Direct perturbation theory for solitons of the derivative nonlinear Schrödinger equation and the modified nonlinear Schrödinger equation

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A direct perturbation theory for solitons of the derivative nonlinear Schrödinger (DNLS) equation is developed based on a closure of eigenfunctions of the linearized DNLS equation around a one-soliton solution. The slow evolution of soliton parameters and the perturbation-induced radiation are obtained. Under the known simple gaugelike transformation, these results are transformed into those for the perturbed modified nonlinear Schrödinger (MNLS) equation describing propagation of femtosecond pulses in optical fibers. A calculation of the perturbation-induced radiation fields for the perturbed DNLS and MNLS equations is also made. Our results for the perturbed MNLS equation can be reduced perfectly to those for the perturbed nonlinear Schrödinger equation in the small nonlinear-dispersion limit.

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I. INTRODUCTION

It is well known that exactly integrable nonlinear differential equations have soliton solutions that travel stationarily and collide elastically. Many wave propagation phenomena can be described by integrable equations in some ideal conditions. Several soliton perturbation theories have been developed to study the effects of small perturbations on integrable equations. In these theories, two categories are mathematically complete because they cannot only obtain the slow evolution of soliton parameters, but also the perturbation-induced radiation. The first one is the perturbation theory based on the inverse scattering transform (IST) [1,2], which has a recent development based on the Riemann-Hilbert problem [3,4]. The second one includes the Green's function perturbation theory [5] and the direct soliton perturbation theory, which are both based on a set of complete eigenfunctions of the linearized equation around a soliton solution. The direct method was further developed in recent years (see, e.g., Ref. [6] and references therein). Results of these two categories of perturbation theories are consistent in general.

The idea of the direct method is very simple: just to expand the perturbed equation around the soliton solution and solve the resulted linear inhomogeneous equations at various orders. Usually, the expansion is considered up to the first-order correction only. If one can find a complete set of eigenfunctions for the homogeneous equation, the first-order correction can be expanded in this complete set. Expansion coefficients can be found by solving some ordinary differential equations. The coefficients for the discrete eigenfunctions would grow secularly and should be suppressed. Suppression of the secular terms leads to the slow evolution equations for soliton parameters. Then only the terms of continuous eigenfunctions appear in the first-order correction. These continuous eigenfunctions are energy radiation from solitons. In general, the direct method is independent of the IST even though connections to IST scattering are still visible and sometimes results of the IST are helpful to find the eigenfunctions. The direct method is simpler to apply. For example, the direct perturbation theory for dark solitons was developed [7] recently where the theory based on IST encountered a difficulty caused by the varying background wave [8]. The direct theory has the further advantage that it can be applied to nonintegrable equations [9], thus it has a wider range of applications.

The key of the direct method is to find a complete set of eigenfunctions of the linearized equation. It is known that derivatives of the soliton solution with respect to soliton parameters are discrete eigenfunctions. But one still needs to know continuous eigenfunctions in order to construct a complete set. A complete set can be constructed by the so-called ‘‘squared Jost solutions’’ [10], or by directly solving the eigenvalue problem [11], or by finding the discrete eigenfunctions using the derivatives of the soliton solution with respect to soliton parameters and then guessing and testing the continuous eigenfunctions. But no applicable general method is available. However, it has been shown recently that, at least for the Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS), and modified Korteweg–de Vries (mKdV) hierarchies, the linearization operators of all equations in the same integrable hierarchy share the same complete set of eigenfunctions [6], that is, whenever a complete set of eigenfunctions for one equation is found it can also be used to develop direct perturbation theories for the rest of the equations in the same hierarchy. This idea is very helpful in finding the eigenfunctions of the linearized derivative NLS (DNLS) equation presented in this paper.

The propagation of picosecond solitons in single-mode nonlinear fibers is well described by the NLS equation. The effects of perturbations such as fiber loss, higher-order dispersion, self-induced Raman scattering, and the nonlinear dispersion (self-steepening) can be studied by perturbation theories. But for femtosecond pulses, the effects of the nonlinear dispersion are so significant that it can no longer be

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treated as a perturbation [12,13]. However, the equation including the nonlinear dispersion [the third term on the left hand side of Eq. (1)], the so-called modified nonlinear Schrödinger (MNLS) equation

$$i\partial_t v+\partial_x^2 v+i\alpha\partial_x(|v|^2 v)+2\beta|v|^2 v=0 \tag{1}$$

is also integrable [14]. A Painlevé analysis suggests that two other models containing both the third-order dispersion term and the nonlinear dispersion term are integrable [15]. For temporal pulses in optical fibers, v is the field amplitude, t is the propagating distance, x is the time measured in a frame moving with the group velocity, α and β are constants representing the relative magnitudes of the nonlinear dispersion term and the nonlinear term. The effects of nonlinear dispersion can be studied exactly while the effects of other perturbations should be studied based on a perturbation theory for the MNLS equation. Such a theory was first developed by Shchesnovich and Doktorov based on the Riemann-Hilbert problem [4]. However, they only obtained the evolution of soliton parameters. To our knowledge, the problem of perturbation-induced radiation has never been solved for the perturbed MNLS soliton. The aim of this paper is not only to provide a direct perturbation theory for the perturbed MNLS equation, but also to solve the problem of perturbation-induced radiation for perturbed MNLS solitons.

It is known that under a simple gaugelike transformation (see, e.g., [16]), Eq. (1) becomes the DNLS equation

$$i\partial_t u+\partial_x^2 u+i\partial_x(|u|^2 u)=0, \tag{2}$$

which is relevant to Alven waves in plasmas [17]. We find that not only the perturbed MNLS equation, but also the linearized equation for the perturbed MNLS equation around a soliton solution can be transformed into that for the DNLS equation with an effective perturbation. Therefore, we can develop a direct perturbation theory for both the DNLS and MNLS equations, beginning with the simpler DNLS equation. However, eigenfunctions for the linearized DNLS equation have never been found in the literature before. The derivatives of the soliton solution with respect to its parameters show that the discrete eigenfunctions are much more complicated than those in the NLS, KdV, and mKdV hierarchies. So, to construct the eigenfunctions with the squared Jost solutions, using the results of IST for the DNLS equation [18] might be a better choice, but we immediately find that the squared Jost solutions are not eigenfunctions. However, similarity between the KdV hierarchy and the DNLS hierarchy [19] suggests that eigenfunctions for the linearized DNLS equation are possibly derivatives of the squared Jost solutions with respect to x, just as the KdV hierarchy. We confirmed this using the Lax pair of the DNLS equation. We also find that the adjoint eigenfunctions are squared Jost solutions with flipped components, similar to the KdV hierarchy. The completeness of these eigenfunctions and adjoint eigenfunctions can be verified directly by complex integration. We note that Gerdjikov, Ivanov, and Kulish also derived a closure relation for squared Jost solutions in [19]. But that closure relation is not the one we need in a direct perturbation theory. After the complete set of eigenfunctions is obtained, we then develop a direct perturbation theory for the DNLS equation by the usual procedure of direct perturbation theories (see, e.g., Ref. [20]). This theory gives the slow evolution of soliton parameters and the evolution of the perturbation-induced radiation. These results are then transformed to the perturbed MNLS equation, in which the evolution of the soliton center and the phase slightly differ from those in Ref. [4]. However, in the small nonlinear dispersion limit, our results for the evolution of soliton parameters can be perfectly reduced to the well-known results for the perturbed NLS equation [1,2]. This fact supports the validity of our results, not Shchesnovich and Doktorov's [4]. Finally, as an example of our theory, we study the slow evolution of a damped MNLS soliton and find that the decay of its amplitude is a little slower than the exponential decay of a NLS soliton, but its energy decay rate is the same as that of a NLS soliton.

II. THE PERTURBED DNLS EQUATION AND ITS LINEARIZATION OPERATOR

The perturbed DNLS equation is

$$i\partial_t u+\partial_x^2 u+i\partial_x(|u|^2 u)=ir(u), \tag{3}$$

where $r(u)$ is the perturbation function, which can be written as $r(u)=\epsilon p(u)$ [ϵ is a small positive parameter representing the amplitude of perturbation $r(u)$]. We expand u around a DNLS soliton solution u_s up to $O(\epsilon)$,

$$u=u_s+\epsilon q, \tag{4}$$

where the one-soliton solution is [16]

$$u_s(x,t)=-4\Delta\sin\gamma e^{2\theta}e^{-i2\varphi}\frac{e^{4\theta}+e^{i\gamma}}{(e^{4\theta}+e^{-i\gamma})^2}, \tag{5}$$

which can be rewritten as

$$u_s(x,t)=-4\eta\frac{\overline{\zeta_1}e^{2\theta}+\zeta_1 e^{-2\theta}}{(\zeta_1 e^{2\theta}+\overline{\zeta_1}e^{-2\theta})^2}e^{-i2(\varphi-3/4\gamma)}. \tag{6}$$

Here $\zeta_1=\Delta\exp(i\gamma/2)$ $(0<\gamma<\pi)$ is the discrete eigenvalue, the bar denotes complex conjugate, and

$$\xi=\mathrm{Re}(\zeta_1^2)=\Delta^2\cos\gamma,\quad \eta=\mathrm{Im}(\zeta_1^2)=\Delta^2\sin\gamma, \tag{7}$$

$$\theta=\eta(x-\hat{x}),\quad \hat{x}=-4\xi t+x_0, \tag{8}$$

$$\varphi=\xi(x-\hat{x})+\hat{\varphi},\quad \hat{\varphi}=-2(\xi^2+\eta^2)t+\varphi_0. \tag{9}$$

There are four independent soliton parameters, ξ, η, x_0, and φ_0. In the presence of perturbation, they will evolve slowly with the "slow time" $\tau=\epsilon t$. In the two-time scale, ∂_t becomes $\partial_t+\epsilon\partial_\tau$. We use a trick proposed in Ref. [20] of replacing t in $\hat{x}$ and $\hat{\varphi}$ to make them only depend on the slow time τ,

$$\hat{x}=-4\xi\tau\epsilon^{-1}+x_0, \tag{10}$$

$$\hat{\varphi}=-2(\xi^2+\eta^2)\tau\epsilon^{-1}+\varphi_0, \tag{11}$$

and expand them as

$$\hat{x}=\hat{x}_{-1}\epsilon^{-1}+x_0+\hat{x}_1\epsilon^1+\cdots, \tag{12}$$

$$\hat{\varphi}=\hat{\varphi}_{-1}\epsilon^{-1}+\varphi_0+\hat{\varphi}_1\epsilon^1+\cdots. \tag{13}$$

Substituting all of the above expansions into Eq. (3), in ϵ^0 order, we have

$$\partial_\tau\hat{x}_{-1}=-4\xi, \tag{14}$$

$$\partial_\tau\hat{\varphi}_{-1}=-2(\xi^2+\eta^2). \tag{15}$$

In ϵ^1 order, combining its complex conjugate, we get a linearized vector equation

$$(\partial_t-\boldsymbol{L})\boldsymbol{q}=\boldsymbol{p}(u_s)-\boldsymbol{s}(u_s), \tag{16}$$

in which $\boldsymbol{L}$ is a 2×2 differential operator,

$$\boldsymbol{L}=\begin{pmatrix} i\partial_x^2-2|u_s|^2\partial_x-2\partial_x(|u_s|^2) & -u_s^2\partial_x-\partial_x(u_s^2) \\ -\overline{u_s^2}\partial_x-\partial_x(\overline{u_s^2}) & -i\partial_x^2-2|u_s|^2\partial_x-2\partial_x(|u_s|^2) \end{pmatrix}, \tag{17}$$

and

$$\boldsymbol{q}=(q\ \ \overline{q})^T, \quad \boldsymbol{p}(u_s)=(p(u_s)\overline{p(u_s)})^T, \tag{18}$$

$$\boldsymbol{s}(u_s)=\frac{\partial\boldsymbol{u}_s}{\partial\xi}\frac{d\xi}{d\tau}+\frac{\partial\boldsymbol{u}_s}{\partial\eta}\frac{d\eta}{d\tau}+\frac{\partial\boldsymbol{u}_s}{\partial x_0}\frac{dx_0}{d\tau}+\frac{\partial\boldsymbol{u}_s}{\partial\varphi_0}\frac{d\varphi_0}{d\tau}, \tag{19}$$

$$\boldsymbol{u}_s=(u_s\ \ \overline{u_s})^T. \tag{20}$$

Here the superscript "T" represents the transpose of a matrix. In the linear space of solutions for $\boldsymbol{L}$, we define the inner product between functions $\boldsymbol{F}$ and $\boldsymbol{G}$ as

$$\langle\boldsymbol{G}|\boldsymbol{F}\rangle=\int_{-\infty}^{+\infty}dx\boldsymbol{G}^T\boldsymbol{F}, \tag{21}$$

and define $\boldsymbol{L}^A$—the adjoint of $\boldsymbol{L}$, as

$$\langle\boldsymbol{G}|\boldsymbol{L}\boldsymbol{F}\rangle=\langle\boldsymbol{L}^A\boldsymbol{G}|\boldsymbol{F}\rangle+i(\boldsymbol{G}^T\boldsymbol{\sigma}_3\boldsymbol{F}_x-\boldsymbol{G}_x^T\boldsymbol{\sigma}_3\boldsymbol{F})|_{-\infty}^{+\infty}. \tag{22}$$

Here

$$\boldsymbol{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \boldsymbol{\sigma}_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \boldsymbol{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

are Pauli spin matrices. According to these definitions, we find that

$$\boldsymbol{L}^A=\begin{pmatrix} i\partial_{xx}+2|u_s|^2\partial_x & \overline{u_s^2}\partial_x \\ u_s^2\partial_x & -i\partial_{xx}+2|u_s|^2\partial_x \end{pmatrix}. \tag{23}$$

The key to solving the inhomogeneous equation (16) is to find a complete set of eigenfunctions for $\boldsymbol{L}$. It is not difficult to show that for any Jost solution

$$w=(w_1\ \ w_2)^T \tag{24}$$

satisfying the Lax equations (A1), the corresponding squared Jost solution

$$\boldsymbol{W}=w\bigcirc w=(w_1^2\ \ w_2^2)^T \tag{25}$$

satisfies equations

$$(\partial_t-\boldsymbol{L})\boldsymbol{W}_x=0 \tag{26}$$

and

$$(\partial_t+\boldsymbol{L}^A)\boldsymbol{\sigma}_1\boldsymbol{W}=0. \tag{27}$$

This is similar to the KdV hierarchy [6]. Thus we can construct eigenfunctions of $\boldsymbol{L}$ and $\boldsymbol{L}^A$ with squared Jost solutions, i.e.,

$$\boldsymbol{L}\Psi(x,t;\zeta)=\partial_t\Psi(x,t;\zeta), \tag{28a}$$

$$\boldsymbol{L}\tilde{\Psi}(x,t;\zeta)=\partial_t\tilde{\Psi}(x,t;\zeta), \tag{28b}$$

$$\boldsymbol{L}^A\Phi(x,t;\zeta)=-\partial_t\Phi(x,t;\zeta), \tag{29a}$$

$$\boldsymbol{L}^A\tilde{\Phi}(x,t;\zeta)=-\partial_t\tilde{\Phi}(x,t;\zeta), \tag{29b}$$

where

$$\Psi(x,t;\zeta)=\partial_x(\psi(x,t;\zeta)\psi(x,t;\zeta)),$$

$$\tilde{\Psi}(x,t;\zeta)=\partial_x(\tilde{\psi}(x,t;\zeta)\tilde{\psi}(x,t;\zeta)), \tag{30a}$$

$$\Phi(x,t;\zeta)=\boldsymbol{\sigma}_1\phi(x,t;\zeta)\phi(x,t;\zeta),$$

$$\tilde{\Phi}(x,t;\zeta)=\boldsymbol{\sigma}_1\tilde{\phi}(x,t;\zeta)\tilde{\phi}(x,t;\zeta). \tag{30b}$$

Here definitions for Jost solutions $\psi(x,t;\zeta)$, $\phi(x,t;\zeta)$, $\tilde{\psi}(x,t;\zeta)$, and $\tilde{\phi}(x,t;\zeta)$ and their explicit expressions for one-soliton potential are given in Appendixes A and B. Although the above results are presented by expanding the perturbed DNLS equation around a soliton solution, actually they are valid for expansions around any arbitrary solution of the DNLS equation.

III. EIGENFUNCTIONS FOR THE ONE-SOLITON CASE

For one-soliton case, it is more convenient to choose θ and t as independent variables instead of x and t. Then we have

$$\partial_t \to \partial_t + 4\xi\eta\partial_\theta, \quad \partial_x \to \eta\partial_\theta, \quad \partial_x^2 \to \eta^2\partial_\theta^2. \tag{31}$$

Variables in the eigenfunctions can be separated as

$$\Psi(x,t;\zeta) = e^{i2\zeta^2\hat{x} + i4\zeta^4 t - i2\hat{\varphi}} e^{-i2\sigma_3\hat{\varphi}} \boldsymbol{\Psi}(\theta,\zeta), \tag{32a}$$

$$\tilde{\Psi}(x,t;\zeta) = e^{-i2\zeta^2\hat{x} - i4\zeta^4 t + i2\hat{\varphi}} e^{-i2\sigma_3\hat{\varphi}} \tilde{\boldsymbol{\Psi}}(\theta,\zeta), \tag{32b}$$

$$\Phi(x,t;\zeta) = e^{-i2\zeta^2\hat{x} - i4\zeta^4 t + i2\hat{\varphi}} e^{i2\sigma_3\hat{\varphi}} \boldsymbol{\Phi}(\theta,\zeta), \tag{33a}$$

$$\tilde{\Phi}(x,t;\zeta) = e^{i2\zeta^2\hat{x} + i4\zeta^4 t - i2\hat{\varphi}} e^{i2\sigma_3\hat{\varphi}} \tilde{\boldsymbol{\Phi}}(\theta,\zeta), \tag{33b}$$

where explicit expressions for $\boldsymbol{\Psi}(\theta,\zeta)$, $\tilde{\boldsymbol{\Psi}}(\theta,\zeta)$, $\boldsymbol{\Phi}(\theta,\zeta)$, and $\tilde{\boldsymbol{\Psi}}(\theta,\zeta)$ can be found in Appendix C. When these expressions are substituted into Eqs. (28) and (29), we obtain the following familiar eigenvalue equations:

$$\mathcal{L}\boldsymbol{\Psi}(\theta,\zeta) = i4(\zeta^2 - \zeta_1^2)(\zeta^2 - \overline{\zeta_1^2})\boldsymbol{\Psi}(\theta,\zeta), \tag{34a}$$

$$\mathcal{L}\tilde{\boldsymbol{\Psi}}(\theta,\zeta) = -i4(\zeta^2 - \zeta_1^2)(\zeta^2 - \overline{\zeta_1^2})\tilde{\boldsymbol{\Psi}}(\theta,\zeta), \tag{34b}$$

$$\mathcal{L}^A\boldsymbol{\Phi}(\theta,\zeta) = i4(\zeta^2 - \zeta_1^2)(\zeta^2 - \overline{\zeta_1^2})\boldsymbol{\Phi}(\theta,\zeta), \tag{35a}$$

$$\mathcal{L}^A\tilde{\boldsymbol{\Phi}}(\theta,\zeta) = -i4(\zeta^2 - \zeta_1^2)(\zeta^2 - \overline{\zeta_1^2})\tilde{\boldsymbol{\Phi}}(\theta,\zeta). \tag{35b}$$

Here

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\theta,\zeta) = e^{i2\sigma_3\hat{\varphi}} \boldsymbol{L} e^{-i2\sigma_3\hat{\varphi}} - 4\xi\eta\partial_\theta I - i4\Delta^4\sigma_3 \\ &= \begin{pmatrix} i\eta^2\partial_\theta^2 - 2\eta[|u_0(\theta)|^2 + 2\xi]\partial_\theta - 2\eta\dfrac{d|u_0(\theta)|^2}{d\theta} - i4\Delta^4 & -\eta u_0^2(\theta)\partial_\theta - \eta\dfrac{du_0^2(\theta)}{d\theta} \\ -\eta\overline{u_0^2(\theta)}\partial_\theta - \eta\dfrac{d\overline{u_0^2(\theta)}}{d\theta} & -i\eta^2\partial_\theta^2 - 2\eta[|u_0(\theta)|^2 + 2\xi]\partial_\theta^2 - 2\eta\dfrac{d|u_0(\theta)|^2}{d\theta} + i4\Delta^4 \end{pmatrix},\end{aligned} \tag{36}$$

I is the unit matrix

$$u_0(\theta) = u_s e^{i2\hat{\varphi}}, \tag{37}$$

and the adjoint operator $\mathcal{L}^A$ is

$$\begin{aligned}\mathcal{L}^A &= \mathcal{L}^A(\theta,\zeta) = e^{-i2\sigma_3\hat{\varphi}} \boldsymbol{L}^A e^{i2\sigma_3\hat{\varphi}} + 4\xi\eta\partial_\theta I - i4\Delta^4\sigma_3 \\ &= \begin{pmatrix} i\eta^2\partial_\theta^2 + 2\eta[|u_0(\theta)|^2 + 2\xi]\partial_\theta - i4\Delta^4 & \eta\overline{u_0^2(\theta)}\partial_\theta \\ \eta u_0^2(\theta)\partial_\theta & -i\eta^2\partial_\theta^2 + 2\eta[|u_0(\theta)|^2 + 2\xi]\partial_\theta + i4\Delta^4 \end{pmatrix}.\end{aligned} \tag{38}$$

In this case, the first-order equation (16) becomes

$$(\partial_t - \mathcal{L})\boldsymbol{Q} = \boldsymbol{P} - \boldsymbol{S}, \tag{39}$$

where

$$\boldsymbol{Q} = e^{i2\sigma_3\hat{\varphi}}\boldsymbol{q}, \quad \boldsymbol{P} = e^{i2\sigma_3\hat{\varphi}}\boldsymbol{p}, \quad \boldsymbol{S} = e^{i2\sigma_3\hat{\varphi}}\boldsymbol{s}. \tag{40}$$

Also, Eq. (22) becomes

$$\begin{aligned}\langle \boldsymbol{G}|\mathcal{L}\boldsymbol{F}\rangle &= \langle \mathcal{L}^A\boldsymbol{G}|\boldsymbol{F}\rangle + i\eta^2(\boldsymbol{G}^T\sigma_3\boldsymbol{F}_\theta - \boldsymbol{G}_\theta^T\sigma_3\boldsymbol{F})\big|_{-\infty}^{+\infty} \\ &\quad - 4\xi\eta\boldsymbol{G}^T\boldsymbol{F}\big|_{-\infty}^{+\infty}.\end{aligned} \tag{41}$$

In Eq. (41), choosing $\boldsymbol{F} = \boldsymbol{\Psi}(\theta,\zeta)$, $\boldsymbol{G} = \boldsymbol{\Phi}(\theta,\zeta')$ and $\boldsymbol{F} = \tilde{\boldsymbol{\Psi}}(\theta,\zeta)$, $\boldsymbol{G} = \tilde{\boldsymbol{\Phi}}(\theta,\zeta')$, respectively, we find the orthogonalities of the continuous eigenfunctions to be

$$\langle \boldsymbol{\Phi}(\zeta')|\boldsymbol{\Psi}(\zeta)\rangle = i2\pi\zeta^2 a^2(\zeta)\,\delta(\zeta^2 - \zeta'^2), \tag{42a}$$

$$\langle \tilde{\boldsymbol{\Phi}}(\zeta')|\tilde{\boldsymbol{\Psi}}(\zeta)\rangle = -i2\pi\zeta^2 \tilde{a}^2(\zeta)\,\delta(\zeta^2 - \zeta'^2). \tag{42b}$$

Nonzero inner products between discrete eigenfunctions are

$$\langle \dot{\boldsymbol{\Phi}}(\zeta_1)|\boldsymbol{\Psi}(\zeta_1)\rangle = \langle \boldsymbol{\Phi}(\zeta_1)|\dot{\boldsymbol{\Psi}}(\zeta_1)\rangle = -\frac{\zeta_1}{2}\,\dot{a}^2(\zeta_1), \tag{43a}$$

$$\langle \dot{\tilde{\boldsymbol{\Phi}}}(\overline{\zeta_1})|\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})\rangle = \langle \tilde{\boldsymbol{\Phi}}(\overline{\zeta_1})|\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\zeta_1})\rangle = \frac{\overline{\zeta_1}}{2}\,\dot{\tilde{a}}^2(\overline{\zeta_1}), \tag{43b}$$

$$\langle \dot{\boldsymbol{\Phi}}(\zeta_1)|\dot{\boldsymbol{\Psi}}(\zeta_1)\rangle = -\tfrac{1}{2}\dot{a}^2(\zeta_1) - \tfrac{1}{2}\zeta_1\dot{a}(\zeta_1)\ddot{a}(\zeta_1), \tag{44a}$$

$$\langle \dot{\tilde{\boldsymbol{\Phi}}}(\overline{\zeta_1})|\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\zeta_1})\rangle = \tfrac{1}{2}\dot{\tilde{a}}^2(\overline{\zeta_1}) + \tfrac{1}{2}\zeta_1\dot{\tilde{a}}(\overline{\zeta_1})\ddot{\tilde{a}}(\overline{\zeta_1}). \tag{44b}$$

Here the dot denotes the derivative with respect to ζ. Explicit expressions for $a(\zeta)$, $\tilde{a}(\zeta)$ and for eigenfunctions can be found in Appendixes B and C.

IV. CLOSURE OF THE SET OF EIGENFUNCTIONS

The closure relation of the set of eigenfunctions is

$$\begin{aligned}
I=&\frac{2[\dot{a}(\zeta_1)+\zeta_1\ddot{a}(\zeta_1)]}{\zeta_1^2\dot{a}^3(\zeta_1)}|\boldsymbol{\Psi}(\zeta_1)\rangle\langle\boldsymbol{\Phi}(\zeta_1)|\\
&-\frac{2}{\zeta_1\dot{a}^2(\zeta_1)}[|\boldsymbol{\Psi}(\zeta_1)\rangle\langle\dot{\boldsymbol{\Phi}}(\zeta_1)|+|\dot{\boldsymbol{\Psi}}(\zeta_1)\rangle\langle\boldsymbol{\Phi}(\zeta_1)|]\\
&-\frac{2[\dot{\tilde{a}}(\overline{\zeta_1})+\overline{\zeta_1}\ddot{\tilde{a}}(\overline{\zeta_1})]}{\overline{\zeta_1}^2\dot{\tilde{a}}^3(\overline{\zeta_1})}|\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})\rangle\langle\tilde{\boldsymbol{\Phi}}(\overline{\zeta_1})|\\
&+\frac{2}{\overline{\zeta_1}\dot{\tilde{a}}^2(\overline{\zeta_1})}[|\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})\rangle\langle\dot{\tilde{\boldsymbol{\Phi}}}(\overline{\zeta_1})|+|\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\zeta_1})\rangle\langle\tilde{\boldsymbol{\Phi}}(\overline{\zeta_1})|]\\
&+\frac{1}{\pi i}\int_\Gamma d\zeta\frac{|\boldsymbol{\Psi}(\zeta)\rangle\langle\boldsymbol{\Phi}(\zeta)|}{\zeta a^2(\zeta)}\\
&-\frac{1}{\pi i}\int_{\tilde{\Gamma}} d\zeta\frac{|\tilde{\boldsymbol{\Psi}}(\zeta)\langle\tilde{\boldsymbol{\Phi}}(\zeta)|}{\zeta\tilde{a}^2(\zeta)},
\end{aligned}\tag{45}$$

or

$$\begin{aligned}
\delta(x-y)=&\frac{2[\dot{a}(\zeta_1)+\zeta_1\ddot{a}(\zeta_1)]}{\zeta_1^2\dot{a}^3(\zeta_1)}\boldsymbol{\Psi}(x,\zeta_1)\boldsymbol{\Phi}^T(y,\zeta_1)\\
&-\frac{2}{\zeta_1\dot{a}^2(\zeta_1)}[\boldsymbol{\Psi}(x,\zeta_1)\dot{\boldsymbol{\Phi}}^T(y,\zeta_1)\\
&+\dot{\boldsymbol{\Psi}}(x,\zeta_1)\boldsymbol{\Phi}^T(y,\zeta_1)]\\
&-\frac{2[\dot{\tilde{a}}(\overline{\zeta_1})+\overline{\zeta_1}\ddot{\tilde{a}}(\overline{\zeta_1})]}{\overline{\zeta_1}^2\dot{\tilde{a}}^3(\overline{\zeta_1})}\tilde{\boldsymbol{\Psi}}(x,\overline{\zeta_1})\tilde{\boldsymbol{\Phi}}^T(y,\overline{\zeta_1})\\
&+\frac{2}{\overline{\zeta_1}\dot{\tilde{a}}^2(\overline{\zeta_1})}[\tilde{\boldsymbol{\Psi}}(x,\overline{\zeta_1})\dot{\tilde{\boldsymbol{\Phi}}}^T(y,\overline{\zeta_1})\\
&+\dot{\tilde{\boldsymbol{\Psi}}}(x,\overline{\zeta_1})\tilde{\boldsymbol{\Phi}}^T(y,\overline{\zeta_1})]\\
&+\frac{1}{\pi i}\int_\Gamma d\zeta\frac{\boldsymbol{\Psi}(x,\zeta)\boldsymbol{\Phi}^T(y,\zeta)}{\zeta a^2(\zeta)}\\
&-\frac{1}{\pi i}\int_{\tilde{\Gamma}} d\zeta\frac{\tilde{\boldsymbol{\Psi}}(x,\zeta)\tilde{\boldsymbol{\Phi}}^T(y,\zeta)}{\zeta\tilde{a}^2(\zeta)},
\end{aligned}\tag{46}$$

where I is a unit matrix, Γ is a path consisting of a line from $i\infty$ to 0 and a line from 0 to ∞ in the first quadrant of the ζ plane, while $\tilde{\Gamma}$ is a path consisting of a line from $i\infty$ to 0 and a line from 0 to $-\infty$ in the second quadrant. Equation (46) can be directly proved by performing complex integrations. Therefore, any solution of Eq. (39) can be expanded in this complete set.

V. EVOLUTION OF THE SOLITON PARAMETERS

Having obtained a complete set of eigenfunctions for the linearization operator $\mathcal{L}$, we can expand the first-order correction $\boldsymbol{Q}$ into it. In the expansion, the terms of discrete eigenfunctions are secular and thus should be suppressed, yielding the secularity conditions

$$\langle\boldsymbol{\Phi}(\zeta_1)|(\boldsymbol{P}-\boldsymbol{S})\rangle=0,\tag{47a}$$

$$\langle\dot{\boldsymbol{\Phi}}(\zeta_1)|(\boldsymbol{P}-\boldsymbol{S})\rangle=0,\tag{47b}$$

$$\langle\tilde{\boldsymbol{\Phi}}(\overline{\zeta_1})|(\boldsymbol{P}-\boldsymbol{S})\rangle=0,\tag{48a}$$

$$\langle\dot{\tilde{\boldsymbol{\Phi}}}(\overline{\zeta_1})|(\boldsymbol{P}-\boldsymbol{S})\rangle=0.\tag{48b}$$

Due to Eqs. (C13) and (C15), it is obvious that Eq. (48) is just the complex conjugate of Eq. (47). As usual [5], $\boldsymbol{S}$ can be expressed as a linear combination of the discrete eigenfunctions (see Appendix D),

$$\begin{aligned}
\boldsymbol{S}=&i2\sin^2\gamma\{-[e^{-i\gamma}\boldsymbol{\Psi}(\zeta_1)-e^{i\gamma}\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})]+\Delta[e^{-i\gamma/2}\dot{\boldsymbol{\Psi}}(\zeta_1)\\
&-e^{i\gamma/2}\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\zeta_1})]\}\frac{d\Delta}{d\tau}+\Delta\sin\gamma\{i2[e^{-i2\gamma}\boldsymbol{\Psi}(\zeta_1)\\
&-e^{i2\gamma}\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})]-\Delta\sin\gamma[e^{-i\gamma/2}\dot{\boldsymbol{\Psi}}(\zeta_1)+e^{i\gamma/2}\dot{\tilde{\boldsymbol{\Psi}}}(\overline{\zeta_1})]\}\frac{d\gamma}{d\tau}\\
&+4\Delta^3\sin^2\gamma[\boldsymbol{\Psi}(\zeta_1)+\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})]\frac{dx_0}{d\tau}\\
&-4\Delta\sin^2\gamma[e^{-i\gamma}\boldsymbol{\Psi}(\zeta_1)+e^{i\gamma}\tilde{\boldsymbol{\Psi}}(\overline{\zeta_1})]\frac{d\varphi_0}{d\tau}.
\end{aligned}\tag{49}$$

Hence,

$$\langle\boldsymbol{\Phi}(\zeta_1)|\boldsymbol{S}\rangle=ie^{-i7/2\gamma}\frac{d\zeta_1}{d\tau},\tag{50}$$

$$\begin{aligned}
\langle\dot{\boldsymbol{\Phi}}(\zeta_1)|\boldsymbol{S}\rangle=e^{-i(7/2)\gamma}\Bigg[&-\frac{1}{\eta}\frac{d\xi}{d\tau}-\frac{i}{4}\frac{1}{\xi^2+\eta^2}\frac{d(\xi^2+\eta^2)}{d\tau}\\
&+2(\xi+i\eta)\frac{dx_0}{d\tau}-2\frac{d\varphi_0}{d\tau}\Bigg],
\end{aligned}\tag{51}$$

$$\epsilon\langle\boldsymbol{\Phi}(\zeta_1)|\boldsymbol{P}\rangle=e^{-i(7/2)\gamma}\zeta_1^2\int_{-\infty}^{+\infty}d\theta e^{2\theta}D(\theta)R_+(\theta,t),\tag{52}$$

$$\epsilon\langle\dot{\mathbf{\Phi}}(\zeta_1)|\boldsymbol{P}\rangle=-ie^{-i(7/2)\gamma}\frac{4\zeta_1^3}{\eta}\int_{-\infty}^{+\infty}d\theta\,\theta e^{2\theta}D(\theta)R_-(\theta,t)+ie^{-i(7/2)\gamma}\frac{2\zeta_1}{\eta}\int_{-\infty}^{+\infty}d\theta[\xi e^{2\theta}r_0(\theta,t)-|\zeta_1|^2e^{-2\theta}\overline{r_0(-\theta,t)}]D(\theta). \tag{53}$$

Here

$$r_0(\theta,t)=e^{i2[\varphi-(3/4)\gamma]}r(u_s), \tag{54}$$

$$R_\pm(\theta,t)=r_0(\theta,t)\pm\overline{r_0(-\theta,t)}, \tag{55}$$

$$D(\theta)=\frac{1}{(\overline{\zeta_1}e^{2\theta}+\zeta_1e^{-2\theta})^2}. \tag{56}$$

Substituting the above expressions into Eq. (48) and after some simple algebra, we find the following evolution equations for a DNLS soliton's parameters:

$$\frac{d\xi}{dt}=-i\int_{-\infty}^{+\infty}d\theta(\zeta_1^3e^{2\theta}-\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_+(\theta,t), \tag{57}$$

$$\frac{d\eta}{dt}=-\int_{-\infty}^{+\infty}d\theta(\zeta_1^3e^{2\theta}\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_+(\theta,t), \tag{58}$$

$$\frac{dx_0}{dt}=-\frac{1}{\eta^2}\int_{-\infty}^{+\infty}d\theta\theta(\zeta_1^3e^{2\theta}+\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_-(\theta,t)+\frac{i}{4\eta}\int_{-\infty}^{+\infty}d\theta(\zeta_1e^{2\theta}+\overline{\zeta_1}e^{-2\theta})D(\theta)R_-(\theta,t), \tag{59}$$

$$\frac{d\left(\varphi_0-\frac{3}{4}\gamma\right)}{dt}=\xi\frac{dx_0}{dt}+i\frac{1}{\eta}\int_{-\infty}^{+\infty}d\theta\theta(\zeta_1^3e^{2\theta}-\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_-(\theta,t)-i\frac{\xi}{4\eta}\int_{-\infty}^{+\infty}d\theta(\zeta_1e^{2\theta}+\overline{\zeta_1}e^{-2\theta})D(\theta)R_-(\theta,t)-i\frac{|\zeta_1|^2}{4\eta}\int_{-\infty}^{+\infty}d\theta(\overline{\zeta_1}e^{2\theta}+\zeta_1e^{-2\theta})D(\theta)R_-(\theta,t). \tag{60}$$

VI. THE PERTURBATION-INDUCED RADIATION

Upon suppression of the discrete terms, the first-order solution $\boldsymbol{Q}$ becomes

$$|\boldsymbol{Q}(\theta,t)\rangle=\int_\Gamma d\zeta f(t,\zeta)|\mathbf{\Psi}(\theta,\zeta)\rangle+\int_{\tilde{\Gamma}}d\zeta\tilde{f}(t,\zeta)|\tilde{\mathbf{\Psi}}(\theta,\zeta)\rangle, \tag{61}$$

which contains only energy radiation. Substituting it into Eq. (39), letting $\langle\mathbf{\Phi}(\zeta)|$ and $\langle\tilde{\mathbf{\Phi}}(\zeta)|$ act on both sides of the equation, we get ordinary differential equations for the expansion coefficients f and $\tilde{f}$ as

$$\partial_tf(t,\zeta)-i4(\zeta^2-\zeta_1^2)(\zeta^2-\overline{\zeta_1^2})f(t,\zeta)=-i\frac{\langle\mathbf{\Phi}(\zeta)|\boldsymbol{P}(t)\rangle}{2\pi\zeta a^2(\zeta)}, \tag{62a}$$

$$\partial_t\tilde{f}(t,\zeta)+i4(\zeta^2-\zeta_1^2)(\zeta^2-\overline{\zeta_1^2})\tilde{f}(t,\zeta)=i\frac{\langle\tilde{\mathbf{\Phi}}(\zeta)|\boldsymbol{P}(t)\rangle}{2\pi\zeta\tilde{a}^2(\zeta)}. \tag{62b}$$

As

$$\langle\tilde{\mathbf{\Phi}}(\zeta)|\boldsymbol{P}(t)\rangle=\overline{\langle\mathbf{\Phi}(\overline{\zeta})|\boldsymbol{P}(t)\rangle}, \tag{63}$$

it is obvious that $\tilde{f}(t,\zeta)=\overline{f(t,\overline{\zeta})}$. With initial conditions $f(0,\zeta)=0$, solution of $f(t,\zeta)$ is

$$f(t,\zeta)=-i\frac{e^{i4(\zeta^2-\zeta_1^2)(\zeta^2-\overline{\zeta_1^2})t}}{2\pi\zeta a^2(\zeta)}\times\int_0^tdt'e^{-i4(\zeta^2-\zeta_1^2)(\zeta^2-\overline{\zeta_1^2})t'}\langle\mathbf{\Phi}(\zeta)|\boldsymbol{P}(t')\rangle. \tag{64}$$

Thus, we have obtained the perturbation-induced radiation from the first row of Eq. (61), which can be rewritten as an integral of $\lambda=\zeta^2$ because the eigenfunctions are functions of λ,

$$\epsilon q(\theta,t)=e^{-i2\hat{\varphi}}\int_{-\infty+i0^+}^{+\infty+i0^+}d\lambda[C(\lambda,t)\mathbf{\Psi}_1(\theta,\lambda)+\overline{C(\overline{\lambda},t)\mathbf{\Psi}_2(\theta,\overline{\lambda})}]. \tag{65}$$

Here

$$\mathbf{\Psi}(\theta,\lambda)=(\mathbf{\Psi}_1(\theta,\lambda)\mathbf{\Psi}_2(\theta,\lambda))^T, \tag{66}$$

$$C(\lambda,t)=-i\frac{e^{i4(\lambda-\lambda_1)(\lambda-\overline{\lambda_1})t}}{4\pi\lambda a^2(\lambda)}\int_0^tdt'e^{-i4(\lambda-\lambda_1)(\lambda-\overline{\lambda_1})t'}\times\langle\mathbf{\Phi}(\lambda)|\boldsymbol{R}(t')\rangle, \tag{67}$$

$\boldsymbol{R}=\epsilon\boldsymbol{P}$, $\lambda_1=\zeta_1^2$, and

$$
\begin{aligned}
\langle \mathbf{\Phi}(\lambda)|\boldsymbol{R}(t)\rangle = & \frac{\Delta^2}{\lambda}\int_{-\infty}^{+\infty} d\theta D(\theta) e^{i2(\xi/\eta)\theta} \\
& \times\{e^{-i(5/2)\gamma}e^{2\theta}[\lambda_1 e^{-i2(\lambda/\eta)\theta-2\theta}r_0(\theta,t) \\
& +\lambda e^{i2(\lambda/\eta)\theta+2\theta}\overline{r_0(-\theta,t)}]-2e^{-i2\gamma}a(\lambda) \\
& \times[\lambda_1 e^{-i2(\lambda/\eta)\theta-i\gamma/2}r_0(\theta,t) \\
& -\lambda e^{i2(\lambda/\eta)\theta+i\gamma/2}\overline{r_0(-\theta,t)}] \\
& +e^{-i(3/2)\gamma}a^2(\lambda)e^{-2\theta} \\
& \times[\lambda_1 e^{-i2(\lambda/\eta)\theta+2\theta-i\gamma}r_0(\theta,t) \\
& +\lambda e^{i2(\lambda/\eta)\theta-2\theta+i\gamma}\overline{r_0(-\theta,t)}]\}. \qquad (68)
\end{aligned}
$$

When $r_0(\theta,t)$ is a function of θ only [see Eq. (54)], $\boldsymbol{R}$ is independent of t, thus radiation formula (65) can be simplified as

$$
\begin{aligned}
\epsilon q(\theta,t) = & \frac{e^{-i2\hat{\varphi}}}{16\pi}\int_{-\infty+i0^+}^{+\infty+i0^+}\frac{d\lambda}{\lambda(\lambda-\lambda_1)(\lambda-\overline{\lambda_1})} \\
& \times\left[\frac{\langle \mathbf{\Phi}(\lambda)|\boldsymbol{R}\rangle}{a^2(\lambda)}\mathbf{\Psi}_1(\theta,\lambda)+\frac{\overline{\langle \mathbf{\Phi}(\overline{\lambda})|\boldsymbol{R}\rangle}}{\overline{a^2(\overline{\lambda})}}\overline{\mathbf{\Psi}_2(\theta,\overline{\lambda})}\right]. \qquad (69)
\end{aligned}
$$

VII. THE PERTURBED MNLS EQUATION

In this section, we transform the previous results for the perturbed DNLS equation to those for the perturbed MNLS equation. Considering a perturbed MNLS equation

$$i\partial_t v+\partial_x^2 v+i\alpha\partial_x(|v|^2 v)+2\beta|v|^2 v = ir(v), \qquad (70)$$

where the perturbation function can also be written as $r(v) = \epsilon p(v)$. When $\epsilon=0$, its one-soliton solution is

$$v_s(x,t) = -4\eta\frac{\overline{\zeta_1}e^{2\theta}+\zeta_1 e^{-2\theta}}{(\zeta_1 e^{2\theta}+\overline{\zeta_1}e^{-2\theta})^2}e^{-i2\varphi}. \qquad (71)$$

Here

$$\theta(x,t) = \eta(\alpha x-\alpha\hat{x}), \quad \hat{x} = -4\alpha(\xi-\rho)t+x_0, \qquad (72)$$

$$\varphi(x,t) = (\xi/\eta)\theta+\hat{\varphi}(t)-\alpha\rho x+2\alpha^2\rho^2 t,$$

$$\hat{\varphi}(t) = -2\alpha^2(\xi^2+\eta^2)t+\varphi_0, \qquad (73)$$

and $\xi=\mathrm{Re}(\zeta_1^2)$, $\eta=\mathrm{Im}(\zeta_1^2)$, and $\rho=\alpha^{-2}\beta$. Around this soliton solution, Eq. (70) can be linearized as

$$[\partial_t-\boldsymbol{L}^M]\boldsymbol{q}^M = \boldsymbol{p}(v_s)-\boldsymbol{s}(v_s) \qquad (74)$$

in which

$$\boldsymbol{L}^M = \begin{pmatrix} i\partial_x^2-2\alpha|v_s|^2\partial_x-2\alpha\partial_x(|v_s|^2)+i4\beta|v_s|^2 & -\alpha v_s^2\partial_x-\alpha\partial(v_s^2)+i2\beta v_s^2 \\ -\alpha\overline{v_s^2}\partial_x-\alpha\partial_x(\overline{v_s^2})-i2\beta\overline{v_s^2} & -i\partial_x^2-2\alpha|v_s|^2\partial_x-2\alpha\partial_x(|v_s|^2)-i4\beta|v_s|^2 \end{pmatrix}, \qquad (75)$$

$$\boldsymbol{q}^M = (q \ \ \overline{q})^T, \quad \boldsymbol{p}(v_s) = (p(v_s) \ \ \overline{p(v_s)})^T, \qquad (76)$$

$$\boldsymbol{s}(v_s) = \frac{\partial \boldsymbol{v}_s}{\partial\xi}\frac{d\xi}{d\tau}+\frac{\partial \boldsymbol{v}_s}{\partial\eta}\frac{d\eta}{d\tau}+\frac{\partial \boldsymbol{v}_s}{\partial x_0}\frac{dx_0}{d\tau}+\frac{\partial \boldsymbol{v}_s}{\partial\varphi_0}\frac{d\varphi_0}{d\tau}, \qquad (77)$$

$$\boldsymbol{v}_s = (v_s\overline{v_s})^T. \qquad (78)$$

Under a gaugelike transformation,

$$v(x,t) = u(X,T)e^{i2\rho X+i4\rho^2 T}, q^M(x,t) = q^D(X,T)e^{i2\rho X+i4\rho^2 T}, \qquad (79)$$

with

$$x = \alpha^{-1}(X+4\rho T), \quad t = \alpha^{-2}T, \qquad (80)$$

$$\partial_t = \alpha^2\partial_T-4\beta\partial_X, \quad \partial_x = \alpha\partial_X, \qquad (81)$$

the perturbed MNLS equation (70) becomes an effective perturbed DNLS equation

$$i\partial_T u+\partial_X^2 u+i\partial_X(|u|^2 u) = i\epsilon p^D, \qquad (82)$$

with an effective perturbation

$$p^D = \alpha^{-2}e^{-i2\rho X-i4\rho^2 T}p[u(X,T)e^{i2\rho X+i4\rho^2 T}]. \qquad (83)$$

For simplicity, we consider the case where the perturbation function $p(v)$ is of the form

$$p(v) = \sum_{k=0}^{n} p_k(|v|^2)\partial_x^k v, \qquad (84)$$

where $p_k(k=1,\ldots,n)$ are complex functions. This form covers almost all physical perturbations in optical fibers such as fiber loss, higher-order dispersion, and Raman effect [12,13]. Then it is clear that p^D is a function of $u(X,T)$ only. The one-soliton solution (71) becomes

$$u_s(X,T)=-4\eta\frac{\overline{\zeta_1}e^{2\theta}+\zeta_1e^{-2\theta}}{(\zeta_1e^{2\theta}+\overline{\zeta_1}e^{-2\theta})^2}e^{-i2\varphi^D}, \tag{85}$$

where

$$\theta(X,T)=\eta(X-\hat{X}),\quad \hat{X}=-4\xi T+X_0,\quad X_0=\alpha x_0, \tag{86}$$

and

$$\varphi^D(X,T)=\xi/\eta\theta+\hat{\varphi}(T),\quad \hat{\varphi}(T)=-2(\xi^2+\eta^2)T+\varphi_0. \tag{87}$$

The linearized equation (74) also becomes that for the DNLS equation,

$$[\partial_T-\boldsymbol{L}]\boldsymbol{q}^D=\boldsymbol{p}^D(u_s)-\boldsymbol{s}(u_s), \tag{88}$$

where

$$\boldsymbol{s}(u_s)=\frac{\partial\boldsymbol{u}_s}{\partial\xi}\frac{d\xi}{dT_1}+\frac{\partial\boldsymbol{u}_s}{\partial\eta}\frac{d\eta}{dT_1}+\frac{\partial\boldsymbol{u}_s}{\partial X_0}\frac{dX_0}{dT_1}+\frac{\partial\boldsymbol{u}_s}{\partial\varphi_0}\frac{d\varphi_0}{dT_1}$$
$$=\alpha^{-2}e^{-i2\rho X-i4\rho^2T}\boldsymbol{s}(v_s), \tag{89}$$

and $T_1=\epsilon T$. We can obtain soliton and radiation evolution equations for this effective perturbed DNLS soliton using results in the precoding two sections. Notice that the term $\frac{3}{4}\gamma$ in the phase of Eq. (6) is absorbed into φ^D in Eq. (85), and the effective perturbation $r^D(u_s)=\epsilon p^D(u_s)$ just shows up in the form of $r_0^D(\theta,T)$, where

$$r_0^D(\theta,T)=e^{i2\varphi^D}r^D(u_s)=\frac{1}{\alpha^2}r_0(\theta,T), \tag{90}$$

and $r_0(\theta,T)=e^{i2\varphi}r(v_s)$. Then by simply substituting Eq. (90) into Eqs. (57)–(60) for the effective perturbed DNLS soliton and letting $T=\alpha^2t$, we will obtain the following equations for the perturbed MNLS soliton's parameters:

$$\frac{d\xi}{dt}=-i\int_{-\infty}^{+\infty}d\theta(\zeta_1^3e^{2\theta}-\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_+(\theta,t), \tag{91}$$

$$\frac{d\eta}{dt}=-\int_{-\infty}^{+\infty}d\theta(\zeta_1^3e^{2\theta}+\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_+(\theta,t), \tag{92}$$

$$\frac{dx_0}{dt}=-\frac{1}{\alpha\eta^2}\int_{-\infty}^{+\infty}d\theta\theta(\zeta_1^3e^{2\theta}+\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_-(\theta,t)$$
$$+\frac{i}{4\alpha\eta}\int_{-\infty}^{+\infty}d\theta(\zeta_1e^{2\theta}+\overline{\zeta_1}e^{-2\theta})D(\theta)R_-(\theta,t), \tag{93}$$

$$\frac{d\varphi_0}{dt}=\alpha\xi\frac{dx_0}{dt}+i\frac{1}{\eta}\int_{-\infty}^{+\infty}d\theta\theta(\zeta_1^3e^{2\theta}-\overline{\zeta_1^3}e^{-2\theta})D(\theta)R_-(\theta,t)$$
$$-i\frac{\xi}{4\eta}\int_{-\infty}^{+\infty}d\theta(\zeta_1e^{2\theta}+\overline{\zeta_1}e^{-2\theta})D(\theta)R_-(\theta,t)$$
$$-i\frac{|\zeta_1|^2}{4\eta}\int_{-\infty}^{+\infty}d\theta(\overline{\zeta_1}e^{2\theta}+\zeta_1e^{-2\theta})D(\theta)R_-(\theta,t). \tag{94}$$

Here $R_\pm(\theta,t)$ and $D(\theta)$ are defined in Eqs. (55) and (56). Further, we obtain the perturbation-induced radiation as

$$\epsilon q^M(\theta,t)=e^{i2(\beta/\alpha)x-i4(\beta^2/\alpha^2)t-i2\hat{\varphi}}$$
$$\times\int_{-\infty+i0^+}^{+\infty+i0^+}d\lambda[C^M(\lambda,t)\boldsymbol{\Psi}_1(\theta,\lambda)$$
$$+\overline{C^M(\overline{\lambda},t)\boldsymbol{\Psi}_2(\theta,\overline{\lambda})}] \tag{95}$$

in which

$$C^M(\lambda,t)=-i\frac{e^{i4\alpha^2(\lambda-\lambda_1)(\lambda-\overline{\lambda_1})t}}{4\pi\lambda a^2(\lambda)}\int_0^t dt'\,e^{-i4\alpha^2(\lambda-\lambda_1)(\lambda-\overline{\lambda_1})t'}$$
$$\times\langle\boldsymbol{\Phi}(\lambda)|\boldsymbol{R}(t')\rangle. \tag{96}$$

When $r_0(\theta,t)$ is a function of θ only, which is true for the class of perturbations (84), radiation expression (95) can be rewritten as

$$\epsilon q^M(\theta,t)=\frac{e^{i2(\beta/\alpha)x-i4(\beta^2/\alpha^2)t-i2\hat{\varphi}}}{16\pi}$$
$$\times\int_{-\infty+i0^+}^{+\infty+i0^+}\frac{d\lambda}{\lambda(\lambda-\lambda_1)(\lambda-\overline{\lambda_1})}$$
$$\times\left[\frac{\langle\boldsymbol{\Phi}(\lambda)|\boldsymbol{R}\rangle}{a^2(\lambda)}\boldsymbol{\Psi}_1(\theta,\lambda)+\frac{\overline{\langle\boldsymbol{\Phi}(\overline{\lambda})|\boldsymbol{R}\rangle}}{\overline{a^2(\overline{\lambda})}}\overline{\boldsymbol{\Psi}_2(\theta,\overline{\lambda})}\right]. \tag{97}$$

When $\alpha\to0$, the perturbed MNLS equation (70) reduces to the well-known perturbed NLS equation. In this limit, our soliton evolution and radiation results above for the perturbed MNLS equation (70) must reduce to the well-known counterpart results for the perturbed NLS equation. This is indeed the case. Notice that as $\alpha\to0$,

$$\xi\to\frac{\mu}{\alpha}+\frac{\beta}{\alpha^2},\quad \eta\to\frac{\nu}{\alpha}, \tag{98}$$

$$\Delta\to\frac{\sqrt{\beta}}{\alpha},\quad \tan\gamma\to\frac{\alpha\nu}{\beta}, \tag{99}$$

and the MNLS soliton reduces to a NLS soliton,

$$v_s=-2\beta^{-1/2}\nu\,\mathrm{sech}(2\theta)e^{-i2\varphi} \tag{100}$$

with

$$\theta=\nu(x-\hat{x}), \quad \hat{x}=-4\mu t+x_0, \tag{101}$$

$$\varphi=\mu(x-\hat{x})+\hat{\varphi}, \quad \hat{\varphi}=-2(\mu^2+\nu^2)t+\varphi_0. \tag{102}$$

In this limit, evolution equations (91)–(94) of the MNLS soliton parameters also reduce to those of the NLS soliton [1,2],

$$\frac{d\mu}{dt}=\beta^{1/2}\int_{-\infty}^{+\infty}d\theta\tanh(2\theta)\mathrm{sech}(2\theta)\mathrm{Im}[e^{i2\varphi}r(v_s)], \tag{103}$$

$$\frac{d\nu}{dt}=-\beta^{1/2}\int_{-\infty}^{+\infty}d\theta\mathrm{sech}(2\theta)\mathrm{Re}[e^{i2\varphi}r(v_s)], \tag{104}$$

$$\frac{dx_0}{dt}=-\frac{\beta^{1/2}}{\nu^2}\int_{-\infty}^{+\infty}d\theta\,\theta\mathrm{sech}(2\theta)\mathrm{Re}[e^{i2\varphi}r(v_s)], \tag{105}$$

$$\frac{d\varphi_0}{dt}=\mu\frac{dx_0}{dt}-\frac{\beta^{1/2}}{\nu}\int_{-\infty}^{+\infty}d\theta\theta\tanh(2\theta)\mathrm{sech}(2\theta)\times\mathrm{Im}[e^{i2\varphi}r(v_s)]+\frac{\beta^{1/2}}{2\nu}\times\int_{-\infty}^{+\infty}d\theta\mathrm{sech}(2\theta)\mathrm{Im}[e^{i2\varphi}r(v_s)]. \tag{106}$$

We notice that our evolution equations (91) and (92) are equivalent to those obtained by Shchesnovich and Doktorov in Ref. [4]. But Eqs. (93) and (94), the evolution of soliton center and phase, are different from those in Ref. [4]. These differences are actually too small to be verified numerically. However, the fact that our equations (91)–(94) can be reduced perfectly to those of the perturbed NLS soliton in the limit $\alpha\to0$ indicates that our equations (93) and (94) are correct, not Shchesnovich and Doktorov's.

VIII. EXAMPLE: A DAMPED MNLS SOLITON

As an example of the present theory, we consider the simplest example, a MNLS soliton under damping, $r=-\Gamma v_s$, corresponding to the fiber loss in an optical fiber, where t is the propagating distance and x is the time measured in the frame moving with the group velocity of the soliton [12,13]. For modern typical fibers, the damping rate Γ generated from fiber loss is so small that r can be treated as a perturbation. In this case,

$$r_0(\theta)=e^{i\hat{\varphi}}r(v_s)=4\eta\Gamma\frac{\overline{\zeta_1}e^{2\theta}+\zeta_1e^{-2\theta}}{(\zeta_1e^{2\theta}+\overline{\zeta_1}e^{-2\theta})^2}, \tag{107}$$

$$R_+(\theta)=2r_0(\theta), \quad R_-(\theta)=0. \tag{108}$$

Substituting all of the above into Eqs. (91)–(94) leads to evolution equations of this damped MNLS soliton's parameters:

$$\frac{d\xi}{dt}=-2\Gamma\left[\frac{(\xi^2-\eta^2)}{\eta}\gamma-\xi\right], \tag{109}$$

$$\frac{d\eta}{dt}=-2\Gamma[2\xi\gamma-\eta], \tag{110}$$

$$\frac{dx_0}{dt}=0, \quad \frac{d\varphi_0}{dt}=0. \tag{111}$$

Then

$$\frac{d\gamma}{dt}=-2\Gamma\gamma, \tag{112}$$

$$\gamma=\gamma(0)e^{-2\Gamma t}, \tag{113}$$

$$\Delta=\Delta(0)e^{\Gamma t}\sqrt{\frac{\sin[\gamma(0)e^{-2\Gamma t}]}{\sin[\gamma(0)]}}, \tag{114}$$

and the energy decay of the damped MNLS soliton is

$$E(t)=\int_{-\infty}^{+\infty}dx|v_s|^2=\frac{4\gamma}{|\alpha|}=E(0)e^{-2\Gamma t}, \tag{115}$$

which is the same as a damped NLS soliton. This result is in agreement with Ref. [4].

The amplitude of the soliton is

$$A=\frac{2\eta}{\Delta\cos\frac{\gamma}{2}}=4\Delta\sin\frac{\gamma}{2}. \tag{116}$$

Let $B=A/4\Delta=\sin\gamma/2$, we have

$$\frac{dB}{dt}=\frac{1}{2}\frac{d\gamma}{dt}\cos\gamma/2=-2\Gamma\sqrt{1-B^2}\sin^{-1}B. \tag{117}$$

The solution of this equation leads to

$$A(t)=4\Delta\sin\left(e^{-2\Gamma t}\sin^{-1}\frac{A(0)}{4\Delta(0)}\right). \tag{118}$$

We see from this formula that, in the beginning, the decay of the amplitude of the damped MNLS soliton is a little bit slower than that of the damped NLS soliton, which simply decays exponentially at decay rate 2Γ. For larger t, the decay of the damped MNLS soliton approaches that of the damped NLS soliton.

As $\alpha\to0$, $A\to2\beta^{-1/2}\nu$,

$$\nu(t)=\nu(0)\exp(-2\Gamma t). \tag{119}$$

This is the well-known result of a damped NLS soliton.

IX. SUMMARY AND DISCUSSION

In summary, we have developed a direct perturbation theory for the perturbed DNLS and MNLS solitons, correcting some mistakes in the evolution equations of MNLS soliton's center and phase in the literature and obtaining perturbation-induced radiation of the perturbed MNLS solitons. Evolution equations for the MNLS soliton parameters perfectly reduce to those for the NLS equation in the small nonlinear dispersion limit. Also, we obtain a complete set of eigenfunctions for the linearized DNLS equations, which may be useful in developing direct perturbation theories for other equations in the DNLS hierarchy.

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APPENDIX A: THE LAX EQUATIONS AND JOST SOLUTIONS FOR THE DNLS EQUATION

The Lax equations for Eq. (2) are [18]

$$\partial_x W(x,t;\zeta)=LW(x,t;\zeta), \tag{A1a}$$

$$\partial_t W(x,t;\zeta)=MW(x,t;\zeta), \tag{A1b}$$

in which the Lax pair are

$$L=-i\zeta^2\sigma_3+\zeta U, \tag{A2a}$$

$$M=-i2\zeta^4\sigma_3+2\zeta^3U-i\zeta^2U^2\sigma_3+\zeta U^3-i\zeta U_x\sigma_3, \tag{A2b}$$

and

$$U=\begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix}. \tag{A3}$$

We define Jost solution $\phi(x,t;\zeta)$ and $\tilde{\phi}(x,t;\zeta)$ of Eqs. (A1) by

$$\phi(x,t;\zeta)\rightarrow\begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp(-i\zeta^2x), \quad x\rightarrow-\infty, \tag{A4}$$

$$\tilde{\phi}(x,t;\zeta)\rightarrow\begin{pmatrix} 0 \\ -1 \end{pmatrix}\exp(i\zeta^2x), \quad x\rightarrow-\infty, \tag{A5}$$

and define $\psi(x,t;\zeta)$, $\tilde{\psi}(x,t;\zeta)$ by

$$\psi(x,t;\zeta)\rightarrow\begin{pmatrix} 0 \\ 1 \end{pmatrix}\exp(i\zeta^2x), \quad x\rightarrow\infty, \tag{A6}$$

$$\tilde{\psi}(x,t;\zeta)\rightarrow\begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp(-i\zeta^2x), \quad x\rightarrow\infty. \tag{A7}$$

The scattering coefficients are defined by

$$\phi(x,t;\zeta)=a(\zeta)\tilde{\psi}(x,t;\zeta)+b(\zeta)\psi(x,t;\zeta), \tag{A8a}$$

$$\tilde{\phi}(x,t;\zeta)=-\tilde{a}(\zeta)\psi(x,t;\zeta)+\tilde{b}(\zeta)\tilde{\psi}(x,t;\zeta). \tag{A8b}$$

Functions $\phi(x,t;\zeta)$, $\psi(x,t;\zeta)$, and $a(\zeta)$ are analytic in the first and third quadrants of the ζ plane, while $\tilde{\phi}(x,t;\zeta)$, $\tilde{\psi}(x,t;\zeta)$, and $\tilde{a}(\zeta)$ are analytic in the second and fourth quadrants. Furthermore

$$\tilde{a}a+\tilde{b}b=1, \tag{A9}$$

$$\tilde{\phi}(\zeta)=-i\sigma_2\overline{\phi(\overline{\zeta})}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\overline{\phi(\overline{\zeta})}, \tag{A10a}$$

$$\tilde{\psi}(\zeta)=i\sigma_2\overline{\psi(\overline{\zeta})}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\psi(\overline{\zeta})}. \tag{A10b}$$

For more details, see Ref. [18].

APPENDIX B: EXPLICIT EXPRESSIONS OF JOST SOLUTIONS FOR THE ONE-SOLITON POTENTIAL IN THE DNLS EQUATION

It is easy to find the explicit expressions for Jost solutions of one-soliton potential in the DNLS equation from the IST for the DNLS equation in Ref. [18]:

$$\begin{aligned}\psi_1(x,t,\zeta)&=\Delta\frac{\exp(2\theta-i2\varphi+i\gamma)}{\exp(4\theta)+\exp(-i\gamma)}\zeta^{-1}\\&\quad\times[1-a(\zeta)]\exp(i\zeta^2x)h^{-1}(t,\zeta)\\&=\psi_1(\theta,\zeta)e^{i\zeta^2\hat{x}+i2\zeta^4t-i2\hat{\varphi}},\end{aligned} \tag{B1a}$$

$$\begin{aligned}\psi_2(x,t,\zeta)&=\frac{\exp(4\theta)+a(\zeta)\exp(i\gamma)}{\exp(4\theta)+\exp(i\gamma)}\exp(i\zeta^2x)h^{-1}(t,\zeta)\\&=\psi_2(\theta,\zeta)e^{i\zeta^2\hat{x}+i2\zeta^4t},\end{aligned} \tag{B1b}$$

$$\begin{aligned}\phi_1(x,t,\zeta)&=\frac{a(\zeta)\exp(4\theta)+\exp(-i\gamma)}{\exp(4\theta)+\exp(-i\gamma)}\exp(-i\zeta^2x)h(t,\zeta)\\&=\phi_1(\theta,\zeta)e^{-i\zeta^2\hat{x}-i2\zeta^4t},\end{aligned} \tag{B2a}$$

$$\begin{aligned}\phi_2(x,t,\zeta)&=\Delta\frac{\exp(2\theta+i2\varphi-i\gamma)}{\exp(4\theta)+\exp(i\gamma)}\zeta^{-1}[1-a(\zeta)]\\&\quad\times\exp(-i\zeta^2x)h(t,\zeta)\\&=\phi_2(\theta,\zeta)e^{-i\zeta^2\hat{x}-i2\zeta^4t+i2\hat{\varphi}}.\end{aligned} \tag{B2b}$$

Here $h(t,\zeta)=\exp(-i2\zeta^4t)$, and

$$a(\zeta)=\exp(-i2\gamma)\frac{\zeta^2-\zeta_1^2}{\zeta^2-\overline{\zeta}_1^2}, \quad \tilde{a}(\zeta)=a^{-1}(\zeta), \tag{B3}$$

$$\psi_1(\theta,\zeta)=\Delta\frac{e^{i(\zeta^2-2\zeta_1^2)/\eta\theta+i\gamma}}{e^{4\theta}+e^{-i\gamma}}\zeta^{-1}[1-a(\zeta)], \tag{B4a}$$

$$\psi_2(\theta,\zeta)=\frac{e^{4\theta}+a(\zeta)e^{i\gamma}}{e^{4\theta}+e^{i\gamma}}e^{i(\zeta^2/\eta)\theta}, \tag{B4b}$$

$$\phi_1(\theta,\zeta)=\frac{a(\zeta)e^{4\theta}+e^{-i\gamma}}{e^{4\theta}+e^{-i\gamma}}e^{-i(\zeta^2/\eta)\theta}, \tag{B5a}$$

$$\phi_2(\theta,\zeta)=\Delta\frac{e^{-i\frac{\zeta^2-2\bar{\zeta}_1^2}{\eta}\theta-i\gamma}}{e^{4\theta}+e^{i\gamma}}\zeta^{-1}[1-a(\zeta)]. \tag{B5b}$$

Solutions $\tilde{\psi}(\theta,\zeta)$ and $\tilde{\phi}(\theta,\zeta)$ can be simply obtained from Eq. (A10).

APPENDIX C: EIGENFUNCTIONS OF THE LINEARIZED DNLS EQUATION AROUND ONE-SOLITON SOLUTIONS

The continuous eigenfunctions of the linearization operator $\mathcal{L}$ are

$$\mathbf{\Psi}(\theta,\zeta)=\frac{\partial}{\partial\theta}\begin{pmatrix}\psi_1^2(\theta,\zeta)\\ \psi_2^2(\theta,\zeta)\end{pmatrix}=2\begin{pmatrix}\left[\frac{i(\zeta^2-2\zeta_1^2)}{\eta}-\frac{4e^{4\theta}}{e^{4\theta}+e^{-i\gamma}}\right]\psi_1^2(\theta,\zeta)\\ \left[i\frac{\zeta^2}{\eta}+\frac{4e^{4\theta}}{e^{4\theta}+a(\zeta)e^{i\gamma}}-\frac{4e^{4\theta}}{e^{4\theta}+e^{i\gamma}}\right]\psi_2^2(\theta,\zeta)\end{pmatrix}, \tag{C1}$$

$$\tilde{\mathbf{\Psi}}(\theta,\zeta)=\sigma_1\overline{\mathbf{\Psi}(\theta,\bar{\zeta})}, \tag{C2}$$

where the corresponding eigenvalues are $\pm i4(\zeta^2-\zeta_1^2)(\zeta^2-\bar{\zeta}_1^2)$ [see Eq. (34)], and $-\infty<\zeta<\infty$.

The discrete eigenfunctions of $\mathcal{L}$ are

$$\mathbf{\Psi}(\theta,\zeta_1)=i\frac{e^{i\gamma}}{2\Delta\sin^2\gamma}e^{i2\sigma_3\hat{\varphi}}\begin{pmatrix}[1+e^{4\theta}\Theta(\theta)]u_s\\ e^{4\theta}\overline{\Theta(\theta)}\overline{u_s}\end{pmatrix}, \tag{C3}$$

$$\tilde{\mathbf{\Psi}}(\theta,\bar{\zeta}_1)=\sigma_1\overline{\mathbf{\Psi}(\theta,\zeta_1)}, \tag{C4}$$

$$\dot{\mathbf{\Psi}}(\theta,\zeta_1)=\frac{1}{\Delta^2\sin^2\gamma}e^{i2\sigma_3\hat{\varphi}}\begin{pmatrix}\left[-i\frac{e^{i\,3/2\,\gamma}}{e^{4\theta}+e^{i\gamma}}-\frac{e^{i\,1/2\,\gamma}}{\tan\gamma}[1+e^{4\theta}\Theta(\theta)]-\frac{2e^{i\,3/2\,\gamma}}{\sin\gamma}\theta[1+e^{4\theta}\Theta(\theta)]\right]u_s\\ \left[-\frac{e^{i\,3/2\,\gamma}}{\sin\gamma}+i\frac{2e^{i\,3/2\,\gamma}}{e^{4\theta}+e^{i\gamma}}-\frac{e^{i\gamma/2}}{\tan\gamma}e^{4\theta}\overline{\Theta(\theta)}-\frac{2e^{i\,3/2\,\gamma}}{\sin\gamma}\theta e^{4\theta}\overline{\Theta(\theta)}\right]\overline{u_s}\end{pmatrix}, \tag{C5}$$

$$\dot{\tilde{\mathbf{\Psi}}}(\theta,\bar{\zeta}_1)=\sigma_1\overline{\dot{\mathbf{\Psi}}(\theta,\zeta_1)}. \tag{C6}$$

Here

$$\Theta(\theta)=\left(\frac{1}{e^{4\theta}+e^{i\gamma}}-\frac{2}{e^{4\theta}+e^{-i\gamma}}\right), \tag{C7}$$

and

$$\mathcal{L}\mathbf{\Psi}(\theta,\zeta_1)=0, \tag{C8a}$$

$$\mathcal{L}\tilde{\mathbf{\Psi}}(\theta,\bar{\zeta}_1)=0, \tag{C8b}$$

$$\mathcal{L}\dot{\mathbf{\Psi}}(\theta,\zeta_1)=-16\zeta_1\eta\mathbf{\Psi}(\theta,\zeta_1), \tag{C9a}$$

$$\mathcal{L}\dot{\tilde{\mathbf{\Psi}}}(\theta,\bar{\zeta}_1)=-16\bar{\zeta}_1\eta\tilde{\mathbf{\Psi}}(\theta,\bar{\zeta}_1). \tag{C9b}$$

The continuous eigenfunctions of the adjoint operator $\mathcal{L}^A$ are

$$\mathbf{\Phi}(\theta,\zeta)=\begin{pmatrix}\phi_2^2(\theta,\zeta)\\ \phi_1^2(\theta,\zeta)\end{pmatrix}, \tag{C10}$$

$$\tilde{\mathbf{\Phi}}(\theta,\zeta)=\sigma_1\overline{\mathbf{\Phi}(\theta,\bar{\zeta})}, \tag{C11}$$

where the corresponding eigenvalues are $\pm i4(\zeta^2-\zeta_1^2)(\zeta^2-\bar{\zeta}_1^2)$ [see Eq. (35)]. The discrete eigenfunctions of $\mathcal{L}^A$ are

$$\mathbf{\Phi}(\theta,\zeta_1)=e^{-i\,7/2\,\gamma}\zeta_1^2e^{i2\sigma_3[\varphi-\hat{\varphi}-(3/4)\gamma]}\begin{pmatrix}e^{2\theta}D(\theta)\\ e^{-2\theta}\overline{D(\theta)}\end{pmatrix}, \tag{C12}$$

$$\tilde{\mathbf{\Phi}}(\theta,\bar{\zeta}_1)=\sigma_1\overline{\mathbf{\Phi}(\theta,\zeta_1)}, \tag{C13}$$

$$\dot{\mathbf{\Phi}}(\theta,\zeta_1)=i\frac{e^{-i(7/2)\gamma}}{\eta}e^{i2\sigma_3[\varphi-\hat{\varphi}-(3/4)\gamma]}\times\begin{pmatrix}(-4\zeta^3\theta e^{2\theta}+2\xi\zeta_1e^{2\theta})D(\theta)\\ (-4\zeta_1^3\theta e^{-2\theta}-2|\zeta_1|^2\zeta_1e^{2\theta})\overline{D(\theta)}\end{pmatrix}, \tag{C14}$$

$$\dot{\tilde{\mathbf{\Phi}}}(\theta,\bar{\zeta}_1)=\sigma_1\overline{\dot{\mathbf{\Phi}}(\theta,\zeta_1)}, \tag{C15}$$

and

$$\mathcal{L}^A \mathbf{\Phi}(\theta,\zeta_1)=0, \tag{C16a}$$

$$\mathcal{L}^A \tilde{\mathbf{\Phi}}(\theta,\overline{\zeta_1})=0, \tag{C16b}$$

$$\mathcal{L}^A \dot{\mathbf{\Phi}}(\theta,\zeta_1)=-16\zeta_1\eta\mathbf{\Phi}(\theta,\zeta_1), \tag{C17a}$$

$$\mathcal{L}^A \dot{\tilde{\mathbf{\Phi}}}(\theta,\overline{\zeta_1})=-16\overline{\zeta_1}\eta\tilde{\mathbf{\Phi}}(\theta,\overline{\zeta_1}). \tag{C17b}$$

Here the definition for $D(\theta)$ is Eq. (56).

APPENDIX D: RELATIONS BETWEEN DERIVATIVES OF THE DNLS SOLITON WITH RESPECT TO ITS PARAMETERS AND DISCRETE EIGENFUNCTIONS OF THE LINEARIZED EQUATION

Here, for simplicity, we use Eq. (5) and choose Δ, γ, x_0, and φ_0 as its four independent parameters to calculate the derivatives of the DNLS soliton solution with respect to its parameters. They are

$$\frac{1}{u_s}\frac{\partial u_s}{\partial\Delta}=\frac{1}{\Delta}-i4\frac{e^{i\gamma}}{\Delta\sin\gamma}\theta+\frac{8}{\Delta}\Theta(\theta)\theta e^{4\theta}, \tag{D1}$$

$$\frac{1}{u_s}\frac{\partial u_s}{\partial\gamma}=\frac{1}{\tan\gamma}+\frac{ie^{i\gamma}}{e^{4\theta}+e^{i\gamma}}+\frac{i2e^{-i\gamma}}{e^{4\theta}+e^{-i\gamma}}+2\frac{e^{i\gamma}}{\sin\gamma}\theta + \frac{4}{\tan\gamma}\Theta(\theta)\theta e^{4\theta}, \tag{D2}$$

$$\frac{1}{u_s}\frac{\partial u_s}{\partial x_0}=i2\Delta^2 e^{i\gamma}-4\Delta^2\sin\gamma e^{4\theta}\Theta(\theta), \tag{D3}$$

$$\frac{1}{u_s}\frac{\partial u_s}{\partial\varphi_0}=-i2. \tag{D4}$$

With such a choice of parameters, Eq. (19) can be rewritten as

$$\boldsymbol{s}(u_s)=\frac{\partial\boldsymbol{u}_s}{\partial\Delta}\frac{d\Delta}{d\tau}+\frac{\partial\boldsymbol{u}_s}{\partial\gamma}\frac{d\gamma}{d\tau}+\frac{\partial\boldsymbol{u}_s}{\partial x_0}\frac{dx_0}{d\tau}+\frac{\partial\boldsymbol{u}_s}{\partial\varphi_0}\frac{d\varphi_0}{d\tau}. \tag{D5}$$

Upon observations on the explicit expressions of the discrete eigenfunctions, we obtain

$$e^{i2\sigma_3\hat{\varphi}}\frac{\partial\boldsymbol{u}_s}{\partial\Delta}=-i2\Delta\sin^2\gamma[e^{-i\gamma}\mathbf{\Psi}(\zeta_1)-e^{i\gamma}\tilde{\mathbf{\Psi}}(\overline{\zeta_1})] + i2\sin^2\gamma[e^{-i(\gamma/2)}\dot{\mathbf{\Psi}}(\zeta_1)-e^{i(\gamma/2)}\dot{\tilde{\mathbf{\Psi}}}(\overline{\zeta_1})], \tag{D6}$$

$$e^{i2\sigma_3\hat{\varphi}}\frac{\partial\boldsymbol{u}_s}{\partial\gamma}=i2\Delta\sin\gamma[e^{-i2\gamma}\mathbf{\Psi}(\zeta_1)-e^{i2\gamma}\tilde{\mathbf{\Psi}}(\overline{\zeta_1})] -\Delta^2\sin^2\gamma[e^{-i(\gamma/2)}\dot{\mathbf{\Psi}}(\zeta_1)+e^{i(\gamma/2)}\dot{\tilde{\mathbf{\Psi}}}(\overline{\zeta_1})], \tag{D7}$$

$$e^{i2\sigma_3\hat{\varphi}}\frac{\partial\boldsymbol{u}_s}{\partial x_0}=4\Delta^3\sin^2\gamma[\mathbf{\Psi}(\zeta_1)+\tilde{\mathbf{\Psi}}(\overline{\zeta_1})], \tag{D8}$$

$$e^{i2\sigma_3\hat{\varphi}}\frac{\partial\boldsymbol{u}_s}{\partial\varphi_0}=-4\Delta\sin^2\gamma[e^{-i\gamma}\mathbf{\Psi}(\zeta_1)+e^{i\gamma}\tilde{\mathbf{\Psi}}(\overline{\zeta_1})]. \tag{D9}$$

Thus Eq. (49) follows.

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