These embedded solitons were obtained in a quasi-1D waveguide array, and they exist inside the Bloch bands whose Bloch modes have opposite parity from the solitons themselves. These embedded solitons form solution families with continuous ranges of power values. In addition, they are fully stable under perturbations. The method of construction in this article is general, and it can be used to obtain multidimensional embedded solitons in diverse physical systems.

Note added in proof. Recently, we learned of work [25,26] where 2D embedded solitons were briefly reported in coupled Bose-Einstein condensates supported by 2D optical lattices [25], and 3D embedded solitons were briefly reported in single-component Bose-Einstein condensates supported by 1D optical lattices [26]. The 2D embedded solitons in Ref. [25] appear to be stable, while the 3D embedded solitons in Ref. [26] appear to be unstable.

ACKNOWLEDGMENTS

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