Perturbation theory for bright spinor Bose-Einstein condensate solitons

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We develop a perturbation theory for bright solitons of the $F=1$ integrable spinor Bose-Einstein condensates (BEC) model. The formalism is based on using the Riemann-Hilbert problem and provides the means to analytically calculate evolution of the soliton parameters. Both rank-one and rank-two soliton solutions of the model are obtained. We prove equivalence of the rank-one soliton and the ferromagnetic rank-two soliton. Taking into account a splitting of a perturbed polar rank-two soliton into two ferromagnetic solitons, it is sufficient to elaborate a perturbation theory for the rank-one solitons only. Treating a small deviation from the integrability condition as a perturbation, we describe the spinor BEC soliton dynamics in the adiabatic approximation. It is shown that the soliton is quite robust against such a perturbation and preserves its velocity, amplitude, and population of different spin components, only the soliton frequency acquires a small shift. Results of numerical simulations agree well with the analytical predictions, demonstrating only slight soliton profile deformation.

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I. INTRODUCTION

Bright and dark solitons in quasi-one-dimensional Bose-Einstein condensates (BECs), observed experimentally [1–4], are expected to be important for various applications in atom optics [5], including atom interferometry, atom lasers, and coherent atom transport. Recent experimental and theoretical advances in BEC soliton dynamics are reviewed in Refs. [6–8].

Spinor BEC of alkali-metal atoms [9,10] with a purely optical confinement, along with the two-component condensate [11–13], represents an example of the condensate with internal degrees of freedom which endow the solitons with vectorial properties. Modulational instability in the spinor BEC model was investigated in Ref. [14], and some exact solutions and their stability were studied in Ref. [15]. Vector gap solitons and self-trapped waves were identified in the spinor BEC model loaded into one-dimensional optical lattice potential [16]. Recently bright-dark soliton complexes in this model have been found [17] by reducing it to the completely integrable Yajima-Oikawa system [18].

Wadati and co-workers found [19] that the three-component nonlinear equations describing the BEC with the hyperfine spin $F=1$ admit the reduction to another integrable model—the 2×2 matrix nonlinear Schrödinger (NLS) equation, after imposing a constraint on the condensate parameters. Both bright [22] and dark [23] solitons possessing properties of true solitons of integrable equations have been found. The formalism of the inverse scattering transform for the matrix NLS equation under nonvanishing boundary conditions was developed in Ref. [24] and extended in Ref. [25] to describe bright spinor BEC soliton dynamics on a finite background. The full-time description of the modulational

instability development in the integrable spinor BEC model was given both numerically [15] and analytically [26].

Integrable models provide a very useful proving ground for testing new analytical and numerical approaches to study such a complicated system as the spinor BEC. At the same time, the integrability conditions impose specific restrictions on the parameters of the model which can conflict with actual experimental settings, despite the fact that the effective interaction between atoms in BEC can be tuned, to some extent, by the optically induced Feshbach resonance [20,21]. Besides, in experiment it is impossible to exactly hold the conditions between parameters which assure integrability of the model. Therefore, sufficiently general analytical results concerning the full (nonintegrable) model with realistic parameters would be of importance.

As a step in this direction, in the present paper we develop a perturbation theory for the integrable spinor BEC model. Evidently, small disturbance of the integrability condition can be considered as a perturbation of the integrable model. Our formalism is based on the Riemann-Hilbert (RH) problem associated with the spinor BEC model. The main advantage of the proposed method is its algebraic nature, as distinct from the method using the Gel'fand-Levitan integral equations [27]. The application of the RH problem for treating perturbed soliton dynamics goes back to Refs. [28,29]. The modern version of the perturbation theory in terms of the RH problem has been developed in a series of papers [30–34], with its most general formulation in Ref. [35]. Another version of the soliton perturbation theory (the direct perturbation theory) has been developed on the basis of expanding perturbed solutions into squared eigenfunctions of the linearized soliton equations [36,37].

As was shown by Wadati and co-workers [19,22], bright solitons in the integrable spinor BEC model can exist in two spin states—ferromagnetic (nonzero total spin) and polar (zero total spin). Energy of the polar soliton is greater than that of the ferromagnetic soliton. Moreover, the polar soliton demonstrates a two-humped profile in a wide range of its parameters. Our numerical simulations revealed that the po-

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lar soliton is unstable under the action of a perturbation of a rather general form and splits into two ferromagnetic solitons. This fact is crucial for the development of a perturbation theory for spinor BEC solitons.

The paper is organized as follows. After formulating the model in Sec. II, we introduce in Sec. III analytic solutions of the associated spectral problem, in order to formulate in Sec. IV the RH problem. Solving this problem, we derive in Sec. V bright soliton solutions of the integrable spinor BEC model, both for the rank-one and rank-two projectors. The rank-one soliton is characterized by the familiar hyperbolic secant profile, while the rank-two soliton has a more complicated form [19]. Two types of the rank-two solutions are exactly ferromagnetic and polar solitons. We prove that the ferromagnetic rank-two soliton is equivalent to the rank-one soliton. In virtue of the fact that the perturbed polar soliton splits into two rank-two ferromagnetic solitons, it is sufficient to develop a perturbation theory for the rank-one soliton only. This is performed in Sec. VI. We derive evolution equations for the soliton parameters which exactly account for the perturbation and serve as the generating equations for iterations. Section VII contains a description of the soliton dynamics in the adiabatic approximation of the perturbation theory. We show analytically that a ferromagnetic soliton is quite robust against a small disturbance of the integrability condition, the only manifestation of the perturbation action is a minor shift of the soliton frequency. Numerical simulations of the perturbed spinor BEC equations are in close agreement with the analytical predictions revealing only a small soliton shape distortion and little perturbation-induced radiation. Section VIII concludes the paper.

II. MODEL

We consider an effective one-dimensional BEC trapped in a pencil-shaped region elongated in the x direction and tightly confined in the transversal directions. The assembly of atoms in the hyperfine spin $F=1$ state is described by a vector order parameter $\vec{\Phi}(x,t)=(\Phi_+(x,t),\Phi_0(x,t),\Phi_-(x,t))^T$, where its components correspond to three values of the spin projection $m_F=1,0,-1$. The functions $\Phi_\pm$ and Φ_0 obey a system of coupled Gross-Pitaevskii equations [22,38],

$$i\hbar\partial_t\Phi_\pm = -\frac{\hbar^2}{2m}\partial_x^2\Phi_\pm + (c_0+c_2)(|\Phi_\pm|^2+|\Phi_0|^2)\Phi_\pm$$
$$+ (c_0-c_2)|\Phi_\mp|^2\Phi_\pm + c_2\Phi_\mp^*\Phi_0^2,$$

$$i\hbar\partial_t\Phi_0 = -\frac{\hbar^2}{2m}\partial_x^2\Phi_0 + (c_0+c_2)(|\Phi_+|^2+|\Phi_-|^2)\Phi_0$$
$$+ c_0|\Phi_0|^2\Phi_0 + 2c_2\Phi_+\Phi_-\Phi_0^*, \qquad (2.1)$$

where the constant parameters $c_0=(g_0+2g_2)/3$ and $c_2=(g_2-g_0)/3$ control the spin-independent and spin-dependent interaction, respectively. The coupling constant g_f ($f=0,2$) is given in terms of the s-wave scattering length a_f in the channel with the total hyperfine spin f,

$$g_f = \frac{4\hbar^2 a_f}{ma_\perp^2}\left(1 - C\frac{a_f}{a_\perp}\right)^{-1}.$$

Here $a_\perp$ is the size of the transverse ground state, m is the atom mass, and $C=-\zeta(1/2)\approx 1.46$.

It was noted in [19] that Eqs. (2.1) are reduced to an integrable system under the constraint

$$c_0 = c_2 \equiv -c < 0. \qquad (2.2)$$

The negative c_2 means that we consider the ferromagnetic ground state of the spinor BEC with attractive interactions. The condition (2.2), being written in terms of g_f as $2g_0 = -g_2 > 0$, imposes a constraint on the scattering lengths, $a_\perp = 3Ca_0a_2/(2a_0+a_2)$. Redefining the function $\vec{\Phi}$ as $\vec{\Phi} \to (\phi_+, \sqrt{2}\phi_0, \phi_-)^T$, normalizing the coordinates as $t \to (c/\hbar)t$ and $x \to (\sqrt{2mc}/\hbar)x$, and accounting for the constraint (2.2), we obtain a reduced system of equations in a dimensionless form,

$$i\partial_t\phi_\pm + \partial_x^2\phi_\pm + 2(|\phi_\pm|^2+2|\phi_0|^2)\phi_\pm + 2\phi_\mp^*\phi_0^2 = 0,$$

$$i\partial_t\phi_0 + \partial_x^2\phi_0 + 2(|\phi_+|^2+|\phi_0|^2+|\phi_-|^2)\phi_0 + 2\phi_+\phi_0^*\phi_- = 0. \qquad (2.3)$$

After arranging the components $\phi_\pm$ and ϕ_0 into a 2×2 matrix Q,

$$Q = \begin{pmatrix} \phi_+ & \phi_0 \\ \phi_0 & \phi_- \end{pmatrix}, \qquad (2.4)$$

we transform Eqs. (2.3) to the integrable matrix NLS equation

$$i\partial_t Q + \partial_x^2 Q + 2QQ^\dagger Q = 0. \qquad (2.5)$$

The matrix NLS equation (2.5) appears as a compatibility condition of the system of linear equations [27],

$$\partial_x\psi = ik[\Lambda,\psi] + \hat{Q}\psi, \qquad (2.6)$$

$$\partial_t\psi = 2ik^2[\Lambda,\psi] + V\psi, \qquad (2.7)$$

where $\Lambda = \mathrm{diag}(-1,-1,1,1)$,

$$\hat{Q} = \begin{pmatrix} 0 & Q \\ -Q^\dagger & 0 \end{pmatrix}, \quad V = 2k\hat{Q} + i\begin{pmatrix} QQ^\dagger & Q_x \\ Q_x^\dagger & -Q^\dagger Q \end{pmatrix}, \qquad (2.8)$$

and k is a spectral parameter. Equation (2.6) (the spectral problem) enables us to determine initial spectral data from the known potential $\hat{Q}_0$, while Eq. (2.7) governs the temporal evolution of the spectral data. A new solution of Eq. (2.5) [and hence of the BEC equations (2.3)] is obtained as a result of the reconstruction of the potential $\hat{Q}$ from the time-dependent spectral data.

III. JOST AND ANALYTIC SOLUTIONS

To determine the spectral data, we introduce matrix Jost solutions $J_\pm(x,k)$ of the spectral problem (2.6) by means of the asymptotes $J_\pm \to \mathbb{1}$ as $x \to \pm\infty$. Since $\mathrm{tr}\,\Lambda=0$, we have

det $J_\pm = 1$ for all t. Being solutions of the first-order equation (2.6), the Jost functions are not independent but are interconnected by the scattering matrix S,

$$J_- = J_+ ESE^{-1}, \quad E = \exp(ik\Lambda x), \quad \det S = 1. \quad (3.1)$$

Besides, the Jost solutions and the scattering matrix obey the involute property. Indeed, since the potential $\hat{Q}$ is anti-Hermitian, we obtain

$$J_\pm^\dagger(k^*) = J_\pm^{-1}(k). \quad (3.2)$$

Similarly for the scattering matrix,

$$S^\dagger(k) = S^{-1}(k). \quad (3.3)$$

Note that the scattering matrix is defined for real k.

For the subsequent analysis, analytic properties of the Jost solutions are of primary importance. Let us represent the matrix Jost solution J as a collection of columns $J = (J^{[1]}, J^{[2]}, J^{[3]}, J^{[4]})$, and consider the first column. Rewriting the spectral equation (2.6) with the corresponding boundary conditions in the form of the Volterra integral equations, we obtain a closed system of equations for entries of the first column,

$$J_{-11} = 1 + \int_{-\infty}^x dx'(\phi_+ J_{-31} + \phi_0 J_{-41})(x'),$$

$$J_{-21} = \int_{-\infty}^x dx'(\phi_0 J_{-31} + \phi_- J_{-41})(x'),$$

$$J_{-31} = -\int_{-\infty}^x dx'(\phi_+^* J_{-11} + \phi_0^* J_{-21})(x')e^{2ik(x-x')},$$

$$J_{-41} = -\int_{-\infty}^x dx'(\phi_0^* J_{-11} + \phi_-^* J_{-21})(x')e^{2ik(x-x')}.$$

The last two integrands point out that the column $J_-^{[1]}$ is analytic in the upper half-plane $\mathbb{C}_+$, where Im $k > 0$, and continuous on the real axis Im $k = 0$. This can be proved in the same way as for the scalar NLS equation, under the condition of sufficiently fast decrease of the potential $\hat{Q}$ at infinity. Similarly we obtain that the column $J_-^{[2]}$ is analytic in $\mathbb{C}_+$ as well, while the two other columns $J_-^{[3]}$ and $J_-^{[4]}$ are analytic in the lower half-plane $\mathbb{C}_-$ and continuous on the real axis Im $k = 0$. As regards the matrix solution J_+, its first and second columns $J_+^{[1]}$ and $J_+^{[2]}$ are analytic in $\mathbb{C}_-$, while the third and fourth ones $J_+^{[3]}$ and $J_+^{[4]}$ are analytic in $\mathbb{C}_+$. Therefore, the matrix function

$$\psi_+ = (J_-^{[1]}, J_-^{[2]}, J_+^{[3]}, J_+^{[4]}) \quad (3.4)$$

solves the spectral equation (2.6) and is analytic as a whole in $\mathbb{C}_+$.

It is not difficult to see from Eqs. (3.1) and (3.4) that the analytic solution ψ_+ can be expressed in terms of the Jost functions and some entries of the scattering matrix,

$$\psi_+ = J_+ ES_+ E^{-1} = J_- ES_- E^{-1}, \quad (3.5)$$

where

$$S_+(k) = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & 1 & 0 \\ s_{41} & s_{42} & 0 & 1 \end{pmatrix}, \quad S_-(k) = \begin{pmatrix} 1 & 0 & s_{31}^* & s_{41}^* \\ 0 & 1 & s_{32}^* & s_{42}^* \\ 0 & 0 & s_{33}^* & s_{43}^* \\ 0 & 0 & s_{34}^* & s_{44}^* \end{pmatrix}.$$

$$(3.6)$$

In writing the expression for S_- we use the involution (3.3). These upper and lower block-triangular matrices $S_\pm$ factorize the scattering matrix [39] $SS_- = S_+$. Besides, it follows from Eq. (3.5) and det $J_\pm = 1$ that

$$\det \psi_+ = m_+^{(2)} = m_-^{(2)*}, \quad (3.7)$$

where $m_+^{(2)}$ ($m_-^{(2)}$) is the second-order principal upper (lower) minor of the scattering matrix.

To obtain the analytic counterpart of ψ_+ in $\mathbb{C}_-$, we consider the adjoint spectral equation

$$\partial_x K_\pm = ik[\Lambda, K_\pm] - K_\pm \hat{Q} \quad (3.8)$$

with the asymptotic conditions $K_\pm \to \mathbb{1}$ at $x \to \pm\infty$. The inverse matrix J^{-1} can serve as a solution of the adjoint equation (3.8). Now we write a closed system of integral equations for rows of the matrices $K_\pm$. For example, the first row $K_{-[1]}$ obeys the equations

$$K_{-11} = 1 + \int_{-\infty}^x dx'(\phi_+^* K_{-13} + \phi_0^* K_{-14})(x'),$$

$$K_{-12} = \int_{-\infty}^x dx'(\phi_0^* K_{-13} + \phi_-^* K_{-14})(x'),$$

$$K_{-13} = -\int_{-\infty}^x dx'(\phi_+ K_{-11} + \phi_0 K_{-12})(x')e^{-2ik(x-x')},$$

$$K_{-14} = -\int_{-\infty}^x dx'(\phi_0 K_{-11} + \phi_- K_{-12})(x')e^{-2ik(x-x')}.$$

It is seen that the row $K_{-[1]}$ is analytic in $\mathbb{C}_-$. Similarly, the second row $K_{-[2]}$ is analytic in $\mathbb{C}_-$, too, and the rows $K_{-[3]}$ and $K_{-[4]}$ are analytic in $\mathbb{C}_+$. For the matrix solution K_+ we find that the rows $K_{+[1]}$ and $K_{+[2]}$ are analytic in $\mathbb{C}_+$, while $K_{+[3]}$ and $K_{+[4]}$ are analytic in $\mathbb{C}_-$. Therefore, the matrix function

$$\psi_-^{-1} = (K_{-[1]}, K_{-[2]}, K_{+[3]}, K_{+[4]})^T \quad (3.9)$$

solves the adjoint equation (3.8) and is analytic as a whole in $\mathbb{C}_-$. Similar to ψ_+, the function ψ_-^{-1} is expressed in terms of the Jost solutions and the scattering matrix,

$$\psi_-^{-1} = ET_+ E^{-1} J_+^{-1} = ET_- E^{-1} J_-^{-1}, \quad (3.10)$$

where the matrices $T_\pm$,

$$T_+ = \begin{pmatrix} s_{11}^* & s_{21}^* & s_{31}^* & s_{41}^* \\ s_{12}^* & s_{22}^* & s_{32}^* & s_{42}^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix},$$

provide one more factorization of the scattering matrix, $T_- = T_+ S$. As in Eq. (3.7), we can write

$$\det \psi_-^{-1} = m_+^{(2)*} = m_-^{(2)}. \tag{3.11}$$

Note that the analytic solutions satisfy the involute property as well,

$$\psi_+^\dagger(k) = \psi_-^{-1}(k^*). \tag{3.12}$$

This property can be taken as a definition of the analytic function ψ_-^{-1} from the known analytic function ψ_+.

IV. RIEMANN-HILBERT PROBLEM

Hence, we constructed two matrix functions ψ_+ and ψ_-^{-1} which are analytic in complementary domains of the complex plane and conjugate on the real line. Indeed, it follows from Eqs. (3.5) and (3.10) that $\psi_\pm$ obey the relation

$$\psi_-^{-1}(k)\psi_+(k) = EG(k)E^{-1}, \quad \mathrm{Im}\, k = 0, \tag{4.1}$$

where

$$G = T_+ S_+ = T_- S_- = \begin{pmatrix} 1 & 0 & s_{31}^* & s_{41}^* \\ 0 & 1 & s_{32}^* & s_{42}^* \\ s_{31} & s_{32} & 1 & 0 \\ s_{41} & s_{42} & 0 & 1 \end{pmatrix}. \tag{4.2}$$

Equation (4.1) determines a matrix Riemann-Hilbert problem, i.e., a problem of the analytic factorization of a nondegenerate matrix G in (4.2), given on the real line, into a product of two matrices which are analytic in complementary domains $\mathbb{C}_\pm$. The RH problem (4.1) needs a normalization condition, which is usually taken as

$$\psi_\pm(x,k) \to \mathbb{1} \quad \text{at } |k| \to \infty. \tag{4.3}$$

The analytic matrix functions $\psi_\pm$ can be treated as a result of a nonlinear mapping between the potential $\hat{Q}(x)$ and a set of the spectral data which uniquely characterizes a solution of the RH problem (4.1) and (4.3). Conversely, the potential can be reconstructed from an asymptotic expansion of $\psi_\pm(x,k)$ for large k. Indeed, writing $\psi_\pm$ as

$$\psi_+(x,k) = \mathbb{1} + k^{-1}\psi_+^{(1)} + O(k^{-2}),$$

$$\psi_-^{-1}(x,k) = \mathbb{1} + k^{-1}\psi_-^{(1)} + O(k^{-2}),$$

and inserting these expansions into Eqs. (2.6) and (3.8), we obtain

$$\hat{Q} = -i[\Lambda, \psi_+^{(1)}] = i[\Lambda, \psi_-^{(1)}]. \tag{4.4}$$

Hence, having solved the RH problem, we can find solutions of the BEC equations.

In general, the matrices ψ_+ and ψ_-^{-1} can have zeros k_j and κ_l in the corresponding domains of analyticity, $\det \psi_+(k_j)$

$= 0$, $k_j \in \mathbb{C}_+$ and $\det \psi_-^{-1}(\kappa_l) = 0$, $\kappa_l \in \mathbb{C}_-$. In virtue of the involution (3.12), we obtain $\kappa_l = k_l^*$ and equal number N of zeros in both half-planes. The corresponding RH problem is said to be nonregular, or the RH problem with zeros. They are zeros of the RH problem that determine soliton solutions of the BEC equations. It is seen from Eqs. (3.6) and (3.7) that zeros of ψ_+ nullify 2×2 minors of ψ_+. Hence, the rank of $\psi_+(k_j)$ can be equal to one or two. It means in turn that there exist one ($|1_j\rangle$) or two ($|1_j\rangle$ and $|2_j\rangle$) four-component eigenvectors that correspond to zero eigenvalue of $\psi_+(k_j)$,

$$\psi_+(k_j)|1_j\rangle = 0 \quad \text{for rank } \psi_+(k_j) = 1,$$

$$\psi_+(k_j)|1_j\rangle = \psi_+(k_j)|2_j\rangle = 0 \quad \text{for rank } \psi_+(k_j) = 2. \tag{4.5}$$

The geometric multiplicity of k_j is equal to the dimension of the null space of $\psi_+(k_j)$ (1 or 2 in our case). In this paper, we only consider the case of zeros k_j with its geometric multiplicity equal to the algebraic multiplicity [which is the order of the zero k_j in $\det \psi_+(k)$]. Note that the solution of the RH problem for the general case of zeros with unequal geometric and algebraic multiplicities was elaborated in Ref. [40].

We will solve the matrix nonregular RH problem with zeros k_1 and k_1^* by means of its regularization, i.e., by extracting from ψ_+ and ψ_-^{-1} rational factors that are responsible for the appearance of zeros. Hence, $\det \psi_+(k_1) = 0$ [and correspondingly $\det \psi_-^{-1}(k_1^*) = 0$]. We need a rational matrix function $\Xi^{-1}(x,k)$ which has a pole in the point k_1. Let us take $\Xi^{-1}(x,k)$ in the form

$$\Xi^{-1}(x,k) = \mathbb{1} + \frac{k_1 - k_1^*}{k - k_1}P^{(r)},$$

where

$$P^{(r)} = \sum_{l,m=1}^{r} |l\rangle (M^{-1})_{lm}\langle m|, \tag{4.6}$$

$\langle m| = |m\rangle^\dagger$ due to involution, and $r = \mathrm{rank}\, \psi_+(k_1)$. $P^{(r)}$ is a projector of rank r, $(P^{(r)})^2 = P^{(r)}$, and entries of the $r\times r$ matrix M are determined by

$$(M)_{lm} = \langle l|m\rangle = \sum_{a=1}^{4} (l)_a^*(m)_a.$$

In the appropriate basis the projector is represented as $P^{(1)} = \mathrm{diag}(1,0,0,0)$ or $P^{(2)} = \mathrm{diag}(1,1,0,0)$. This yields

$$\det \Xi^{-1} = \left(\frac{k - k_1^*}{k - k_1}\right)^r.$$

Therefore, the product $\psi_+(x,k)\Xi^{-1}(x,k)$ is regular in k_1. In the same way, the regularization of ψ_-^{-1} in the point k_1^* is performed by the rational function

$$\Xi(x,k) = \mathbb{1} - \frac{k_1 - k_1^*}{k - k_1^*}P^{(r)}, \tag{4.7}$$

which provides the product $\Xi \psi_-^{-1}$ to be regular in k_1^*. Therefore, the analytic functions are factorized as

$$\psi_+(k) = \widetilde{\psi}_+(k)\Xi(k), \quad \psi_-^{-1}(k) = \Xi^{-1}(k)\widetilde{\psi}_-^{-1}, \quad (4.8)$$

with holomorphic functions $\widetilde{\psi}_\pm$ which determine the regular (without zeros) RH problem,

$$\widetilde{\psi}_-^{-1}(k)\widetilde{\psi}_+(k) = \Xi(k)EG(k)E^{-1}\Xi^{-1}(k), \quad k \in \mathbb{R}. \quad (4.9)$$

For several pairs of zeros (k_j, k_j^*), $j > 1$, the regularization of the RH problem can be performed in the same step-by-step manner, with the appropriate definition of the eigenvectors within each step. However, for practical calculation of N-soliton effects it is much more convenient to expand a product of rational factors into simple fractions, thereby transforming the product-type expression into a sum-type one [32,40].

It is easy to find the coordinate dependence of the eigenvectors. Indeed, differentiating (4.5) in x and in t with $j=1$ and accounting Eqs. (2.6) and (2.7) gives ($|l_1\rangle \equiv |l\rangle$)

$$\partial_x|l\rangle = ik_1\Lambda|l\rangle, \quad \partial_t|l\rangle = 2ik_1^2|l\rangle, \quad (4.10)$$

with $l=1$ for rank one, and $l=1,2$ for rank two. Hence,

$$|l\rangle = \exp(ik_1\Lambda x + 2ik_1^2\Lambda t)|l^{(0)}\rangle, \quad (4.11)$$

where $|l^{(0)}\rangle$ is the coordinate-free four-dimensional vector.

Zeros k_j and vectors $|l_j^{(0)}\rangle$ comprise the discrete data of the RH problem that determine the soliton content of a solution of the BEC equations. The continuous data are characterized by the off-block-diagonal parts of the matrix $G(k)$ (4.2), $k \in \mathbb{R}$, and are responsible for the radiation components. In the following section we concretize the above relations to obtain one-soliton solutions of Eqs. (2.3).

V. SOLITON SOLUTIONS

A. Rank-one soliton

To obtain the rank-one soliton solution of the BEC equations (2.3), we consider the single pair k_1 and k_1^* of zeros and the eigenvector $|1\rangle$. In accordance with Eq. (4.11), the eigenvector takes the form

$$|1\rangle = (e^{-ik_1x-2ik_1^2t}n_1, e^{-ik_1x-2ik_1^2t}n_2, e^{ik_1x+2ik_1^2t}n_3, e^{ik_1x+2ik_1^2t}n_4)^T, \quad (5.1)$$

where n_a, $a=1,\ldots,4$ are complex numbers. The RH data are purely discrete, $N=1$, $G(k)=1$, $\widetilde{\psi}_\pm=1$. Hence, the solution of the RH problem is given by the rational function Ξ in (4.7) with the projector $P^{(1)}$. The reconstruction formula (4.4) is simplified to

$$\hat{Q} = -2\nu[\Lambda, P^{(1)}], \quad (5.2)$$

where we set $k_1 = \mu + i\nu$ and the projector $P^{(1)}$ in (4.6) is explicitly written as

$$P^{(1)} = \frac{1}{2}[(|n_1|^2 + |n_2|^2)(|n_3|^2 + |n_4|^2)]^{-1/2}$$

$$\times \widetilde{P}e^{-2i\mu x-4i(\mu^2-\nu^2)t} \operatorname{sech} z.$$

Here

$$\widetilde{P}_{ab} = n_a n_b^*, \quad z = 2\nu(x + 4\mu t) + \rho, \quad e^{2\rho} = \frac{|n_1|^2 + |n_2|^2}{|n_3|^2 + |n_4|^2}.$$

Hence, it follows from Eq. (5.2) that the soliton solution is given by

$$Q = 2\nu\Pi^{(1)}e^{-2i\mu x-4i(\mu^2-\nu^2)t} \operatorname{sech} z \quad (5.3)$$

with the polarization matrix

$$\Pi^{(1)} = \frac{1}{2}[(|n_1|^2 + |n_2|^2)(|n_3|^2 + |n_4|^2)]^{-1/2}\begin{pmatrix} n_1n_3^* & n_1n_4^* \\ n_2n_3^* & n_2n_4^* \end{pmatrix}.$$

Note that $n_2n_3^* = n_1n_4^*$ due to the structure of the matrix Q in (2.4). Besides, the matrix $\Pi^{(1)}$ obeys automatically two conditions,

$$\det \Pi^{(1)} = 0, \quad |\Pi_{11}^{(1)}|^2 + |\Pi_{22}^{(1)}|^2 + 2|\Pi_{12}^{(1)}|^2 = 1.$$

Moreover, it is not difficult to show that the matrix $\Pi^{(1)}$ depends only on two essential real parameters. Indeed, the rank-one soliton (5.3) can be represented as

$$Q = 2\nu\begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix}e^{i\varphi} \operatorname{sech} z, \quad (5.4)$$

where

$$\cos\theta = \frac{|n_1|}{|n_1|^2 + |n_2|^2} = \frac{|n_3|}{|n_3|^2 + |n_4|^2}, \quad \chi = \arg(n_3 - n_4),$$

$$\varphi = -2\mu x - 4(\mu^2 - \nu^2)t + \phi_\alpha,$$

$$\varphi_\alpha = \arg(n_1 - n_4) = \arg(n_2 - n_3).$$

The soliton amplitude is determined by the parameter ν, and its velocity is equal to 4μ. The parameters ρ and ϕ_α give the initial position of the soliton center and its initial phase, respectively. The angle θ determines the normalized population of atoms in different spin states, while the phase factor $e^{i\chi}$ is responsible for the relative phases between the components $\phi_\pm$ and ϕ_0.

It should be noted for future use that the constant soliton parameters acquire in general a slow t dependence in the presence of perturbation. This results in a modification of the equations for coordinates

$$z = 2\nu[x - \xi(t)], \quad \varphi = -\frac{\mu}{\nu}z + \delta(t),$$

$$\xi(t) = -\frac{1}{2\nu}(8\int^t dt'\,\mu(t')\nu(t') + \rho(t)),$$

$$\delta(t) = -2\mu\xi(t) - 4\int^t dt'[\mu^2(t') - \nu^2(t')] + \varphi_\alpha(t).$$

$$(5.5)$$

B. Rank-two soliton

As before, we begin with the pair k_1 and k_1^* of zeros, but now we have two linearly independent eigenvectors,

$$|1\rangle = (e^{-ik_1 x - 2ik_1^2 t} p_1, e^{-ik_1 x - 2ik_1^2 t} p_2, e^{ik_1 x + 2ik_1^2 t} p_3, e^{ik_1 x + 2ik_1^2 t} p_4)^T,$$

$$|2\rangle = (e^{-ik_1 x - 2ik_1^2 t} q_1, e^{-ik_1 x - 2ik_1^2 t} q_2, e^{ik_1 x + 2ik_1^2 t} q_3, e^{ik_1 x + 2ik_1^2 t} q_4)^T,$$
$$(5.6)$$

with p_a and q_a, $a = 1, \ldots, 4$, being complex numbers. The rational function Ξ is given by Eq. (4.7) with the rank-two projector $P^{(2)}$. This projector is written in accordance with Eq. (4.6) as

$$P^{(2)} = \sum_{l,m=1}^{2} |m\rangle (M^{-1})_{ml} \langle l|$$
$$= (\det M)^{-1} (M_{22}|1\rangle\langle 1| - M_{12}|1\rangle\langle 2|$$
$$- M_{21}|2\rangle\langle 1| + M_{11}|2\rangle\langle 2|).$$

In this case

$$M = \begin{pmatrix} A_1 e^{z'} + B_1 e^{-z'} & A_3 e^{z'} + B_3 e^{-z'} \\ A_3^* e^{z'} + B_3^* e^{-z'} & A_2 e^{z'} + B_2 e^{-z'} \end{pmatrix},$$

$z' = 2\nu(x + 4\mu t)$, and

$$A_1 = |p_1|^2 + |p_2|^2, \quad B_1 = |p_3|^2 + |p_4|^2,$$

$$A_2 = |q_1|^2 + |q_2|^2, \quad B_2 = |q_3|^2 + |q_4|^2,$$

$$A_3 = p_1^* q_1 + p_2^* q_2, \quad B_3 = p_3^* q_3 + p_4^* q_4.$$

Introducing the notations (to reproduce literally the results of Ref. [19])

$$p_1 q_2 - p_2 q_1 = e^{\rho + i\sigma}, \quad p_3 q_2 - p_2 q_3 = \beta^*, \quad p_1 q_4 - p_4 q_1 = \gamma^*,$$

$$p_1 q_3 - p_3 q_1 = p_4 q_2 - p_2 q_4 = \alpha^*, \quad (5.7)$$

we write explicitly the projector $P^{(2)}$ as

$$P^{(2)} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12}^* & P_{22} & P_{14} & P_{24} \\ P_{13}^* & P_{14}^* & 1 - P_{11} & -P_{12}^* \\ P_{14}^* & P_{24}^* & -P_{12} & 1 - P_{22} \end{pmatrix},$$

where

$$P_{11} = Z^{-1}(|\alpha|^2 + |\gamma|^2 + e^{2z}), \quad P_{22} = Z^{-1}(|\alpha|^2 + |\beta|^2 + e^{2z}),$$

$$P_{12} = -Z^{-1}(\alpha^*\beta + \alpha\gamma^*), \quad P_{14} = e^{i\varphi} Z^{-1}(\alpha e^z - \alpha^* \mathcal{D} e^{-z}),$$

$$P_{13} = e^{i\varphi} Z^{-1}(\beta e^z + \gamma^* \mathcal{D} e^{-z}), \quad P_{24} = e^{i\varphi} Z^{-1}(\gamma e^z + \beta^* \mathcal{D} e^{-z}).$$
$$(5.8)$$

$$\mathcal{D} = \det \Pi^{(2)}, \quad \varphi = -2\mu x - 4(\mu^2 - \nu^2) t + \sigma,$$

$$Z = \det M = 1 + e^{2z} + |\mathcal{D}|^2 e^{-2z}. \quad (5.9)$$

$\Pi^{(2)}$ is the polarization matrix $\Pi^{(2)} = \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix}$ subjected to the normalization condition [19]

$$2|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1. \quad (5.10)$$

As a result, we immediately find from Eq. (5.2) with $P^{(2)}$ the rank-two soliton solution of the BEC equations (2.3) [19],

$$Q(x,t) = 4\nu e^{i\varphi} Z^{-1}(\Pi^{(2)} e^z + \sigma_2 \Pi^{(2)\dagger} \sigma_2 \mathcal{D} e^{-z}), \quad (5.11)$$

where σ_2 is the Pauli matrix. Notice that the soliton solution of the matrix NLS equation was previously obtained in Ref. [27] by means of the Gelfand-Levitan integral equations, while our derivation is purely algebraic. The soliton (5.11) was also derived by Gerdjikov and co-workers via the dressing procedure [41].

We will distinguish between two featured cases of $\det \Pi^{(2)}$, namely, $\det \Pi^{(2)} = 0$ and $\det \Pi^{(2)} \neq 0$. These cases display different spin properties. Indeed, the spin density vector $\vec{f}(x,t) = \text{tr}(Q^\dagger \vec{\sigma} Q)$, where $\vec{\sigma}$ is the set of the Pauli matrices, is given in general by a spatially odd function

$$\vec{f}(x,t) = \left(\frac{4\nu}{Z}\right)^2 (e^{2z} - |\mathcal{D}|^2 e^{-2z}) \begin{pmatrix} \alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma \\ i(\bar{\alpha}\beta - \alpha\bar{\beta} + \alpha\bar{\gamma} - \bar{\alpha}\gamma) \\ |\beta|^2 - |\gamma|^2 \end{pmatrix},$$
$$(5.12)$$

with absolute value being of the form

$$|\vec{f}| = \left(\frac{4\nu}{Z}\right)^2 |e^{2z} - |\mathcal{D}|^2 e^{-2z}| (1 - 4|\mathcal{D}|^2)^{1/2}. \quad (5.13)$$

Therefore, the total spin vector $\vec{F} = \int dx \vec{f}(x,t)$ is zero. However, for $\mathcal{D} = 0$, as it follows from Eq. (5.12), the absolute value of the total spin vector is nonzero, $|\vec{F}| = 4\nu \neq 0$. In accordance with this property, the case $\mathcal{D} = 0$ corresponds to the ferromagnetic state, while the case $\mathcal{D} \neq 0$ is usually referred to as a polar state. In fact, a true polar state corresponds to the condition $|\mathcal{D}| = 1/2$, when, as it is seen from Eq. (5.13), the spin density is zero everywhere, not only the total spin [22].

It follows from Eq. (5.11) that the ferromagnetic state has the hyperbolic secant form

$$Q^f = 2\nu \Pi^{(2)} e^{i\varphi} \text{ sech } z, \quad (5.14)$$

where entries of the polarization matrix obey the normalization condition (5.10) and in addition the constraint $\beta\gamma - \alpha^2 = 0$. These two condition are sufficient to reduce the matrix $\Pi^{(2)}$ to the two-parameter form (5.4) with the identifications

$$\cos \theta = \frac{|p_3|}{(|p_3|^2 + |p_4|^2)^{1/2}}, \quad \chi = \arg(p_3 - p_4),$$

$$\varphi_\alpha = \arg(p_1 - p_3) = \arg(p_4 - p_2).$$

Therefore, the rank-two ferromagnetic soliton is completely equivalent to the rank-one soliton (5.4). Introducing the atom number density $n(x,t)$ and energy density $e(x,t)$,

$$n(x,t) = \text{tr}(Q^\dagger Q), \quad e(x,t) = c \text{ tr}(Q_x^\dagger Q_x - Q^\dagger Q Q^\dagger Q),$$

as well as their total counterparts $N_T = \int dx n(x,t)$ and $E_T = \int dx e(x,t)$, we obtain explicitly the total number of atoms and total energy in the ferromagnetic state,
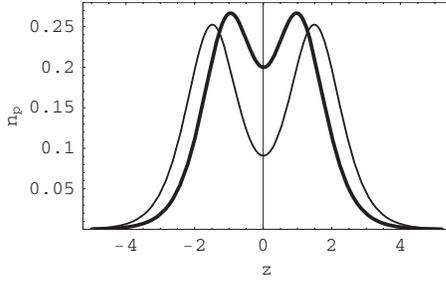
FIG. 1. Profiles of the atom number density function of the polar state: $|\mathcal{D}|=1/8$ (thick line), $|\mathcal{D}|=1/20$ (thin line), $\nu=0.5$.

$$N_T^f = 4\nu, \quad E_T^f = 4cN_T^f(\mu^2 - \nu^2/3).$$

In turn, the total number of atoms in the polar state and its energy are given by

$$N_T^p = 8\nu, \quad E_T^p = 4cN_T^p(\mu^2 - \nu^2/3).$$

The energy difference between both states with equal amount of atoms is $E_T^f - E_T^p = -(1/16)c(N_T^f)^3 < 0$. Hence, the ferromagnetic state is energetically preferable, from the viewpoint of stability, as compared with the polar state.

The atom number density of the polar soliton is described by the function

$$n_p(z) = \left(\frac{4\nu}{Z}\right)^2 (e^{2z} + 4|\mathcal{D}|^2 + |\mathcal{D}|^2 e^{-2z}). \quad (5.15)$$

Figure 1 demonstrates typical profiles of the atom number density function (5.15) for different $|\mathcal{D}|$. The two-humped structure becomes more pronounced with decreasing $|\mathcal{D}|$. Such a state can be treated as a pair of two ferromagnetic solitons with antiparallel spins [22]. Previous analysis of stability of multihumped vector solitons for the cubic nonlinearity revealed that they are always unstable [42,43]. Hence, we can suggest that the most likely scenario of the polar soliton evolution under the action of a perturbation would be its splitting into a pair of ferromagnetic solitons. Indeed, extensive simulations of the perturbed polar soliton behavior demonstrates unambiguously such a splitting. An example of such a behavior is depicted in Fig. 2, where we consider a
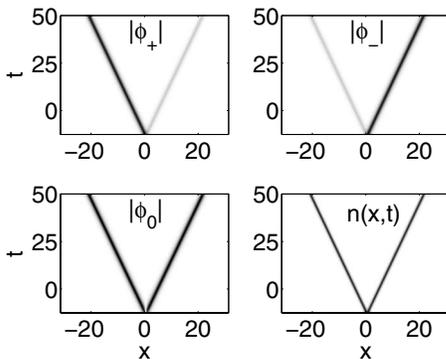

FIG. 2. Splitting of the components $\phi_\pm$ and ϕ_0 of a perturbed polar soliton and of the atom number density $n(x,t)$. Here $|\beta|^2 = 0.7$, $|\alpha|^2 = |\gamma|^2 = 0.1$. The perturbation is of the form (6.1) and (7.2) with $\epsilon = 0.1$.

disturbance of the integrability condition (2.2) as a perturbation with a small parameter $\epsilon = c_0 - c_2$ [see Eq. (7.3) below for a functional form of the perturbation]. It is seen that all of the components of the polar soliton split under the action of the perturbation.

Let us summarize the main conclusions concerning the soliton solutions which will play the key role in studying soliton perturbations. First, we derived the rank-one soliton solution with the hyperbolic secant profile. Second, rank-two solutions were obtained and classified as ferromagnetic and polar solitons. The polar soliton is perturbatively unstable and splits into two rank-two ferromagnetic solitons. Third, we proved equivalence of the rank-one soliton and rank-two ferromagnetic soliton. Therefore, it is sufficient to elaborate a perturbation theory for the more familiar type of solitons—the rank-one soliton (5.4). This will be done in the following section.

VI. PERTURBATION THEORY FOR THE BRIGHT SPINOR BEC SOLITON

In this section we perform a general analysis of the perturbed spinor BEC equations

$$i\partial_t \phi_\pm + \partial_x^2 \phi_\pm + 2(|\phi_\pm|^2 + 2|\phi_0|^2)\phi_\pm + 2\phi_\mp^* \phi_0^2 = \epsilon R_\pm,$$

$$i\partial_t \phi_0 + \partial_x^2 \phi_0 + 2(|\phi_+|^2 + |\phi_0|^2 + |\phi_-|^2)\phi_0 + 2\phi_+ \phi_0^* \phi_- = \epsilon R_0. \quad (6.1)$$

Here $R_\pm$ and R_0 determine a functional form of a perturbation, and ϵ is a small parameter. To distinguish between the integrable and perturbative contributions, we will assign the symbol $\delta/\delta t$ to the latter. Hence,

$$i\frac{\delta \hat{Q}}{\delta t} = \epsilon \hat{R}, \quad \hat{R} = \begin{pmatrix} 0 & R \\ R^\dagger & 0 \end{pmatrix}, \quad R = \begin{pmatrix} R_+ & R_0 \\ R_0 & R_- \end{pmatrix}.$$

In general, a perturbation causes a slow evolution of the RH data. Indeed, a perturbation leads to a variation $\delta \hat{Q}$ of the potential entering the spectral equation (2.6), and in turn to a variation of the Jost solutions,

$$\delta J_{\pm x} = ik[\Lambda, \delta J_\pm] + \delta \hat{Q} J_\pm + \hat{Q} \delta J_\pm.$$

Solving this equation gives

$$\delta J_\pm = J_\pm E \left(\int_{\pm\infty}^x dx' E^{-1} J_\pm^{-1} \delta \hat{Q} J_\pm E\right) E^{-1}.$$

As a result, we find from Eqs. (3.1), (3.5), and (3.10) a variation of the scattering matrix,

$$\frac{\delta S}{\delta t} = -i\epsilon S_+ \int_{-\infty}^\infty dx E^{-1} \psi_+^{-1} \hat{R} \psi_+ E S_-^{-1}$$

$$= -i\epsilon T_+^{-1} \int_{-\infty}^\infty dx E^{-1} \psi_-^{-1} \hat{R} \psi_- E T_-.$$

Here $S_\pm$ and $T_\pm$ are the matrices defined in Sec. III. Notice that they are the analytic solutions $\psi_\pm$ that enter naturally into this equation. Let us denote

$$\Upsilon_{\pm}(a,b) = \int_a^b dx E^{-1}\psi_{\pm}^{-1}\hat{R}\psi_{\pm}E,$$

$$\Upsilon_{\pm}(k) \equiv \Upsilon_{\pm}(-\infty,\infty). \tag{6.2}$$

Then

$$\frac{\delta S}{\delta t} = -i\epsilon S_+\Upsilon_+(k)S_-^{-1} = -i\epsilon T_+^{-1}\Upsilon_-(k)T_-.$$

The matrices $\Upsilon_{\pm}$ are interrelated by means of the matrix G entering the RH problem (4.1):

$$\Upsilon_-(k) = G\Upsilon_+(k)G^{-1}. \tag{6.3}$$

Eventually, variations of the analytic solutions follow from Eqs. (3.5) and (3.10):

$$\frac{\delta\psi_+}{\delta t} = -i\epsilon\psi_+EH_+E^{-1}, \quad \frac{\delta\psi_-^{-1}}{\delta t} = i\epsilon EH_-E^{-1}\psi_-^{-1}.$$

Here $H_{\pm}$ are the evolution functionals [30,35] that are defined in terms of $\Upsilon_{\pm}$,

$$H_+ = \Upsilon_+(k)M_1 - \Upsilon_+(x,\infty),$$

$$H_- = M_1\Upsilon_-(k) - \Upsilon_-(x,\infty), \quad M_1 = \text{diag}(1,1,0,0), \tag{6.4}$$

and contain all essential information about a perturbation. In particular, the evolution equations for $\psi_{\pm}$ gain additional terms caused by the perturbation and expressed in terms of $H_{\pm}$,

$$\partial_t\psi_+ = 2ik^2[\Lambda,\psi_+] + V\psi_+ - i\epsilon\psi_+EH_+E^{-1},$$

$$\partial_t\psi_-^{-1} = 2ik^2[\Lambda,\psi_-^{-1}] - \psi_-^{-1}V + i\epsilon EH_-E^{-1}\psi_-^{-1}. \tag{6.5}$$

Besides, the evolution equation for the matrix G of the RH problem has the form

$$\partial_t G = 2ik^2[\Lambda,G] - i\epsilon(GH_+ - H_-G). \tag{6.6}$$

In fact, this equation gives the evolution of the continuous RH data. Note that the involution (3.12) connects H_+ with H_-, $H_- = H_+^{\dagger}$, $k \in \mathbb{R}$.

Now we consider a single rank-one soliton and derive perturbation-induced evolution equations for the discrete RH data, i.e., for the zero k_1 and the eigenvector $|1\rangle$. It is more convenient to work with the vector $|n\rangle = (n_1,n_2,n_3,n_4)^T$ which is constant in the absence of perturbation and acquires slow t dependence under the action of a perturbation. We start from the equation

$$\psi_+(k_1)|1\rangle = \psi_+(k_1)\exp[(ik_1x+2i\int dtk_1^2)\Lambda]|p\rangle = 0$$

which is valid irrespectively of the presence of a perturbation. Here the integral in the exponent accounts for a possible perturbation-induced time dependence of the zero k_1. Taking the total derivative in t, we obtain

$$\{\partial_t[\psi_+(k)e^{ik\Lambda x+2i\int dtk_1^2\Lambda}] + \partial_k[\psi_+(k)e^{ik\Lambda x+2i\int dtk_1^2\Lambda}]\partial_t k\}_{|k_1}$$

$$+ \psi_+(k_1)e^{ik_1\Lambda x+2i\int dtk_1^2\Lambda}\partial_t|p\rangle = 0.$$

The first term with $\partial_t\psi_+$ is given by Eq. (6.5) which contains the evolution functional H_+. Recall that the evolution functional $H_+(k)$ is defined in terms of Υ_+ in (6.4) which in turn depends on ψ_+^{-1}. Hence, the function H_+ is meromorphic in $\mathbb{C}_+$ with the simple pole in k_1, where ψ_+ has zero,

$$H_+(k) = H_+^{(\text{reg})}(k) + \frac{1}{k-k_1}\text{Res}[H_+(k),k_1].$$

Here $H_+^{(\text{reg})}$ stands for the regular part of H_+ in the point k_1. Following now the method developed in Refs. [30,33,35], we find that the perturbed evolution of the vector $|n\rangle$ is given by

$$\partial_t|n\rangle = i\epsilon e^{-2i\int dtk_1^2\Lambda}H_+^{(\text{reg})}(k_1)e^{2i\int dtk_1^2\Lambda}|n\rangle. \tag{6.7}$$

Since the left-hand side of Eq. (6.7) is evidently x independent, we can consider this equation for $x \to +\infty$, where H_+ has only two nonzero columns [see Eq. (6.4)],

$$H_+(x \to +\infty) = (\Upsilon_+^{[1]},\Upsilon_+^{[2]},0,0). \tag{6.8}$$

Hence, Eq. (6.7), written in components, takes the form

$$\partial_t n_1 = i\epsilon(X_{11}n_1 + X_{12}n_2),$$

$$\partial_t n_2 = i\epsilon(X_{21}n_1 + X_{22}n_2),$$

$$\partial_t n_3 = i\epsilon(X_{31}n_1 + X_{32}n_2),$$

$$\partial_t n_4 = i\epsilon(X_{41}n_1 + X_{42}n_2), \tag{6.9}$$

where for simplicity we use the notation $X_{ab} = \Upsilon_{+ab}^{(\text{reg})}(k_1)$, $a,b = 1,2$, and $X_{ab} = \Upsilon_{+ab}^{(\text{reg})}(k_1)e^{-4i\int dtk_1^2}$ for $a = 3,4$ and $b = 1,2$. Here $\Upsilon_+^{(\text{reg})}$ is the regular part of Υ_+ in the point k_1,

$$\Upsilon_+^{(\text{reg})}(k_1) = \int dx E^{-1}(k_1)\{\hat{R}(\mathbb{1} - P^{(1)}) + P^{(1)}\hat{R}P^{(1)}$$

$$+ 2\nu x[\Lambda,P^{(1)}\hat{R}(\mathbb{1} - P^{(1)})]\}E(k_1). \tag{6.10}$$

This relation follows from $\Upsilon_+(k) = \int dx E^{-1}\Xi^{-1}\hat{R}\Xi E$. Note that in virtue of the specific symmetry of the matrix $\hat{Q}$ (2.8) the entries Υ_{+32} and Υ_{+41} are equal, as well as the entries Υ_{+14} and Υ_{+23}.

Now we can derive the evolution equation for the parameters θ and χ entering the polarization matrix of the soliton solution (5.4). Indeed, these parameters are defined in terms of n_a which in turn obey Eqs. (6.9). Simple calculation gives

$$\partial_t \cos\theta = \frac{i\epsilon}{2}[e^{\rho+i\varphi_\alpha}(X_{31}e^{-i\chi}\cos\theta + X_{41}\sin\theta)$$

$$- e^{\rho-i\varphi_\alpha}(X_{31}^*e^{i\chi}\cos\theta + X_{41}^*\sin\theta)], \tag{6.11}$$

$$\partial_t \chi = \frac{\epsilon}{2} \{ e^{\rho + i\varphi_\alpha} [X_{31} e^{-i\chi} - X_{42} e^{i\chi} + (\tan\theta - \cot\theta) X_{41}]$$

$$- e^{\rho - i\varphi_\alpha} [X_{31}^* e^{i\chi} - X_{42}^* e^{-i\chi} + (\tan\theta - \cot\theta) X_{41}^*] \}.$$
$$(6.12)$$

Just in the same way we obtain evolution equations for the parameters φ_α and ρ which are also expressed in terms of n_a,

$$\partial_t \varphi_\alpha = \frac{\epsilon}{2} [X_{11} + X_{11}^* + (X_{12} e^{-i\chi} + X_{12}^* e^{i\chi}) \tan\theta$$

$$- e^{\rho + i\varphi_\alpha} (X_{41} \cot\theta + X_{42} e^{i\chi})$$

$$- e^{\rho - i\varphi_\alpha} (X_{41}^* \cot\theta + X_{42}^* e^{-i\chi})], \qquad (6.13)$$

$$\partial_t \rho = \frac{i\epsilon}{2} \{ (X_{11} - X_{11}^*) \cos^2\theta + (X_{22} - X_{22}^*) \sin^2\theta$$

$$+ [(X_{12} - X_{21}^*) e^{i\chi} - (X_{12}^* - X_{21}) e^{-i\chi}] \sin\theta \cos\theta$$

$$+ e^{\rho + i\varphi_\alpha} (X_{31} e^{-i\chi} \cos^2\theta + 2X_{41} \sin\theta \cos\theta$$

$$+ X_{42} e^{i\chi} \sin^2\theta) - e^{\rho - i\varphi_\alpha} (X_{31}^* e^{i\chi} \cos^2\theta$$

$$+ 2X_{41}^* \sin\theta \cos\theta + X_{42}^* e^{-i\chi} \sin^2\theta) \}. \qquad (6.14)$$

In fact, Eqs. (6.11)–(6.13) are greatly simplified when calculating the functions X_{ab} for a specific perturbation. This will be demonstrated in the next section.

To derive evolution equation for k_1 (and hence for the soliton amplitude ν and velocity μ), we start from the equation $\det\psi_+(k_1) = 0$. Taking the total derivative in t yields

$$\partial_t [\det\psi_+(k)]_{|k_1} + [\partial_k \det\psi_+(k)]_{|k_1} \partial_t k_1 = 0.$$

In accordance with Eqs. (4.8) and (4.7) we can write

$$\det\psi_+(k) = \frac{k - k_1}{k - k_1^*} \det\widetilde{\psi}_+(k),$$

where $\det\widetilde{\psi}_+(k_1) \neq 0$ because $\widetilde{\psi}_+(k)$ is a solution of the regular RH problem (4.9). Accounting now for the relation

$$\partial_t \det\psi_+(k) = -i\epsilon \operatorname{tr} H_+ \det\psi_+(k),$$

we eventually obtain a simple evolution equation for the zero k_1,

$$\partial_t k_1 = \frac{i}{2}\epsilon \operatorname{tr} \operatorname{Res}[H_+(k), k_1] = \frac{i}{2}\epsilon \operatorname{Res}[\Upsilon_{+11}(k) + \Upsilon_{+22}(k), k_1].$$
$$(6.15)$$

Summarizing, Eqs. (6.6), (6.11)–(6.13), and (6.15) determine perturbation-induced evolution of the RH data. It should be stressed that these equations are exact because we did not yet refer to smallness of ϵ anywhere. At the same time, these equations cannot be directly applied because $\Upsilon_\pm$ entering them depend on unknown solutions $\psi_\pm$ of the spectral problem with the perturbed potential $\hat{Q}$. To proceed further, we develop, owing to the smallness of ϵ, the adiabatic approximation of the general perturbation theory.

VII. ADIABATIC APPROXIMATION

In the framework of the adiabatic approximation, we assume that the perturbed soliton adjusts its shape to the unperturbed one at the cost of slow evolution of its parameters. Hence, only the discrete RH data are relevant in this approximation, and we can set $\widetilde{\psi}_+ = \mathbb{1}$ for the solution of the regular RH problem (4.9). Therefore, $\psi_+ = \Xi$. In other words, it is the rational function Ξ that completely determines soliton dynamics in the adiabatic approximation. In particular, we have

$$\Upsilon_+ = \int_{-\infty}^{\infty} dx E^{-1} \Xi^{-1} \hat{R} \Xi E, \quad \Xi^{-1}(k) = \Xi^\dagger(k). \quad (7.1)$$

As an important example, we consider a perturbation caused by a small disturbance of the integrability condition (2.2). In this case we introduce a small parameter as $\epsilon = c_0 - c_2$, while the functional form of the perturbations $R_{\pm,0}$ has the form

$$R_{\pm,0} = (|\phi_+|^2 + 2|\phi_0|^2 + |\phi_-|^2)\phi_{\pm,0}. \quad (7.2)$$

Inserting the explicit expressions for the soliton components $\phi_{\pm,0}$ (5.4) into this equation gives

$$R_+ = (2\nu)^3 e^{i(\varphi - \chi)} \cos^2\theta \operatorname{sech}^3 z,$$

$$R_- = (2\nu)^3 e^{i(\varphi + \chi)} \sin^2\theta \operatorname{sech}^3 z,$$

$$R_0 = (2\nu)^3 e^{i\varphi} \cos\theta \sin\theta \operatorname{sech}^3 z. \quad (7.3)$$

Matrix elements of Υ_+ which are the main ingredients of the evolution equations for the soliton parameters are found from Eqs. (7.1) and (4.7), and the projector $P^{(1)}$ is calculated by means of the simple formula

$$P^{(1)} = \frac{|1\rangle\langle 1|}{\langle 1|1\rangle}, \quad \langle 1| = |1\rangle^\dagger,$$

which follows from Eq. (4.6). The eigenvector $|1\rangle$ is given by Eq. (5.1). As a result, matrix elements of the projector are as follows ($P_{ba} = P_{ab}^*$):

$$P_{11}^{(1)} = \frac{1}{2} e^z \cos^2\theta \operatorname{sech} z, \quad P_{12}^{(1)} = \frac{1}{2} e^{z-i\chi} \cos\theta \sin\theta \operatorname{sech} z,$$

$$P_{13}^{(1)} = \frac{1}{2} e^{i(\varphi - \chi)} \cos^2\theta \operatorname{sech} z, \quad P_{24}^{(1)} = \frac{1}{2} e^{i(\varphi + \chi)} \sin^2\theta \operatorname{sech} z,$$

$$P_{14}^{(1)} = P_{23}^{(1)} = \frac{1}{2} e^{i\varphi} \cos\theta \sin\theta \operatorname{sech} z, \quad P_{22}^{(1)} = \frac{1}{2} \sin^2\theta \operatorname{sech} z,$$

$$P_{33}^{(1)} = \frac{1}{2} e^{-z} \cos^2\theta \operatorname{sech} z, \quad P_{44}^{(1)} = \frac{1}{2} e^{-z} \sin^2\theta \operatorname{sech} z,$$

$$P_{34}^{(1)} = \frac{1}{2} e^{-z+i\chi} \cos\theta \sin\theta \operatorname{sech} z.$$

Now we easily obtain from Eqs. (7.1) and (7.2) that

$$\operatorname{Res}[\Upsilon_{+11}(k) + \Upsilon_{+22}(k), k_1] = 0.$$

Therefore, $\partial_t k_1 = 0$ in accordance with Eq. (6.15), which means that the soliton amplitude and velocity preserve their
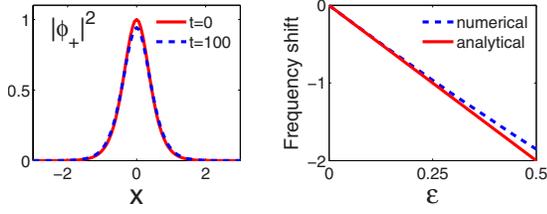
FIG. 3. (Color online) Left-hand panel: Evolution of the perturbed ϕ_+ component profile obtained numerically. Here $\epsilon = 0.1$. Right-hand panel: Comparison of the analytically predicted frequency shift of the perturbed soliton with that obtained numerically from Eqs. (6.1) and (7.2) with $\epsilon = 0.1$. In both panels $|\alpha| = |\beta| = |\gamma| = 0.5$.

initial values. Finding evolution of the other soliton parameters demands knowledge of the regular part of $\Upsilon_+(k_1)$. Calculation due to Eq. (6.10) gives

$$X_{11} = X_{12} = X_{21} = X_{22} = 0,$$

$$X_{31} = (2\nu^2)\exp(-\rho - i\varphi_\alpha + i\chi)\cos^2\theta,$$

$$X_{42} = (2\nu^2)\exp(-\rho - i\varphi_\alpha - i\chi)\sin^2\theta,$$

$$X_{41} = X_{32} = (2\nu^2)\exp(-\rho - i\varphi_\alpha)\cos\theta\sin\theta.$$

Substituting these functions into Eqs. (6.11)–(6.13), we obtain

$$\rho = \text{const}, \quad \theta = \text{const}, \quad \chi = \text{const}, \quad \varphi_\alpha(t) = \varphi_\alpha(0) - 4\epsilon\nu^2 t.$$

As a result, within the adiabatic approximation, the only manifestation of the perturbation caused by a small deviation from the integrability condition (2.2) consists in a small shift of the soliton frequency equal to $4\epsilon\nu^2$. Hence, a ferromagnetic soliton is a pretty robust object against a small disturbance of the integrability condition.

This conclusion has been checked by comparison with direct simulations of the perturbed Eqs. (6.1). The left-hand panel of Fig. 3 demonstrates the evolution of the perturbed ϕ_+ component profile. We see a small profile distortion. Very little energy radiation is emitted to the far field. The same results are valid for the other two components. It is seen from the right-hand panel that there is a good agreement of the predicted linear dependence of the frequency shift on ϵ with that obtained numerically.

VIII. CONCLUSION

In the present paper we have developed a perturbation theory for bright solitons of the integrable spinor BEC model. This model is equivalent to the 2×2 matrix NLS equation and is naturally associated with the matrix RH problem. We have demonstrated the efficiency of the formalism based on the RH problem, for solving both integrable and nearly integrable versions of the spinor BEC model. We have obtained the rank-one and rank-two soliton solutions of the model. Depending on the spin properties, the rank-two soliton can be of the ferromagnetic type or of the polar type. We have proven that the ferromagnetic soliton is equivalent to the rank-one soliton. As regards the polar soliton, its profile is characterized by a two-humped structure in a wide region of the soliton parameters. We have observed from numerical experiments that the polar soliton is unstable under the action of a perturbation and splits into a pair of ferromagnetic solitons. Owing to this fact, the problem to construct a perturbation theory for the spinor BEC solitons has been reduced to that for the rank-one solitons.

We have derived perturbation-induced evolution equations for the soliton parameters. In the adiabatic approximation of the perturbation theory these equations have been applied to a practically important case of a perturbation caused by a small deviation of the model parameters from those in the integrable case. We have shown a considerable stability of the ferromagnetic soliton in the presence of such a perturbation. Namely, the soliton preserves its amplitude, velocity and spin properties, a small frequency shift being the only manifestation of the perturbed environment. At the same time, the polar soliton solution of the integrable model has a restrictive area of applicability due to its instability and splitting under perturbations.

Three more points deserve a special comment. First, instability of a perturbed polar soliton and its splitting into ferromagnetic ones have been observed numerically. Analytical study of this phenomenon demands a separate consideration and can be performed, for example, by a stability analysis as developed in Ref. [43]. Second, we have restricted ourselves to the study of the adiabatic approximation of the general perturbation theory. Our equations permit us to go beyond this approximation and take into account the soliton shape distortion effects. However, quantitative characteristics of the first-order effects are too small to be verified experimentally, at least at present. Examples of practical calculations in the first-order approximation can be found in Ref. [33]. Third, the formalism developed here for the single perturbed soliton can be straightforwardly generalized to the case of N weakly interacting solitons arranged into a trainlike configuration. Analysis of the soliton train dynamics by the soliton perturbation theory can be found in Refs. [44,45] for optical solitons and in Ref. [46] for scalar bright BEC solitons.

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