Adiabatic interaction of N ultrashort solitons: Universality of the complex Toda chain model

V. S. Gerdjikov*
Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria

E. V. Doktorov†
B. I. Stepanov Institute of Physics, 220072 Minsk, Belarus

J. Yang‡
Department of Mathematics and Statistics, University of Vermont, Burlington, Vermont 05401

Using the Karpman-Solov'ev method we derive the equations for the two-soliton adiabatic interaction for solitons of the modified nonlinear Schrödinger equation (MNSE). Then we generalize these equations to the case of N interacting solitons with almost equal velocities and widths. On the basis of this result we prove that the N MNSE-soliton train interaction ($N>2$) can be modeled by the completely integrable complex Toda chain (CTC). This is an argument in favor of universality of the complex Toda chain that was previously shown to model the soliton train interaction for nonlinear Schrödinger solitons. The integrability of the CTC is used to describe all possible dynamical regimes of the N-soliton trains that include asymptotically free propagation of all N solitons, N-soliton bound states, various mixed regimes, etc. It allows also to describe analytically the manifolds in the $4N$-dimensional space of initial soliton parameters that are responsible for each of the regimes mentioned above. We compare the results of the CTC model with the numerical solutions of the MNSE for two and three-soliton interactions and find a very good agreement.

I. INTRODUCTION

The analytical description of the dynamics of picosecond solitons in single-mode nonlinear fibers is based on the nonlinear Schrödinger equation (NSE) [1,2]. The NSE serves as a very good integrable model admitting comprehensive investigation in the framework of the inverse spectral transform (IST) [3]. IST provides the complete analytical description of the soliton interaction in a generic case of asymptotically free N solitons moving with pairwise different velocities [3,4]. On the other hand, the practically important case, especially in a soliton-based fiber transmission, deals with the so-called N-soliton trains, i.e., with an ordered sequence of N ($N \geq 2$) solitons that are spaced apart almost equally and have almost (or exactly) equal amplitudes and velocities. In a number of recent papers [5–8], an effective formalism was developed for studying the dynamics of well-separated NSE solitons within the N-soliton train. This approach is based on a generalization of the two-soliton quasiparticle method by Karpman and Solov'ev [9] to the case of N solitons. In the framework of this approach, the soliton interaction is governed by a dynamical system for $4N$ soliton parameters. Such an approximation is called adiabatic because interaction between the solitons is displayed as a slow deformation of their parameters, a possible presence of radiation being ignored. It is important to realize that the above generalization from two to N solitons is nontrivial because of lack of the superposition principle for the nonlinear dynamical system.

Under some additional restrictions imposed on the soliton parameters, which ensure the validity of the adiabatic approximation, the above dynamical system is reduced to the complex Toda chain (CTC) equations with N nodes [10–12]. Extensive use of the fact that the CTC is a completely integrable model permits to classify soliton parameter regions with different asymptotic regimes of the N-soliton train [5–8]. It was also shown in Ref. [8] that the CTC can be associated with any equation from the NSE hierarchy.

One of the purposes in the optical fiber soliton communication is to achieve a bit rate as high as possible. A natural way in this direction is the use of shorter optical pulses. It should be noted, however, that when dealing with ultrashort optical pulses with duration ≤ 100 fs, the NSE should be modified to take into account some additional effects, such as the nonlinearity dispersion, the intrapulse Raman scattering, and the higher-order dispersion [1]. As a rule, the extra terms added to the NSE violate its integrability. On the other hand, if these additional terms are small, the IST-based soliton perturbation theory is usualy treated as the relevant method to account for their influence on the soliton behavior [13–15].

It is remarkable that adding a term accounting for the nonlinearity dispersion to the NSE preserves the integrability of the equation. In other words, the modified nonlinear Schrödinger equation (MNSE)

$$iu_t + \tfrac{1}{2}u_{xx} + i\alpha(|u|^2 u)_x + |u|^2 u = 0, \qquad (1)$$

is still integrable by means of IST, though the associated spectral problem (the so called Wadati-Konno-Ichikawa spectral problem [16], or quadratic bundle) does not belong to the familiar Zakharov-Shabat class. The parameter α in

*Email address: gerjikov@inrne.bas.bg
†Email address: doktorov@dragon.bas-net.by
‡Email address: jyang@emba.uvm.edu

Eq. (1) governs the strength of the nonlinearity dispersion. The case $\alpha=0$ corresponds to NSE. Thereby, the effect of the nonlinearity dispersion is considered nonperturbatively in Eq. (1). Moreover, we have to stress that it is the completely integrable model (1) that should be considered as a true starting point for analytical investigation of subpicosecond soliton dynamics. Indeed, it was shown in Ref. [17] that numerical simulation of the soliton propagation according to the MNSE (1) revealed various kinds of dynamical behavior that cannot be accounted for by treating the nonlinearity dispersion term of the MNSE (1) as a perturbation term in the NSE. Analogous idea in treating the perturbed NSE was developed by Kodama and Hasegawa in Refs. [18,19]. There the NSE with perturbations like the third order dispersion, nonlinear gain, and nonlinear dispersion was treated as a perturbed higher-order NSE.

The relevance of Eq. (1) to the problem of ultrashort pulse propagation in fibers was demonstrated in Refs. [20,21]. MNSE (1) is also used in plasma physics [22] and is relevant for description of a deformed continuous Heisenberg ferromagnet [23]. It is the Alfvén waves in magnetized plasma where the first successful application of IST to the quadratic bundle was achieved on an example of the derivative NSE [24], which is Eq. (1) without the last term. Both equations are interrelated by a gauge-like transformation, see, for example, Refs. [25–27]. The soliton solutions and the Hamiltonian structures of the MNSE were obtained for the first time in Refs. [26,27]. N-soliton solutions were further rederived by different methods: by IST using the above relation with the derivative NSE [28], by Bäcklund and Darboux transformations [29], by technique of determinant calculations [30], by the Hirota method [31], and by the $\bar{\partial}$ method [32]. It should be noted that the solutions obtained in these papers refer to the general case of asymptotically free solitons and being exact were too complicated for practical use.

Recently, a novel parametrization for the MNSE solitons was proposed within the framework of the Riemann-Hilbert formulation of IST [33]. The convenient parametrization of the MNSE soliton facilitated the development of an effective adiabatic soliton perturbation theory for the MNSE that is able to take into account nonzero terms in the right-hand side of Eq. (1), see [33].

The next natural step is to derive dynamical equations of the Karpman-Solov'ev type for the adiabatic evolution of the soliton parameters for the MNSE N-soliton train. Several questions arise in the process of solving this problem. Is it possible to associate an N-node chain model, like the CTC, with this dynamical system? Will this chain model be different from the CTC and, therefore, is the CTC valid only for the NSE hierarchy? How well do the numerical simulations of the MNSE with adiabatic N-soliton train initial conditions agree with the chainlike model predictions? All these questions will be answered below.

The purpose of this paper is to derive a dynamical system for the $4N$ soliton parameters for the MNSE N-soliton train. To this end we will generalize to the quadratic bundle the similar investigations performed for the NSE. In the next section we apply the Karpman-Solov'ev approach to the MNSE (1) and derive the dynamical system for the two-

soliton train. In Sec. III we show how this result can be generalized to the MNSE N-soliton train for $N>2$. We show in Sec. IV that after some additional assumptions the corresponding dynamical system for the soliton parameters acquires the form of the CTC. Thus we find that the CTC is characteristic not only for the NSE hierarchy [8], but has wider field of applications. This is an argument in favor of its universality.

In Sec. V we show how the integrability of the CTC can be used to determine the dynamical regimes of the N-soliton trains. We demonstrate on the examples of $N=2$ and $N=3$ how one can describe analytically the manifolds in the $4N$-dimensional space of initial soliton parameters that are responsible for the: (i) N-soliton bound state regime; (ii) asymptotically free regimes; (iii) various mixed regimes, etc. Although the analysis follows closely the ideas developed in Refs. [5–7] the description of the corresponding manifolds differs from the ones for the NSE soliton trains. The reason for this lies in the fact that the CTC fields $Q_j(t)$ are parametrized in a different way; in particular, $\operatorname{Im} Q_j$ depend not only on the soliton phases δ_j (as is the case for the NSE) but also on the soliton amplitudes.

In Sec. VI predictions of the CTC model are compared with the numerical results from the MNSE and find an excellent match for most regimes with $N=2$ and $N=3$. We found some disagreement between the CTC and numerical MNSE solution in the regimes when CTC predicts a very slow soliton separation.

II. TWO-SOLITON INTERACTIONS FOR THE MNSE

First of all we summarize the basic results concerning the soliton solution of the MNSE (1) [33]. This equation admits the Lax representation

$$\Phi_x = -\frac{2i}{\alpha}\left(k^2 - \frac{1}{4}\right)[\sigma_3, \Phi] + 2ikQ\Phi, \qquad (2)$$

$$\Phi_t = -\frac{4i}{\alpha^2}\left(k^2 - \frac{1}{4}\right)^2 [\sigma_3, \Phi] + \left(\frac{4i}{\alpha}k^3 Q + 2ik^2 Q^2 \sigma_3 - \frac{i}{\alpha}kQ\right.$$

$$\left. + kQ_x\sigma_3 - 2i\alpha kQ^3\right)\Phi.$$

Here the Hermitian matrix

$$Q = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix},$$

stands for the potential of the spectral problem (2), k is a spectral parameter. There exist various parametrizations of the soliton solution of the MNSE, the first one having been proposed in Refs. [26,27]. We follow here the parametrization given in Ref. [33], which was proven to be useful for practical calculations and admits a simple (though nontrivial) reduction to the NSE for $\alpha\to 0$. The one-soliton solution of MNSE related to the discrete eigenvalues $\pm k_1$, $\pm \bar{k}_1$ of the spectral equation (2) has the form

$$u_s(z,t) = i\frac{\nu}{\alpha}\frac{ke^{-z}+\bar{k}e^z}{(ke^z+\bar{k}e^{-z})^2}e^{i\phi}, \qquad (3)$$

where for notational simplicity we have dropped the index 1. Here $k=k_R-ik_I$, $k_I>0$, $\lambda\equiv 4k^2-1=\mu-i\nu$,

$$z=-\frac{\nu}{\alpha}[x-\xi(t)], \quad \phi=\frac{\mu}{\nu}z+\delta(t),$$

$$\xi(t)=-\frac{\mu}{\alpha}t+\xi_{(0)}, \quad \delta(t)=\frac{1}{2\alpha^2}(\mu^2+\nu^2)t+\delta_{(0)}. \quad (4)$$

It should be stressed that the soliton (3) is not of the hyperbolic-secant type with a real argument, characteristic for the NSE. It is specified by four real parameters μ, ν, $\xi_{(0)}$, and $\delta_{(0)}$ with $(-\mu/\alpha)$ being the soliton velocity, (α/ν) is its width, $\delta_{(0)}$ and $\xi_{(0)}$ are initial phase and position of the soliton. To carry out the limit reduction to the NSE, one should decompose the spectral parameter in the following manner:

$$k=\frac{1}{2}-\frac{\alpha}{2}(\mu^{\mathrm{NSE}}+i\nu^{\mathrm{NSE}})+O(\alpha^2), \quad \alpha\to 0, \qquad (5)$$

which gives $(-\mu/\alpha)\to 2\mu^{\mathrm{NSE}}$ and $(\alpha/\nu)\to(2\nu^{\mathrm{NSE}})^{-1}$, as should be.

If there is a small perturbation in a system described by the MNSE, we will deal with a perturbed MNSE

$$iu_t+\tfrac{1}{2}u_{xx}+i\alpha(|u|^2u)_x+|u|^2u=r(x,t), \qquad (6)$$

where $r(x,t)$ describes a functional form of the perturbation. In what follows we will restrict ourselves to the adiabatic approximation of the soliton perturbation theory. In other words, we suppose that a perturbation causes a slow variation of the soliton parameters only. The evolution equations for the perturbation-induced soliton parameters are given in Ref. [33]. Here we write them in terms of the parameters (4). The key equation has a very simple form

$$\frac{dk}{dt}=\frac{i}{2}\alpha k^2\int_{-\infty}^{\infty}\frac{R_+e^z}{(ke^{-z}+\bar{k}e^z)^2}dz, \qquad (7)$$

where $R_{\pm}=\exp[-i\phi(z,t)]r(z,t)\pm\exp[i\phi(-z,t)]\bar{r}(-z,t)$. Taking into account Eq. (4), we obtain

$$\frac{d\mu}{dt}=2i\alpha\int_{-\infty}^{\infty}\frac{k^3e^z-\bar{k}^3e^{-z}}{(ke^{-z}+\bar{k}e^z)^2}R_+dz, \qquad (8)$$

$$\frac{d\nu}{dt}=-2\alpha\int_{-\infty}^{\infty}\frac{k^3e^z+\bar{k}^3e^{-z}}{(ke^{-z}+\bar{k}e^z)^2}R_+dz. \qquad (9)$$

Evolution of ξ and δ is given by the following formulas:

$$\frac{d\xi}{dt}=-\frac{\mu}{\alpha}-\frac{1}{\alpha\nu}\left(\int_0^t dt'\,\mu(t')\right)\frac{d\nu}{dt}$$
$$+\frac{4\alpha^2}{\nu^2}\int_{-\infty}^{\infty}\frac{dzR_-}{(ke^{-z}+\bar{k}e^z)^2}\left[z(k^3e^z+\bar{k}^3e^{-z})\right.$$
$$\left.+\frac{i\nu}{8}(ke^z+\bar{k}e^{-z})\right], \qquad (10)$$

$$\frac{d\delta}{dt}=\frac{\mu^2+\nu^2}{2\alpha^2}+\frac{1}{\alpha^2\nu}\left(\int_0^t dt'\,\mu(t')\right)\left(\mu\frac{d\nu}{dt}-\nu\frac{d\mu}{dt}\right)$$
$$+\frac{4i\alpha}{\nu}\int_{-\infty}^{\infty}\frac{dzR_-}{(ke^{-z}+\bar{k}e^z)^2}\left[|k^2|(ke^{-z}+\bar{k}e^z)\right.$$
$$\left.-\frac{1}{8}(\bar{\lambda}ke^z-\lambda\bar{k}e^{-z})+\frac{iz}{\nu}(\bar{\lambda}k^3e^z+\lambda\bar{k}^3e^{-z})\right]. \qquad (11)$$

It should be noted that for the symmetric perturbations obeying the condition $\exp[-i\phi(z,t)]r(z,t)=\exp[i\phi(-z,t)]\bar{r}(-z,t)$, i.e., $R_-=0$, the complicated integrals in the right-hand sides of Eqs. (10) and (11) disappear.

Now we have all the necessary information to derive the Karpman-Solov'ev-like dynamical system of equations for the adiabatic interaction of two well-separated MNSE solitons. Below we will formulate more precisely, the condition of sufficient separability of solitons. We suppose that a two-soliton solution to the MNSE (1) is well approximated by the sum of two MNSE solitons

$$u(x,t)=u_1(z_1,t)+u_2(z_2,t), \qquad (12)$$

where $u_j(z_j,t)$, $j=1,2$, is given by Eq. (3) with

$$z_j=-\frac{\nu_j}{\alpha}(x-\xi_j),$$

$$\phi_j=\frac{\mu_j}{\nu_j}z_j+\delta_j.$$

$$\xi_j(t)=-\frac{1}{\alpha}\int_0^t dt'\,\mu_j(t')+\xi_{j0},$$

$$\delta_j(t)=\frac{1}{2\alpha^2}\int_0^t dt'[\mu_j^2(t')+\nu_j^2(t')]+\delta_{j0},$$

where we took into account the possible evolution of μ_j and ν_j. Now, by substituting Eq. (12) into the MNSE (1), it is easy to see that, because of the nonlinearity, each soliton feels the presence of the other one and the interaction is described by the perturbed MNSE

$$iu_{jt}+\tfrac{1}{2}u_{jxx}+i\alpha(|u_j|^2u_j)_x+|u_j|^2u_j=r_j, \qquad (13)$$

where

$$r_j = -i\alpha(2|u_j|^2 u_{3-j} + u_j^2 \bar{u}_{3-j})_x - (2|u_j|^2 u_{3-j} + u_j^2 \bar{u}_{3-j}). \tag{14}$$

It should be stressed that the perturbation (14) arises effectively as a result of treating two-soliton solution as the sum (12) of the one-soliton ones.

Now we formulate the conditions that provide the representation (12) as a two-soliton solution of the MNSE (1). At first we express z_2 in terms of z_1,

$$z_2 = \left(1 + \frac{\nu_2 - \nu_1}{\nu_1}\right) z_1 + \frac{\nu_2}{\alpha}(\xi_2 - \xi_1).$$

We suppose that solitons have almost equal widths, i.e.,

$$\frac{|\nu_2 - \nu_1|}{\nu_0} \ll 1, \tag{15}$$

where $\nu_0 = (1/2)(\nu_1 + \nu_2)$. Hence, we have

$$z_2 - z_1 \simeq \frac{\nu_0}{\alpha}(\xi_2 - \xi_1). \tag{16}$$

Calculation of the overlap integral $\left|\int_{-\infty}^{\infty} u_1(z_1,t) u_2(z_2,t) dx\right|$ (or, equivalently, $\int_{-\infty}^{\infty} dx |u_1(z_1,t) u_2(z_2,t)|$) gives an expression containing the factor $\exp[-(\nu_0/\alpha)(\xi_2 - \xi_1)] \equiv \epsilon$ for $\xi_2 > \xi_1$. Just this exponential factor determines a measure of overlapping neighboring solitons. We take in the following:

$$\frac{\nu_0}{\alpha}|\xi_2 - \xi_1| \gg 1 \tag{17}$$

(or $\epsilon \ll 1$), which means weak overlapping between the solitons.

Let us consider now the phase difference $\phi_2 - \phi_1 = (\mu_2/\nu_2)z_2 - (\mu_1/\nu_1)z_1 + \delta_2 - \delta_1$. Accounting for Eqs. (15) and (16) we may write

$$\phi_2 - \phi_1 = \frac{1}{\nu_2}\left[\mu_2 - \left(1 + \frac{\nu_2 - \nu_1}{\nu_1}\right)\mu_1\right]z_1 + \frac{\mu_2}{\alpha}\frac{\nu_0}{\nu_2}(\xi_2 - \xi_1)$$

$$+ \delta_2 - \delta_1.$$

Since we consider solitons moving with small relative velocities we assume

$$\frac{|\mu_2 - \mu_1|}{\nu_0} \ll 1. \tag{18}$$

Then the phase difference will not contain the z dependence. Furthermore,

$$\frac{\mu_2}{\alpha}\frac{\nu_0}{\nu_2}(\xi_2 - \xi_1) = \frac{\mu_2}{\alpha}\left(1 + \frac{\nu_0 - \nu_2}{\nu_2}\right)(\xi_2 - \xi_1).$$

As the last condition we suppose

$$|\nu_j - \nu_0|(\xi_2 - \xi_1) \ll 1, \tag{19}$$

hence, the phase difference takes the form

$$\phi_2 - \phi_1 = \frac{\mu_0}{\alpha}(\xi_2 - \xi_1) + \delta_2 - \delta_1.$$

Therefore, the conditions (15), (17), (18), (19) provide a possibility to consider a two-soliton solution of the MNSE (1) as the sum of the form (12).

To derive the Karpman-Solov'ev-like equations for the soliton parameters, we use Eqs. (7)–(11) with the perturbation (14). Accounting for the above conditions, we obtain after simple but tedious calculations $(j=1,2)$,

$$\frac{d\lambda_j}{dt} = (-1)^j 4 k_j \left(\frac{w_j \nu_j}{\alpha}\right)^2 \frac{w_{3-j} \nu_{3-j}}{\bar{k}_{3-j}} e^{-\Delta - i\psi}, \tag{20}$$

$$\frac{d\mu_j}{dt} = (-1)^j \frac{2}{\alpha^2} \nu_j^2 \nu_{3-j} \times \left(w_j^2 w_{3-j} \frac{k_j}{\bar{k}_{3-j}} e^{-i\psi} \right.$$
$$\left. + \bar{w}_j^2 \bar{w}_{3-j} \frac{\bar{k}_j}{k_{3-j}} e^{i\psi}\right) e^{-\Delta}, \tag{21}$$

$$\frac{d\nu_j}{dt} = (-1)^{j+1} \frac{2i}{\alpha^2} \nu_j^2 \nu_{3-j} \left(w_j^2 w_{3-j} \frac{k_j}{\bar{k}_{3-j}} e^{-i\psi} \right.$$
$$\left. - \bar{w}_j^2 \bar{w}_{3-j} \frac{\bar{k}_j}{k_{3-j}} e^{i\psi}\right) e^{-\Delta}, \tag{22}$$

$$\frac{d\xi_j}{dt} = -\frac{1}{\alpha}\mu_j + (-1)^j \frac{2i}{\alpha^3}\nu_j \nu_{3-j}\left(\int_0^t dt' \mu_j(t')\right)$$

$$\times \left(w_j^2 w_{3-j} \frac{k_j}{\bar{k}_{3-j}} e^{-i\psi} - \bar{w}_j^2 \bar{w}_{3-j} \frac{\bar{k}_j}{k_{3-j}} e^{i\psi}\right) e^{-\Delta}$$

$$+ \frac{i}{\alpha}\nu_{3-j}\left([(1 + \bar{w}_j^2)(1 - 2\bar{w}_j^2)\right.$$

$$+ 4is_j] w_j^2 w_{3-j} \frac{k_j}{\bar{k}_{3-j}} e^{-i\psi} - [(1 + w_j^2)(1 - 2w_j^2)$$

$$\left. - 4is_j] \bar{w}_j^2 \bar{w}_{3-j} \frac{\bar{k}_j}{k_{3-j}} e^{i\psi}\right) e^{-\Delta}, \tag{23}$$

$$\frac{d\delta_j}{dt} = \frac{1}{2\alpha^2}(\mu_j^2 + \nu_j^2) - (-1)^j \frac{2i}{\alpha^4} \nu_j \nu_{3-j}\left(\int_0^t dt' \mu_j(t')\right)$$

$$\times\left(\lambda_j w_j^2 w_{3-j}\frac{k_j}{\bar{k}_{3-j}}e^{-i\psi} - \bar{\lambda}_j \bar{w}_j^2 \bar{w}_{3-j}\frac{\bar{k}_j}{k_{3-j}}e^{i\psi}\right)e^{-\Delta}$$

$$-\frac{i}{\alpha^2}\bar{\lambda}_j\nu_{3-j}\left([(1+\bar{w}_j^2)(1-2\bar{w}_j^2)\right.$$

$$+4is_j]w_j^2 w_{3-j}\frac{k_j}{\bar{k}_{3-j}}e^{-i\psi}$$

$$-[(1+w_j^2)(1-2w_j^2)-4is_j]$$

$$\left.\times \bar{w}_j^2 \bar{w}_{3-j}\frac{\bar{k}_j}{k_{3-j}}e^{i\psi}\right)e^{-\Delta}. \tag{24}$$

Above we used the notations

$$w_j = \frac{k_j}{\bar{k}_j} = \exp(-2is_j), \quad \Delta = \frac{\nu_0}{\alpha}|\xi_2 - \xi_1|,$$

$$\psi = \frac{\mu_0}{\alpha}(\xi_2 - \xi_1) + \delta_2 - \delta_1, \quad s_j = \frac{1}{2}\arctan\frac{\nu_j}{1+\mu_j},$$

where the last relation follows from $\lambda_j = 4k_j^2 - 1 = \mu_j - i\nu_j$.

Equations (21)–(24) are the analog of the Karpman-Solov'ev equations in the case of the adiabatic interaction of two well-separated MNSE solitons and reduce to the NSE dynamical system in the limit (5).

The dynamical system (21)–(24) is rather complicated and needs further simplification to perform its analytical investigation. Integrable approximation is of special importance, and finding such an approximation is one of our purposes. But first of all we will generalize these equations to the case of N MNSE solitons.

III. N-SOLITON TRAIN INTERACTIONS FOR THE MNSE

Since the Karpman-Solov'ev-like dynamical system is nonlinear, it does not allow the superposition principle. It is physically clear because in the case of the N-soliton train with $N \geq 3$ a middle soliton will be influenced by its neighbors from both sides. Hence, it is not possible to describe the interaction of $N \geq 3$ solitons within the framework of two-soliton interaction like (21)–(24).

The first remark we should keep in mind is that the interaction force between the solitons is of the order of their overlap. Therefore, we can take into account only the nearest-neighbor interaction. Indeed, for the N-soliton train we assume that

$$u = \sum_{j=1}^N u_j, \tag{25}$$

where u_j is the MNSE soliton (3) whose center of mass is located at ξ_j. Assume that $\xi_1 < \xi_2 < \ldots < \xi_N$. Inserting this *ansatz* into the cubic term of the MNSE (1) gives

$$|u^2|u = \sum_{j=1}^N |u_j^2|u_j + \sum_{j\neq l}(|u_j^2|u_l + 2u_j^2\bar{u}_l) + \sum_{j\neq l\neq n}u_j\bar{u}_l u_n. \tag{26}$$

Straightforward analysis shows that the integrals in Eqs. (7)–(11) corresponding to each type of terms in Eq. (26) are of the following order of magnitude:

$$|u_k|^2 u_m, \quad u_k^2 \bar{u}_m \leftrightarrow O(\epsilon^{-|k-m|}),$$

$$u_j\bar{u}_l u_n, \quad j < l < n \leftrightarrow O(\epsilon^{-|j-l|-|l-n|}).$$

Here ϵ is of the order of $\exp[-(\nu_0/\alpha)|\xi_j - \xi_l|]$ for $|j-l|=1$. Because we keep only terms of the order of ϵ, we see that it is enough to take into account only terms with $|j-l|=1$. In other words, the "triple" terms like $u_j\bar{u}_l u_n$ can be neglected. Quite analogous is the situation with the cubic terms containing x derivative.

Second, as in Sec. II, we pose the conditions

$$|\nu_j - \nu_l| \ll \nu_0, \quad |\mu_j - \mu_l| \ll \nu_0,$$

$$\nu_0|\xi_{0j} - \xi_{0l}| \gg 1, \quad |\nu_j - \nu_l||\xi_{0j} - \xi_{0l}| \ll 1,$$

where $\nu_0 = N^{-1}\Sigma_{j=1}^N \nu_j$, $\mu_0 = N^{-1}\Sigma_{j=1}^N \mu_j$. They mean that we consider the chainlike configuration of N solitons with equal or nearly equal velocities and widths. Substituting the soliton solutions (3) into the perturbation

$$r_j = -\sum_{l=j\pm1}[i\alpha(2|u_j|^2 u_l + u_j^2 \bar{u}_l)_x + (2|u_j|^2 u_l + u_j^2 \bar{u}_l)], \tag{27}$$

and calculating the integrals in Eq. (7), we obtain

$$\frac{d\lambda_j}{dt} = -k_j\left(\frac{2w_j\nu_j}{\alpha}\right)^2\sum_{l=j\pm1}s_{lj}\frac{w_l\nu_l}{\bar{k}_l}e^{-|\Delta_{lj}|}e^{-is_{lj}\psi_{lj}}, \tag{28}$$

where

$$s_{lj} = \begin{cases} 1 & \text{for } l=j+1, \\ -1 & \text{for } l=j-1, \end{cases} \quad \Delta_{lj} = \frac{\nu_0}{\alpha}\left|\xi_l - \xi_j\right|,$$

$$\psi_{lj} \equiv \psi_l - \psi_j = \frac{\mu_0}{\alpha}(\xi_l - \xi_j) + \delta_l - \delta_j. \tag{29}$$

The corresponding formula for μ_j and ν_j follow from Eq. (28) as real and imaginary parts. It is not difficult to derive also the equations for the rest two parameters ξ_j and δ_j, generalizing those in Eqs. (23) and (24) for the two-soliton interaction. Keeping in mind, however, our aim to formulate the equations for the adiabatic interaction of the MNSE solitons in the form tractable analytically, it is sufficient to represent the equations for ξ_j and δ_j in the following form:

$$\frac{d\xi_j}{dt} = -\frac{\mu_j}{\alpha} + O(\epsilon), \tag{30}$$

$$\frac{d\delta_j}{dt} = \frac{\mu_j^2 + \nu_j^2}{2\alpha^2} + O(\epsilon). \tag{31}$$

Indeed, let us impose the conditions on the scattering data of the spectral problem (2), which correspond to the adiabatic approximation. Just as for the NSE, we require that the eigenvalues of the Lax operator are clustered around their mean value,

$$|\lambda_j - \lambda_0|^2 \simeq O(\epsilon), \quad \lambda_0 = \frac{1}{N}\sum_{j=1}^{N} \lambda_j.$$

Thus we obtain the estimates in Eqs. (30) and (31), which mean that we can neglect the perturbation-induced evolution of the parameters ξ_j and δ_j as compared to their main (unperturbed) evolution. At the same time s_j and w_j characterize the initial conditions and it is important to take them into account in the right-hand side of Eq. (28).

IV. DERIVATION OF THE COMPLEX TODA CHAIN MODEL

The next important step towards deriving a model of N MNSE-soliton interactions tractable analytically consists in a careful account for the terms of the order of ϵ. First note that because the right-hand side of Eq. (28) is of the order of ϵ, we may approximate k_j by $|k_0|e^{-is_j}$, where k_0 is the mean value

$$k_0 = \frac{1}{N}\sum_{j=1}^{N} k_j.$$

Thereby we neglect terms such as $|\nu_0 - \nu_j|\epsilon$ and $|\mu_0 - \mu_j|\epsilon$, which due to Eqs. (15) and (18) are of the higher order in ϵ. Hence, Eq. (28) is written as follows:

$$\frac{d\lambda_j}{dt} = \frac{4\nu_0^3}{\alpha^2}(e^{Q_{j+1}-Q_j}f_j - e^{Q_j-Q_{j-1}}g_j), \tag{32}$$

where

$$Q_{j+1} - Q_j = -\frac{\nu_0}{\alpha}(\xi_{j+1} - \xi_j) - i\left[\pi + \frac{\mu_0}{\alpha}(\xi_{j+1} - \xi_j) + \delta_{j+1}\right.$$
$$\left. - \delta_j + 4s_{j+1} + 4s_j\right], \tag{33}$$

$$f_j = \exp[i(s_{j+1} - s_j)], \quad g_j = \exp[i(s_{j-1} - s_j)]. \tag{34}$$

The recurrent relation (33) can be solved for Q_j with the result

$$Q_j = -\frac{\nu_0}{\alpha}\xi_j - i\left[j\pi + \frac{\mu_0}{\alpha}\xi_j + \delta_j + \delta_0 + \sum_{k=1}^{j-1} 8s_k + 4s_j\right],$$

$$\delta_0 = \frac{1}{N}\sum_{j=1}^{N} \delta_j. \tag{35}$$

The next step is to derive the evolution equation for Q_j (35). It should be noted first of all that up to terms of the order of ϵ,

$$\frac{d\delta_0}{dt} = \frac{1}{2\alpha^2 N}\sum_{j=1}^{N} (\mu_j^2 + \nu_j^2)$$

$$= \frac{1}{2\alpha^2 N}\sum_{j=1}^{N} [\mu_0^2 + \nu_0^2 + 2\mu_0(\mu_j - \mu_0)$$

$$+ 2\nu_0(\nu_j - \nu_0) + O(\epsilon)]$$

$$= \frac{1}{2\alpha^2}[\mu_0^2 + \nu_0^2 + O(\epsilon)].$$

Then, in view of Eqs. (30) and (31), we get

$$\frac{dQ_j}{dt} = \frac{i}{\alpha^2}(\mu_0 - i\nu_0)\mu_j - \frac{i}{2\alpha^2}(\mu_j^2 + \nu_j^2 + \mu_0^2 + \nu_0^2)$$

$$= \frac{i}{2\alpha^2}[-(\mu_j - \mu_0)^2 - (\nu_j - \nu_0)^2 - 2i\nu_0(\mu_j - i\nu_j)]$$

$$= \frac{\nu_0}{\alpha^2}\lambda_j + O(\epsilon). \tag{36}$$

Finally, keeping only the leading-order terms and replacing $f_j \simeq 1$ and $g_j \simeq 1$ we find from Eqs. (33) and (36)

$$\frac{d^2Q_j}{dt^2} = 4\left(\frac{\nu_0}{\alpha}\right)^4 (e^{Q_{j+1}-Q_j} - e^{Q_j-Q_{j-1}}), \tag{37}$$

i.e., the CTC model. Hence we see that the CTC model arises naturally as the integrable limit of the Karpman-Solov'ev-like equations describing the adiabatic interaction of N MNSE solitons within the train of solitons with near velocities and widths.

As we will see in the next section the interactions of the MNSE solitons are substantially different from the ones of the NSE. As it can be seen from Eq. (33), the dependence of Q_j on the soliton parameters is different from that for the NSE case. An important point here is that $\mathrm{Im}\, Q_j$ depends explicitly also on the amplitudes of the solitons through $s_j = \arctan[\nu_j/\alpha(1 + \mu_j)]$.

V. DYNAMICAL REGIMES OF THE N-SOLITON TRAINS

It is well known that the CTC is a completely integrable dynamical system. Most of the results concerning the CTC such as the Lax representation, the integrals of motion, explicit solutions, etc., are direct consequences of the classical results by Toda and Moser [34–36] on the real Toda chain (RTC). However, there is a qualitative difference between the RTC and the CTC when one tries to analyze the dynami-

cal regimes of the two systems, see Refs. [5,7,37,38].

Indeed, the Lax representation of the CTC (37) is of the form

$$\frac{dL}{d\tau}=[L,M], \qquad (38)$$

$$L(\tau)=\sum_{j=1}^{N}[b_jE_{j,j}+a_j(E_{j,j+1}+E_{j+1,j})], \qquad (39)$$

$$M(\tau)=\sum_{j=1}^{N}a_j(E_{j+1,j}-E_{j,j+1}), \qquad (40)$$

where

$$\tau=c_0t, \quad c_0=2\nu_0^2/\alpha^2, \quad a_j=\tfrac{1}{2}\exp(Q_{j+1}-Q_j)/2,$$

$$b_j=-\tfrac{1}{2}dQ_j/d\tau=-\lambda_j/4\nu_0, \quad (E_{i,j})_{ln}=\delta_{il}\delta_{jn};$$

see Eq. (36). In fact, without loss of generality we can assume that tr$L=0$. This can be achieved by subtracting $\zeta_0\mathbb{1}$ from L, where $\zeta_0=\Sigma_{j=1}^{N}\zeta_j/N=\Sigma_{j=1}^{N}b_j/N$. Note that ζ_0 is obviously an integral of motion for the CTC, i.e., $d\zeta_0/d\tau=0$.

The explicit solution to the CTC is given by

$$q_k(\tau)=q_1(0)+\ln\frac{A_k(\tau)}{A_{k-1}(\tau)}, \qquad (41)$$

where $A_0\equiv1$,

$$A_1(\tau)=\sum_{k=1}^{N}r_k^2e^{-2\zeta_k\tau}, \qquad (42)$$

$$A_k(\tau)=\sum_{1\leqslant l_1<\ldots<l_k\leqslant N}(r_{l_1}\ldots r_{l_k})^2$$
$$\times W^2(l_k,l_{k-1},\ldots,l_1)\exp[-2(\zeta_{l_1}+\ldots+\zeta_{l_k})\tau], \qquad (43)$$

and

$$A_N=W^2(N,N-1,\ldots,2,1)\exp[-2(\zeta_1+\cdots+\zeta_N)\tau]\prod_{k=1}^{N}r_k^2. \qquad (44)$$

Here ζ_j are the eigenvalues of the Lax matrix L, $W(l_k,\cdots,l_1)$ denotes the Vandermonde determinant:

$$W(l_k,\ldots,l_1)=\prod_{\substack{s>p\\s,p\in\{l_1,\cdots,l_k\}}}(2\zeta_s-2\zeta_p), \qquad (45)$$

and r_j are the first components $r_j=\vec{v}_{j,1}$ of the eigenvectors

$$L\vec{v}_j=\zeta_j\vec{v}_j, \qquad (46)$$

normalized by

$$(\vec{v}_j,\vec{v}_j)\equiv\sum_{k=1}^{N}(\vec{v}_{j,k})^2=1. \qquad (47)$$

Due to the fact that L is a symmetric matrix we find also

$$\sum_{j=1}^{N}r_j^2=1. \qquad (48)$$

Using the explicit solution for $Q_j(t)$ we can estimate the asymptotic behavior of $Q_j(\tau)$ for $\tau\rightarrow\infty$.

Such an analysis for the RTC, i.e., when Q_j, a_j, and b_j are real, shows that (i) r_j and ζ_j are real valued, (ii) $\zeta_j\neq\zeta_k$ for $j\neq k$. Therefore, one finds that for $\tau\rightarrow\infty$ each ''particle'' Q_j moves uniformly with a velocity $2\zeta_j$ [35,36]. Since ζ_j are pairwise different we conclude that the only possible dynamical regime is the asymptotically free (AFR) one.

The same considerations applied to the CTC lead, however, to a qualitatively different results. Indeed, now r_j and $\zeta_j=\kappa_j+i\eta_j$ become complex valued and there are no restrictions on the eigenvalues ζ_j. Then evaluating the limits of $Q_j(\tau)$ for $\tau\rightarrow\infty$ we find that the asymptotic velocity of Q_j is determined by $2\kappa_j=2\mathrm{Re}\,\zeta_j$. As a result we have much wider spectra of dynamical regimes. The reason for that is also in the fact that CTC can be viewed as a dynamical system of N ''complex'' particles that are characterized not only by their positions $\mathrm{Re}\,Q_j$ and velocities $v_j=\mathrm{Re}\,b_j$, but also by their phases and phase velocities; the latter are related to $\mathrm{Im}\,Q_j$ and $\mathrm{Im}\,b_j$. Physically speaking these ''complex'' particles have, just like the bright NLE solitons, an internal degree of freedom. This makes the interaction between the particles more complicated and as a result the number of the possible dynamical regimes increases substantially.

The AFR that takes place if $\kappa_j\neq\kappa_k$ for $j\neq k$ is just one of the options. Another option is $\kappa_1=\kappa_2=\ldots=\kappa_N=0$, which corresponds to a bound state regime (BSR) of all N ''complex'' particles (solitons) in the train. There is also a large class of intermediate or mixed regimes (MR) for which only several of the parameters κ_j are equal. For example, if $\kappa_1=\kappa_2=\kappa_3>\kappa_4\cdots>\kappa_N$ then the first three particles (solitons) will form a bound state while the rest $N-3$ particles will be asymptotically free.

Note that this variety of regimes exist in the generic case when the eigenvalues ζ_j of L are pairwise different; so in the previous case we assume that $\eta_1\neq\eta_2\neq\eta_3$. One may consider also degenerate regimes (when two or more of the eigenvalues ζ_j become equal) and singular regimes [when one or more of the functions $Q_j(\tau)$ develop singularities for finite τ].

There is also another important consequence from the integrability of CTC. From the Lax representation one easily finds that the eigenvalues ζ_j are the integrals of motion for the CTC, i.e., ζ_j are time independent. Therefore, we can evaluate them, for example, at the initial moment $t=0$ using for this the initial values of the soliton parameters. Then, knowing ζ_j and, more specifically, κ_j we can predict the asymptotic regime of the corresponding N-soliton train.

We can also answer another question: describe the set of initial soliton parameters for which the corresponding N-soliton train will develop a specific dynamic regime. In other words, we can describe the set of initial soliton parameters for which we will have, say, an N-soliton bound state regime. To describe the BSR all we need to do is to solve the corresponding characteristic equation

$$\det(L - \zeta \mathbb{1}) = 0, \tag{49}$$

and impose the condition $\kappa_1 = \kappa_1 = \cdots = \kappa_N = 0$. Since the coefficients of Eq. (49) and consequently κ_j will be expressed in terms of the initial soliton parameters, we will have a set of nonlinear equations describing the BSR. Analogously, if we need to describe the AFR we must solve for $\kappa_j \neq \kappa_k$ for $k \neq j$.

We will show how this can be done analytically for the simplest nontrivial cases with $N=2$ and $N=3$. For generic values of N this can always be done by numeric means; one needs only to solve algebraic equation (49) of order N.

Let us briefly describe the manifolds of soliton parameters responsible for each of the dynamical regimes for $N=2$ and $N=3$. As it is clear from the above considerations, we have to solve the characteristic equation (49) and to express the eigenvalues ζ_j of L in terms of the soliton parameters.

A. $N=2$ case

For simplicity, from now on we shall consider trains with zero initial velocities, $\mu_j(0) = 0$, i.e., in the relevant moving coordinate system. The matrix

$$L_0 \equiv L(t=0) = \begin{pmatrix} b & a \\ a & -b \end{pmatrix},$$

with $\operatorname{tr} L = 0$ is built from the initial soliton parameters

$$a = -\frac{i}{2} \exp\left(-\frac{\nu_0}{2\alpha} r_0 - \frac{i}{2}\Gamma\right), \quad b = \frac{i}{4}d,$$

where

$$r_0 = \xi_{2(0)} - \xi_{1(0)}, \quad \Gamma = \delta_{2(0)} - \delta_{1(0)} + 4s_1 + 4s_2,$$

$$d = (\nu_{1(0)} - \nu_0)/\nu_0. \tag{50}$$

Then

$$\zeta_{1,2} = \pm \sqrt{b^2 + a^2} = \pm \frac{i\Delta_{cr,2}}{4}\sqrt{y_0^2 + e^{-i\Gamma}}, \tag{51}$$

with

$$\Delta_{cr,2} = 2 \exp[-\nu_0 r_0/(2\alpha)], \quad y_0 = \frac{d}{\Delta_{cr,2}}. \tag{52}$$

Obviously if $\Gamma \neq 0,\pi$ then $\operatorname{Re} \zeta_{1,2} \neq 0$ and we will have an AFR. If $\Gamma = 0$, then $\operatorname{Re} \zeta_{1,2} = 0$ and we have a BSR. If $\Gamma = \pi$, then $\operatorname{Re} \zeta_{1,2} = 0$, i.e., we will have a BSR only provided

$$|d| > \Delta_{cr,2}. \tag{53}$$

If $|d| < \Delta_{cr,2}$, both roots $\zeta_{1,2}$ become real and we go into the AFR.

It was already noted that the conditions $\Gamma = 0,\pi$ involve, besides the phases δ_j, also the amplitudes of the solitons through s_j. In particular, for $\nu_0 = \alpha = 1$ and $\mu_j = 0$ we have $s_1 + s_2 = \pi/4$. Therefore, two such MNSE solitons *attract* each other and form a bound state provided $\delta_2 - \delta_1 = \pi$ and *repulse* each other (which leads to AFR) for $\delta_2 - \delta_1 = 0$. Such behavior of the two-soliton interaction is quite to the contrary to that known for the NSE two-soliton interaction.

The explicit solution to the CTC with $N=2$ is of the form

$$Q_1(t) = -Q_2(t) = \ln \frac{\cosh(2\zeta_1 c_0 t - \gamma_1)}{2\zeta_1},$$

$$\gamma_1 = \ln \frac{r_1}{r_2}, \tag{54}$$

where ζ_1 is expressed in terms of the soliton parameters (51) and

$$\gamma_1 = \frac{1}{2} \ln \frac{\sqrt{y_0^2 + e^{-i\Gamma}} + y_0}{\sqrt{y_0^2 + e^{-i\Gamma}} - y_0}. \tag{55}$$

Obviously for $\Gamma = 0$ the solution $Q_1(t)$,

$$Q_1(t) = \ln \frac{2 \cos(Y_0 c_0 t/2 + i\gamma_{10})}{iY_0}, \tag{56}$$

$$Y_0 = \Delta_{cr,2}\sqrt{y_0^2 + 1}, \quad \gamma_{10} = \frac{1}{2} \ln \frac{\sqrt{y_0^2 + 1} + y_0}{\sqrt{y_0^2 + 1} - y_0},$$

becomes a periodic function of $t = \tau/c_0$ with period depending on y_0:

$$T_{2s;1} = \frac{4\pi}{c_0 \Delta_{cr,2}\sqrt{y_0^2 + 1}}. \tag{57}$$

Analogously for $\Gamma = \pi$ from Eq. (55) we have

$$Q_1(t) = -Q_2(t) = \ln \frac{2\cosh(i\Delta_{cr,2}\sqrt{y_0^2 - 1}\, c_0 t/2 - \gamma_{11})}{i\Delta_{cr,2}\sqrt{y_0^2 - 1}}, \tag{58}$$

$$\gamma_{11} = \frac{1}{2} \ln \frac{\sqrt{y_0^2 - 1} + y_0}{\sqrt{y_0^2 - 1} - y_0}.$$

The solution is periodic only if $y_0 > 1$ and the period is

$$T_{2s;2} = \frac{4\pi}{c_0 \Delta_{cr,2}\sqrt{y_0^2 - 1}}. \tag{59}$$

As a conclusion, the BSR for $N=2$ provides periodic solutions. For $\Gamma = \pi$, $y_0 < 1$ we have AFR and the solution is not periodic.

The final remark in this section is that for $y_0 \to 0$ the solution (54) becomes singular and blows up periodically with

period $4\pi/(c_0\Delta_{\mathrm{cr},2})$. In this limit we have two "equal" solitons with amplitudes $\nu_j=\nu_0=1$ with phase difference π.

B. $N=3$ case

For the case of the three-soliton train with zero initial velocities the matrix L_0 has the form

$$L_0=\begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \end{pmatrix},\quad \mathrm{tr}\,L_0=0,$$

with

$$a_j=-\frac{i}{2}\exp\left(-\frac{\nu_0}{2\alpha}r_0-\frac{i}{2}\Gamma_j\right),\quad b_j=\frac{i}{4}d_j,$$

where

$$d_j=\frac{\nu_{j(0)}-\nu_0}{\nu_0},\quad r_0=\xi_{2(0)}-\xi_{1(0)}=\xi_{3(0)}-\xi_{2(0)},$$

$$\Gamma_j=\delta_{j+1(0)}-\delta_{j(0)}+4s_{j+1}+4s_j. \tag{60}$$

Then the characteristic equation takes the form

$$\zeta^3+p\zeta+q=0, \tag{61}$$

where

$$p=-\tfrac{1}{16}(d_1d_2+d_2d_3+d_1d_3)+\tfrac{1}{4}e^{-r_0\nu_0/\alpha}(e^{-i\Gamma_1}+e^{-i\Gamma_2}),$$

$$q=\frac{i}{64}d_1d_2d_3-\frac{i}{16}e^{-r_0\nu_0/\alpha}(d_1e^{-i\Gamma_2}+d_3e^{-i\Gamma_1}). \tag{62}$$

It is natural to make use of the well known Cardano formulas for solving cubic equations. We first consider the cases when p and q are real. The roots of Eq. (61) are given by

$$\zeta_1=A+B,\quad \zeta_2=\omega A+\omega^2 B,\quad \zeta_3=\omega^2 A+\omega B, \tag{63}$$

where

$$A=\sqrt[3]{-\frac{q}{2}+\sqrt{Q}},\quad B=\sqrt[3]{-\frac{q}{2}-\sqrt{Q}}, \tag{64}$$

$$Q=\frac{q^2}{4}+\frac{p^3}{27},\quad \omega=\exp\left(\frac{2\pi i}{3}\right).$$

If both p and q are real, then so is Q. Here we have four subcases corresponding to qualitatively different sets of roots for real p and q.

(i) $Q<0$. This is possible only if $p<p_{\mathrm{cr}}$, $p_{\mathrm{cr}}=-3(q^2/4)^{1/3}$. Then $A=B^*$ and all three roots ζ_j become real $\zeta_j=\kappa_j$ and pairwise different,

$$\kappa_1=2|A|\cos\Omega_0,\quad \kappa_{2,3}=2|A|\cos\left(\Omega_0\pm\frac{2\pi}{3}\right), \tag{65}$$

with $\Omega_0\neq 0,\pi$. Obviously this leads to AFR. If $\Omega_0=0$ or π then $\kappa_2=\kappa_3$ and a MR follows.

(ii) $Q>0$ and $q\neq 0$. Here both A and B are real and formula (63) shows that one root ζ_1 is real, while the other two are complex conjugate:

$$\mathrm{Re}\,\zeta_1=-2\,\mathrm{Re}\,\zeta_2=-2\,\mathrm{Re}\,\zeta_3,\quad\text{or}\quad \kappa_1=-2\kappa_2=-2\kappa_3, \tag{66}$$

which corresponds to a MR.

(iii) $Q>0$ and $q=0$. Now $p>0$, the cubic equation (61) simplifies and is trivially solved by

$$\zeta_1=0,\quad \zeta_{2,3}=\pm\sqrt{-p}. \tag{67}$$

All the roots have zero real parts that obviously corresponds to BSR.

(iv) $Q=0$. If p and q are nonzero, all the roots are real and pairwise different,

$$\zeta_1=3q/p,\quad \zeta_2=\zeta_3=-3q/2p,$$

we have AFR. If p and q are zero, we get a degenerate case with all three zero roots.

The symmetry in the eigenvalues leads also to a symmetry in the solutions of the CTC. Therefore, the configuration (67) corresponds to a particular type of BSR's. This is due to the fact that we restricted so far both q and p to be real. Of course this is not necessary; moreover, from Eq. (62) we see that generically both q and p are complex. If we want to specify the soliton parameters that are responsible for the BSR we may also use Viette formulas which show that the characteristic equation (61) will have purely imaginary roots if p is real and negative and q is purely imaginary. That is why we will consider also the configuration (v) below.

(v) $p=\bar{p},q=-\bar{q}$. In this case we have two qualitatively different possibilities depending on whether Q is positive or negative.

Note that since $q=-\bar{q}$ we should modify our reasoning as compared to the above analysis. Indeed, with $q=iq'$, q' real and $Q\geq 0$ we find that $A=-\bar{B}$. Therefore, from Eqs. (63) and (64) we have that all the roots ζ_k satisfy $\zeta_k=-\bar{\zeta}_k$, i.e., are purely imaginary and BSR takes place.

Analogously, if $Q<0$ then the roots ζ_k satisfy $\zeta_1=-\bar{\zeta}_1$ and $\zeta_3=-\bar{\zeta}_2$ which leads to AFR.

Hence, we revealed two possibilities to realize bound state regime: subcase (iii) and subcase (v) with $Q>0$.

Let us now briefly describe the sets of soliton parameters relevant to each of the regimes mentioned above. For definiteness we will use two configurations of soliton widths

$$d_1=-d_3,\quad d_2=0,\quad W1 \tag{68}$$

$$d_1=d_3,\quad d_2=-2d_1,\quad W2. \tag{69}$$

The condition that p is real immediately means that

$$\Gamma_1=-\Gamma_2\equiv\Phi. \tag{70}$$

Then

$$p = -\tfrac{1}{16}(d_1 d_2 + d_1 d_3 + d_2 d_3) + \tfrac{1}{2} e^{-\nu_0 r_0/\alpha} \cos \Phi, \quad (71)$$

$$q = \frac{i}{64} d_1 d_2 d_3 - \frac{i}{16} e^{-\nu_0 r_0/\alpha} (d_1 e^{i\Phi} + d_3 e^{-i\Phi}), \quad (72)$$

$$\Phi = \delta_2 - \delta_1 + 4s_1 + 4s_2. \quad (73)$$

Choosing the sets of widths to be $W1$ and $W2$ we get, respectively,

$$p^{(1)} = \frac{d_1^2}{16} + \frac{\epsilon_0^2}{2} \cos \Phi,$$

$$q^{(1)} = \frac{d_1 \epsilon_0^2}{8} \sin \Phi, \quad (74)$$

where $\epsilon_0 = \exp[-\nu_0 r_0/(2\alpha)]$, and

$$p^{(2)} = \frac{3d_1^2}{16} + \frac{\epsilon_0^2}{2} \cos \Phi,$$

$$q^{(2)} = -\frac{i d_1^3}{32} - \frac{i d_1 \epsilon_0^2}{8} \cos \Phi. \quad (75)$$

1. Case I: q=0

The characteristic equation (61) has the roots

$$\zeta_1 = 0, \quad \zeta_{2,3} = \pm \sqrt{-p}. \quad (76)$$

From Eq. (74) we get that for the $W1$ configuration the condition $q^{(1)} = 0$ holds provided

$$\Phi = k\pi, \quad k = 0,1, \quad (77)$$

which means that

$$p^{(1)} = \frac{d_1^2}{16} + (-1)^k \frac{\epsilon_0^2}{2}. \quad (78)$$

As a consequence we find that $p^{(1)} > 0$ for $k = 0$; for $k = 1$ we get that $p^{(1)} > 0$ only provided $|d_1|$ is greater than the critical value

$$|d_1| > \Delta_{cr,3}, \quad \Delta_{cr,3} = 2\sqrt{2} \epsilon_0. \quad (79)$$

In all these cases $\zeta_{2,3}$ are purely imaginary, i.e., these sets of parameters lead to BSR.

Note that Eq. (77) means

$$\delta_2 = \delta_1 + k\pi - 4s_1 - 4s_2, \quad k = 0,1. \quad (80)$$

If instead of Eq. (79) we have $|d_1| < \Delta_{cr,3}$ then $p^{(1)} < 0$ and the roots $\zeta_{2,3}$ become real. That means that taking d_1 below the critical value we will see a transition from BSR to AFR.

The same considerations applied to the $W2$ configuration lead to different results. From Eq. (75) we see that $q^{(2)} = 0$ holds if

$$\cos \Phi = -\frac{d_1^2}{4\epsilon_0^2}, \quad (81)$$

which implies that

$$|d_1| \leq 2\epsilon_0 = \frac{\Delta_{cr,3}}{\sqrt{2}} \quad (82)$$

and

$$p^{(2)} = \frac{d_1^2}{16} \geq 0. \quad (83)$$

Such configurations obviously lead to BSR. If $|d_1|$ is chosen to be greater than the critical value in the right-hand side of Eq. (82) we find that then $q^{(2)}$ becomes purely imaginary; such situation is considered below.

Let us briefly treat also the case of "equal" solitons, i.e., $d_j = 0$. Then obviously $q = 0$, $s_1 = s_2 = s_3 = \pi/8$, and $p = (\epsilon_0^2/2) \cos \Phi$. As a result we find that if

$$-\frac{\pi}{2} < \Phi < \frac{\pi}{2}, \quad \text{i.e.,} \quad \frac{\pi}{2} < \delta_2 - \delta_1 < \frac{3\pi}{2}, \quad (84)$$

then $p > 0$ and we have BSR; if

$$\frac{\pi}{2} < \Phi < \frac{3\pi}{2}, \quad \text{i.e.,} \quad -\frac{\pi}{2} < \delta_2 - \delta_1 < \frac{\pi}{2}, \quad (85)$$

then $p < 0$ and AFR follows.

2. Case II p=0

In this case the characteristic equation (61) has as roots

$$\zeta_k = \sqrt[3]{-q} \, \omega^k, \quad \omega = e^{2\pi i/3}, \quad k = 0,1,2. \quad (86)$$

If in addition q is real then Eq. (86) leads to a MR; otherwise we get AFR.

For the $W1$ configuration $p^{(1)} = 0$ means

$$\cos \Phi = -\frac{d_1^2}{(\Delta_{cr,3})^2}; \quad (87)$$

this is possible only if $|d_1| \leq \Delta_{cr,3}$. From Eq. (74) we get that $q^{(1)}$ is real and such configuration leads to MR.

For the $W2$ configuration $p^{(2)} = 0$ holds if

$$\cos \Phi = -\frac{3d_1^2}{8\epsilon_0^2} = -\frac{3d_1^2}{(\Delta_{cr,3})^2}, \quad (88)$$

which is possible only if $|d_1| \leq \Delta_{cr,3}/\sqrt{3}$. From Eq. (75) we find that $q^{(2)}$ is purely imaginary, i.e., AFR follows.

3. Case III: $p = \bar{p}$ and $q = -\bar{q} \neq 0$.

This is possible only for the $W2$ configuration, so p and q are given by Eq. (75). The resolvent of the cubic equation (61) in this case takes the form

$$Q = \frac{(p^{(2)})^3}{27} + \frac{(q^{(2)})^2}{4} = \frac{\epsilon_0^6}{8} \left[\left(y^2 + \frac{c}{3} \right)^3 - y^2 \left(y^2 + \frac{c}{2} \right)^2 \right]$$

$$= \frac{\epsilon_0^6}{8} \frac{c^2}{12} \left(y^2 + \frac{4c}{9} \right), \tag{89}$$

where $y = d_1/\Delta_{cr,3}$ and $c = \cos \Phi$.

It is easy to check that $Q(y,c)$ is non-negative for all $c > -9y^2/4$ and vanishes for $c = 0$ and $c = -9y^2/4$. We have to keep in mind also that $|c| \leq 1$. Therefore, if $9y^2/4 > 1$ then $Q \geq 0$ in the whole interval $-1 \leq c \leq 1$. Following the arguments in (v) above we conclude that this configurations leads to BSR.

If we choose

$$|d_1| < \tfrac{2}{3} \Delta_{cr,3}, \tag{90}$$

then there will be an interval for Φ (73),

$$\varphi_{cr} \leq \Phi \leq 2\pi - \varphi_{cr}, \quad \varphi_{cr} = \arccos \left(-\frac{9d_1^2}{4(\Delta_{cr,3})^2} \right), \tag{91}$$

for which $Q < 0$; i.e., if Eq. (91) holds we have AFR.

If Φ belongs to the complementary interval

$$-\varphi_{cr} \leq \Phi \leq \varphi_{cr}, \tag{92}$$

then $Q(y,c) \geq 0$ and we have BSR.

An interested reader can easily extend these studies to other relevant configurations of soliton parameters.

VI. THE CTC VERSUS NUMERICAL SOLUTIONS OF MNSE

It is our aim here to compare the predictions of the CTC model with the numerical solutions of the MNSE. Since the full numerical investigation of the problem is a voluminous and ambitious task we limit ourselves with $N = 2$ and $N = 3$ soliton trains and fix up $\alpha = 1$ and the average width $\nu_0 = 1$.

With this choice of $\alpha = 1$ the derivative term in the MNSE cannot be treated as a perturbation to the NSE. With this choice we are able to exhibit the differences between the MNSE and NSE N-soliton train interactions. As we mentioned above, the dependence of the soliton interaction of the MNSE solitons on the soliton phase difference is qualitatively different from the one of the NSE solitons.

Indeed, let us start with $N = 2$ soliton trains. The formulas from Sec. V A with $\alpha = 1$ and $\nu_0 = 1$ show that ''equal'' solitons (i.e., solitons with equal widths) with phase difference $\delta_2 - \delta_1 = \pi$ (or $\Gamma = 0$) attract each other. In fact, this choice of the soliton parameters corresponds to $y_0 = 0$ and according to Eqs. (54), (55) the solution to the CTC becomes singular. From Fig. 1 we see that apart from a small neighborhood around the singular points the CTC gives a good description of the two-soliton train of the MNSE; the singular points match rather well with the points at which the two solitons are closest to each other. The distance to the first singular points matches $T_{2s,1}/4$ with $T_{2s,1}$ given by formula (57) with $y_0 = 0$.
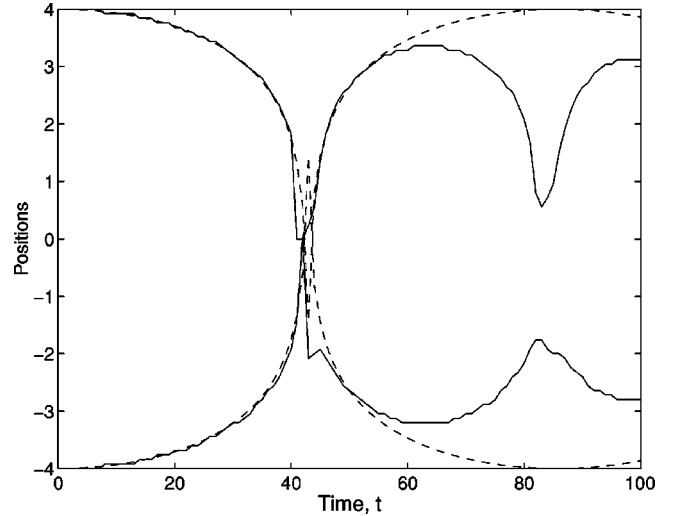

FIG. 1. Two-soliton interactions and their comparison with the CTC model. Solid curve, numerical results; dashed curve, predictions from the Toda chain model. $\nu_1 = \nu_2 = 1.0$, $\delta_1 = 0$, and $\delta_2 = \pi$.

Choosing the solitons to have different widths leads to $\gamma_{10} \neq 0$ in Eq. (56) and removes the singularity of the corresponding solution of the CTC system even if $\Gamma = 0$. This can be seen from Fig. 2 that corresponds to a BSR. Of course now the match between the MNSE simulation and the CTC solution is better than in the previous case.

The situation changes if we consider solitons with phase differences such that $\Gamma = \pi$. There we find a threshold value for $d_1 = -d_2 = (\nu_1 - \nu_0)/\nu_0$, see Eq. (53). Whenever $d_1 < \Delta_{cr,2}$ we get an AFR [see Fig. 3(a)] while for $d_1 > \Delta_{cr,2}$ we get an BSR [see Fig. 3(b)].

Let us now consider the three-soliton interactions. The choices of the soliton parameters illustrates each of the three main configurations outlined in Section V B above.
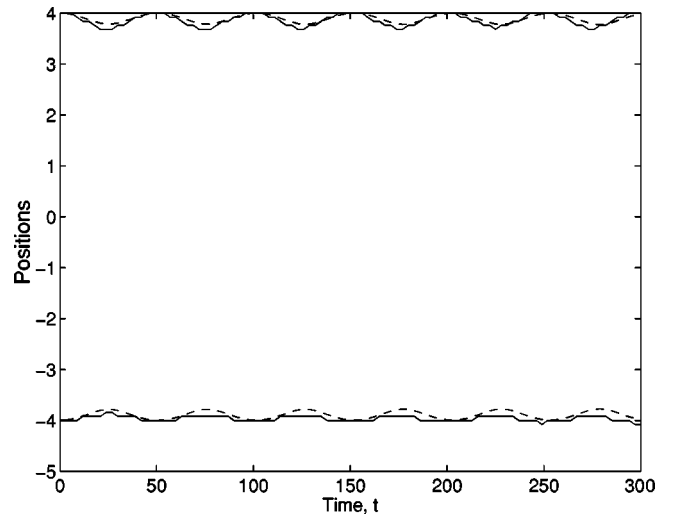
Figure 4 provides examples of three-soliton configura-

FIG. 2. Two-soliton interactions and their comparison with the CTC model. Solid curve, numerical results; dashed curve, predictions from the Toda chain model. $\nu_1 = 0.95$, $\nu_2 = 1.05$, $\delta_1 = 0$, and $\delta_2 = \pi$.
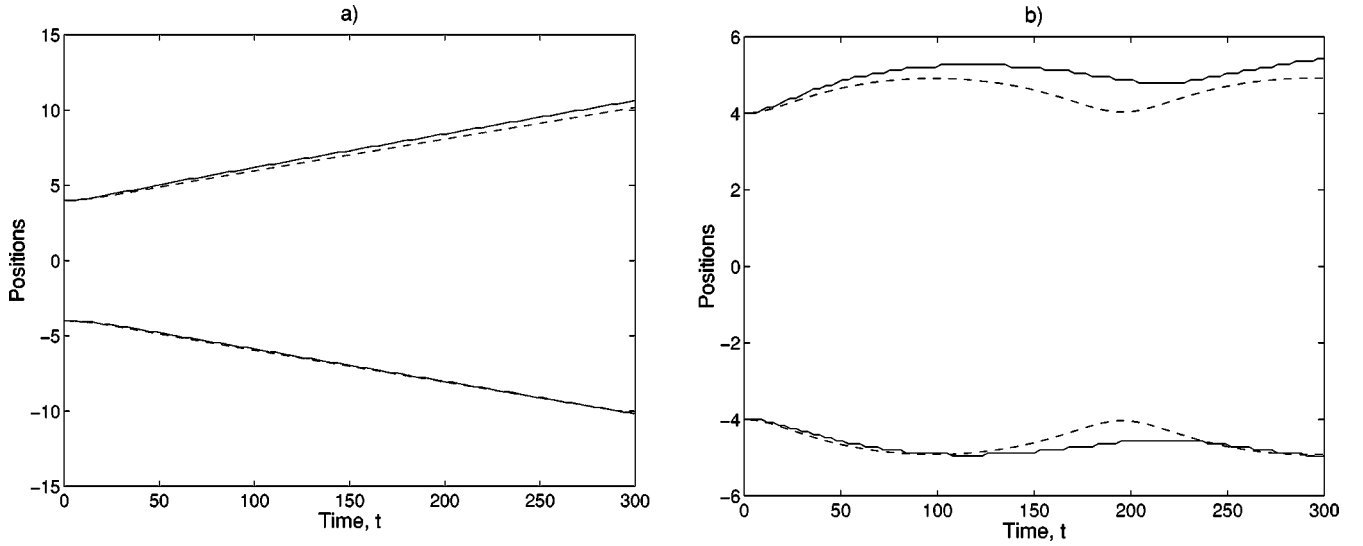
FIG. 3. Two-soliton interactions and their comparison with the CTC model. Solid curve, numerical results; dashed curve, predictions from the Toda chain model. (a) $\nu_1 = 0.97$, $\nu_2 = 1.03$, $\delta_1 = 0$, and $\delta_2 = 0$; (b) $\nu_1 = 0.96$, $\nu_2 = 1.04$, $\delta_1 = 0$, and $\delta_2 = 0$.

tions with $q = 0$ characteristic for case I. Both sets of parameters are such that $\Phi = \pi$. Besides on Fig. 4(a) we have $d_1 < \Delta_{cr,3}$ and as a consequence an AFR must follow. In Fig. 4(b) we have $d_1 > \Delta_{cr,3}$ for which the CTC model predicts a BSR; the match with the simulation here is not that good.

Figure 5 shows a three-soliton configurations with $p = 0$ characteristic for case II. In Fig 5(b) the set of widths is $W1$ and $d_1 < \Delta_{cr,3}$ and, therefore, a MR follows.

In Fig. 6 we used $W2$ set of soliton widths and a choice of parameters characteristic for case III, i.e., p is real while q is purely imaginary. In Fig. 6(a) $Q > 0$ with BSR, and in Fig. 6(b), we have $Q < 0$ and AFR.

VII. CONCLUSIONS

In this paper we extend the formalism by Karpman and Solov'ev proposed to describe the NSE two-soliton interac-

tion [9] and generalized to arbitrary number of solitons [5–8], to the case of the modified nonlinear Schrödinger equation. The aim of our paper was twofold. First, we would like to investigate a possibility to apply an integrable chain-like model to capture adiabatic dynamics of MNSE solitons within the N-soliton train. Because a functional form of the MNSE soliton is not of the familiar hyperbolic-secant type with a real argument, we might expect an existence of some important features as compared with the NSE case. We show that, under specific well-defined conditions, the dynamical system of $4N$ equations for soliton parameters is reduced to the completely integrable complex Toda chain model with N nodes. This is a strong argument in favor of universality of the CTC model for N-soliton interactions. Though the same CTC arises also for the NSE, there are a few peculiarities inherent in the MNSE solitons. In particular, we found out
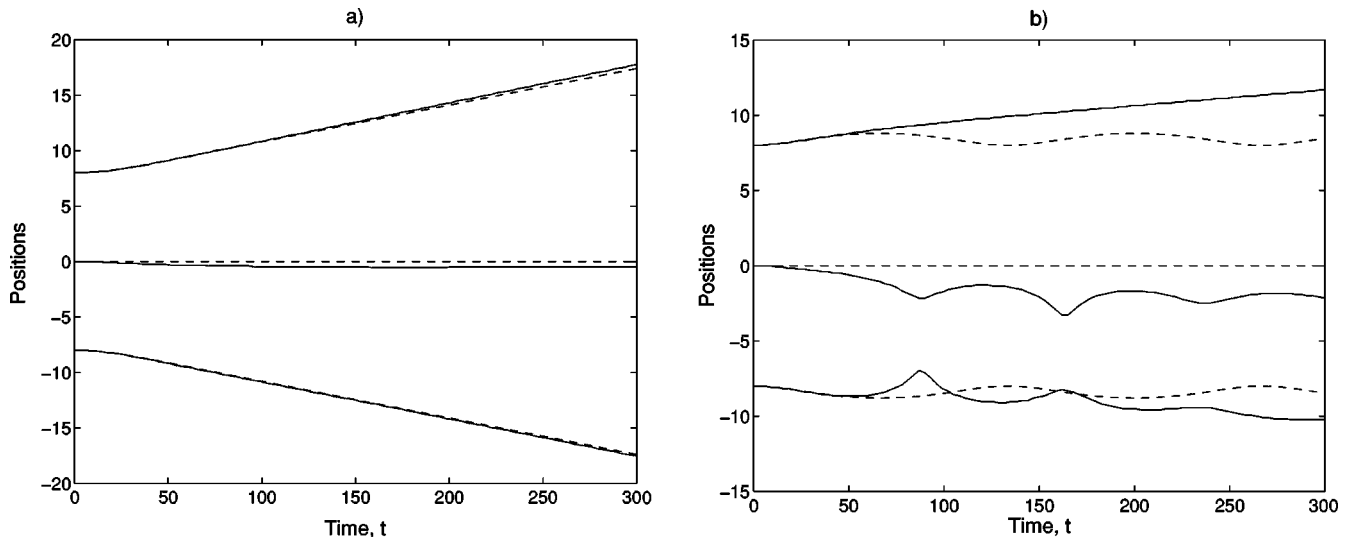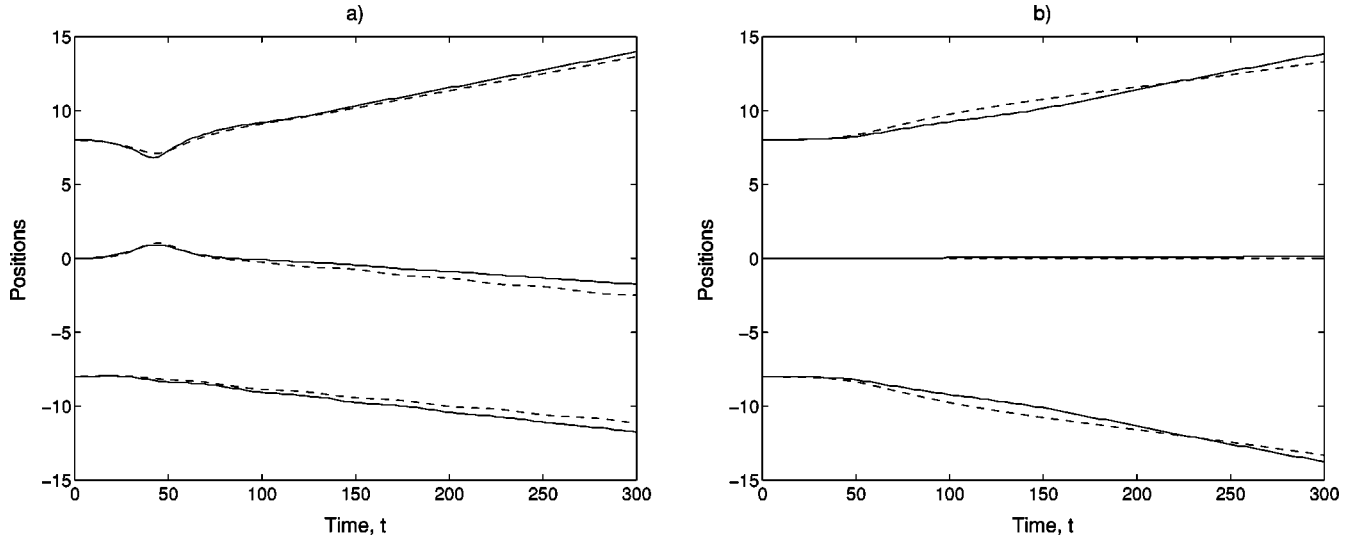

FIG. 4. Three-soliton interactions and their comparison with the CTC model. Solid curve, numerical results; dashed curve, predictions from the Toda chain model. (a) $\nu_1 = 1.04$, $\nu_2 = 1.0$, $\nu_3 = 0.96$, $\delta_1 = 0$, $\delta_2 = -0.0392$, and $\delta_3 = 0.0016$; (b) $\nu_1 = 1.07$, $\nu_2 = 1.0$, $\nu_3 = 0.93$, $\delta_1 = 0$, $\delta_2 = -0.0676$, and $\delta_3 = 0.0049$.

FIG. 5. Three-soliton interactions and their comparison with the CTC model. Solid curve, numerical results; dashed curve, predictions from the Toda chain model. (a) $\nu_1 = 1.04$, $\nu_2 = 1.0$, $\nu_3 = 0.96$, $\delta_1 = 0$, $\delta_2 = 2.1703$, and $\delta_3 = 0.0016$; (b) $\nu_1 = 1.02$, $\nu_2 = 0.96$, $\nu_3 = 1.02$, $\delta_1 = 0$, $\delta_2 = -1.0862$, and $\delta_3 = 0.0420$.

more complicated phase behavior of the *N*-soliton train. Using the integrability of the CTC, we are able to predict various asymptotic regimes of the MNSE *N*-soliton train evolution. Besides, we point out the sets of the initial soliton parameters corresponding to each of the dynamical regimes. Numerical simulations of the MNSE two- and three-soliton interactions are in very good agreement with the CTC-based predictions. Evidently, the results obtained can be extended to treat also multicomponent (vector) generalizations of both the NSE (see, e.g., Refs. [39–41] and references therein) and MNSE [42–44]. Work in this direction is now in progress. We note that in nonintegrable wave systems, Toda-chain type equations may still be derived for the adiabatic interaction of *N* nearly identical solitary waves, but such equations

are generally nonintegrable as well [10–12,40].

Second, we consider the MNSE as a true starting integrable model to describe subpicosecond pulse evolution in nonlinear media. Strictly speaking, to justify a relevance of our results to actual ultrashort pulses, we should also account in our model at least two additional effects, the third-order dispersion and intrapulse Raman scattering. These effects break the integrability of the MNSE, and we are faced with a truly perturbed CTC. Following the lines of recently established interrelations between the perturbed NSE and perturbed CTC [8], we can extend the above formalism to account for small actual perturbations that act along with the effective perturbation (14). The corresponding results will be published elsewhere. The single MNSE soliton dynamics in
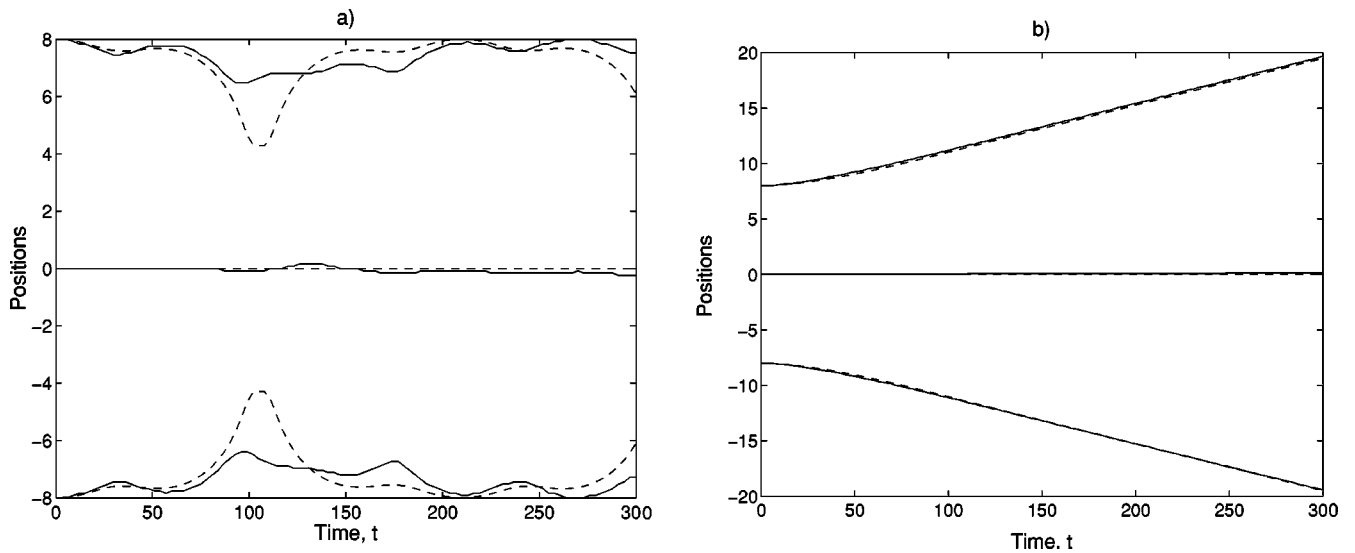

FIG. 6. Three-soliton interactions and their comparison with the CTC model. Solid curve, numerical results; dashed curve, predictions from the Toda chain model. (a) $\nu_1 = 1.02$, $\nu_2 = 0.96$, $\nu_3 = 1.02$, $\delta_1 = 0$, $\delta_2 = 3.142$, and $\delta_3 = 0.0420$, and (b) $\nu_1 = 1.02$, $\nu_2 = 0.96$, $\nu_3 = 1.02$, $\delta_1 = 0$, $\delta_2 = 0.0$, and $\delta_3 = 0.0420$.

the presence of the intrapulse Raman scattering is discussed in a recent paper [45].

The MNSE is not the only candidate to describe subpicosecond optical pulse dynamics. The other equations studied in this respect are of more general form

$$
i\frac{\partial q}{\partial T} + \frac{1}{2}\frac{\partial^2 q}{\partial X^2} + |q|^2 q + i\varepsilon\left(\beta_1\frac{\partial^3 q}{\partial X^3} + \beta_2|q|^2\frac{\partial q}{\partial X} + \beta_3 q\frac{\partial|q|^2}{\partial X}\right) = 0, \tag{93}
$$

due to the presence of third-order dispersion and additional types of nonlinearities. If we assume that $\beta_1:\beta_2:\beta_3 = 1:6:0$ we obtain Hirota equation [46] while for $\beta_1:\beta_2:\beta_3 = 1:6:3$ we get another integrable nonlinear equation known as the Sasa-Satsuma equation [47–49]. It is rather natural to study the N-soliton interactions also for these equations. Although it may seem that Eq. (93) is similar to Eq. (1), the method of solution is much more complicated. The corresponding Lax operator is provided by a 3 $\times 3$ matrix-valued operator with special $\mathbb{Z}_2$-symmetric potential. As a result the equation (93) like the sine-Gordon equation, has two types of soliton solutions: ''simple'' solitons and breathers. That is why the study of the soliton interactions of Eq. (93) requires substantial efforts and will be addressed in subsequent papers.

Recently we were informed [50] that the CTC model arises also in the case of the soliton-train propagation in a system governed by the classical Thirring model [51,52]. This seems natural in view of the facts that: (i) CTC describes the adiabatic soliton interactions for all nonlinear equations of the NSE hierarchy; (ii) the massive Thirring model is just another representative of the MNSE hierarchy.

There remain several natural questions that will be addressed in sequels of this paper. The first one is the limit $\alpha \to 0$ in which we should recover the results for the NSE N-soliton trains. We have proved that the Karpman-Solov'ev-like equations for MNSE N solitons transform under this limit to the known NSE formulas. The second one concerns the treatment of the perturbed versions of the MNSE and the corresponding perturbed CTC model; for the NSE such perturbed CTC models have been briefly analyzed in Ref. [8].

[1] G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic Press, San Diego, 1995).

[2] A. Hasegawa and Y. Kodama, *Solitons in Optical Communications*, 2nd ed. (Oxford University Press, Oxford, UK, 1995).

[3] S. P. Novikov, S. V. Manakov, L. P. Pitaevski, and V. E. Zakharov, *Theory of Solitons, the Inverse Scattering Method* (Consultant Bureau, New York, 1984).

[4] L. A. Takhtadjan and L. D. Faddeev, *Hamiltonian Approach to Soliton Theory* (Springer-Verlag, Berlin, 1986).

[5] V.S. Gerdjikov, D.J. Kaup, I.M. Uzunov, and E.G. Evstatiev, Phys. Rev. Lett. **77**, 3943 (1996).

[6] V.S. Gerdjikov, I.M. Uzunov, E.G. Evstatiev, and G.L. Diankov, Phys. Rev. E **55**, 6039 (1997).

[7] V.S. Gerdjikov, E.G. Evstatiev, D.J. Kaup, G.L. Diankov, and I.M. Uzunov, Phys. Lett. A **241**, 323 (1998).

[8] V.S. Gerdjikov and I.M. Uzunov, Physica D **152-153**, 355 (2001); e-print solv-int/0004031.

[9] V.I. Karpman and V.V. Solov'ev, Physica D **3**, 487 (1981).

[10] J. M Arnold, URL Proc. URSI Electromagnetic Theory Symposium (1995).

[11] J.M. Arnold, J. Opt. Soc. Am. A **15**, 1450 (1998).

[12] J.M. Arnold, Phys. Rev. E **60**, 979 (1999).

[13] D.J. Kaup, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **31**, 121 (1976).

[14] V. I. Karpman and E. M. Maslov, Zh. Éksp. Teor. Fiz **73**, 537 (1977) [Sov. Phys. JETP **46**, 281 (1977)].

[15] Y.S. Kivshar and B.A. Malomed, Rev. Mod. Phys. **61**, 768 (1989).

[16] M. Wadati, K. Konno, and Y.H. Ichikawa, J. Phys. Soc. Jpn. **46**, 1865 (1979).

[17] K. Ohkuma, Y.H. Ichikawa, and Y. Abe, Opt. Lett. **12**, 516 (1987).

[18] Y. Kodama, J. Stat. Phys. **39**, 597 (1985).

[19] Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. **QE-23**, 510 (1987).

[20] N. Tzoar and M. Jain, Phys. Rev. A **23**, 1266 (1981).

[21] D. Anderson and M. Lisak, Phys. Rev. A **27**, 1393 (1983).

[22] K. Mio, T. Ogino, K. Minami, and S. Takeda, J. Phys. Soc. Jpn. **41**, 265 (1976).

[23] K. Porsezian, K.M. Tamizhmani, and M. Lakshmanan, Phys. Lett. A **124**, 159 (1987).

[24] D.J. Kaup and A.C. Newell, J. Math. Phys. **19**, 798 (1978).

[25] D. Mihalache, N. Truta, N.-C. Panoiu, and D.-M. Baboiu, Phys. Rev. A **47**, 3190 (1993).

[26] V.S. Gerdjikov and M. Ivanov, Bulg. J. Phys. **10**, 13 (1983).

[27] V.S. Gerdjikov and M. Ivanov, Bulg. J. Phys. **10**, 130 (1983).

[28] V.A. Vysloukh and I.V. Cherednik, Theor. Math. Phys. **78**, 24 (1989).

[29] A. Rangwala and J.A. Rao, J. Math. Phys. **31**, 1126 (1990).

[30] Z.-Y. Chen and N.-N. Huang, Phys. Rev. A **41**, 4066 (1990).

[31] S.-L. Lin and W.-Z. Wang, Phys. Rev. E **48**, 3054 (1993).

[32] E.V. Doktorov and V.S. Shchesnovich, J. Math. Phys. **36**, 7009 (1995).

[33] V.S. Shchesnovich and E.V. Doktorov, Physica D **129**, 115 (1999).

[34] M. Toda, *Theory of Nonlinear Lattices* (Springer Verlag, Berlin, 1989).

[35] J. Moser, Lect. Notes Phys. **38**, 467 (1975).

[36] J. Moser, Adv. Math. **16**, 197 (1975).

[37] V.S. Gerdjikov, E.G. Evstatiev, and R.I. Ivanov, J. Phys. A **31**, 8221 (1998).

[38] V.S. Gerdjikov, E.G. Evstatiev, and R.I. Ivanov, J. Phys. A **33**, 975 (2000).

[39] J. Yang, Phys. Rev. E **59**, 2393 (1999).

[40] J. Yang, Phys. Rev. E **64**, 026607 (2001).

[41] V. S Gerdjikov, e-print nlin.SI/0009036.

[42] M. Hisakado, T. Iizuka, and M. Wadati, J. Phys. Soc. Jpn. **63**, 2887 (1994).

[43] M. Hisakado and M. Wadati, J. Phys. Soc. Jpn. **64**, 408 (1995).

[44] E.V. Doktorov, S.Y. Sakovich, and R.A. Vlasov, J. Phys. Soc. Jpn. **65**, 876 (1996).

[45] A.A. Afanas'ev, E.V. Doktorov, R.A. Vlasov, and V.M. Volkov, Opt. Commun. **153**, 83 (1998).

[46] R. Hirota, J. Math. Phys. **14**, 805 (1973).

[47] N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).

[48] D. Mihalache, L. Torner, F. Moldoveanu, N.-C. Panoiu, and N. Truta, J. Phys. A **26**, L757 (1993).

[49] D. Mihalache, L. Torner, F. Moldoveanu, N.-C. Panoiu, and N. Truta, Phys. Rev. E **48**, 4699 (1993).

[50] V. S. Shchesnovich (unpublished).

[51] W. E. Thirring, Ann. Phys. (N.Y.) **3**, 91 (1958).

[52] E.A. Kuznetsov and A.V. Mikhailov, Theor. Math. Phys. **30**, 193 (1977).