Necessity of $\mathcal{PT}$ symmetry for soliton families in one-dimensional complex potentials

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A B S T R A C T

For the one-dimensional nonlinear Schrödinger equation with a complex potential, it is shown that if this potential is not parity-time ($\mathcal{PT}$) symmetric, then no continuous families of solitons can bifurcate out from linear guided modes, even if the linear spectrum of this potential is all real. Both localized and periodic non-$\mathcal{PT}$-symmetric potentials are considered, and the analytical conclusion is corroborated by explicit examples. Based on this result, it is argued that $\mathcal{PT}$-symmetry of a one-dimensional complex potential is a necessary condition for the existence of soliton families.

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1. Introduction

Nonlinear wave systems used to be divided into two main categories: conservative systems and dissipative systems. In the former category, the system has no energy gain or loss, and solitary waves (or solitons in short) exist as continuous families, parameterized by their propagation constants. A well known example is the nonlinear Schrödinger equation with or without a real potential [1,2]. In the latter category, the system has energy gain and loss, and solitons generally exist as isolated solutions at certain discrete propagation-constant values, where the energy gain and loss on the soliton are exactly balanced (such solitons are often referred to as dissipative solitons in the literature) [3]. A typical example in this latter category is the Ginzburg–Landau equation or short-pulse lasers (see also [4,5]).

However, a recent discovery is that, in dissipative but parity-time ($\mathcal{PT}$) symmetric systems [6], solitons can still exist as continuous families, parameterized by their propagation constants [7–24]. This exact balance of continually deformed wave profiles in the presence of gain and loss is very remarkable.

The existence of soliton families in conservative and $\mathcal{PT}$-symmetric systems can be intuitively understood as follows. In both cases, the linear spectrum of the system is all-real or partly-real [6,25]. This means that the system supports linear guided modes. Then under nonlinearity, these linear guided modes can bifurcate out, leading to continuous families of solitons. In typical dissipative systems (such as the Ginzburg–Landau equation), however, the linear spectrum is all complex. Since there are no linear guided

modes, soliton-family bifurcation from linear modes then is not possible. As a result, it is understandable that only isolated solitons can exist in such typical dissipative systems.

It turns out that non-$\mathcal{PT}$-symmetric dissipative systems can also possess all-real or partly-real linear spectra. Indeed, for the one-dimensional (1D) linear Schrödinger operator, various non-$\mathcal{PT}$-symmetric complex potentials with all-real spectra have been constructed by the supersymmetry method [26–28]. Other non-$\mathcal{PT}$-symmetric dissipative systems with partly-real spectra have been reported as well [4,29,30]. In such non-$\mathcal{PT}$-symmetric dissipative systems, since linear guided modes exist, then an important question is: can continuous families of solitons bifurcate out from them? If they do, then the underlying non-$\mathcal{PT}$-symmetric dissipative system would allow much more flexibility in steering nonlinear localized modes (such as optical solitons) with continuous ranges of intensities, and this flexibility could have potential physical applications.

In this article, we investigate the existence of soliton families in the 1D nonlinear Schrödinger (NLS) equation with a non-$\mathcal{PT}$-symmetric complex potential. This NLS system governs paraxial nonlinear light propagation in a medium with non-$\mathcal{PT}$-symmetric refractive-index and gain-loss landscape [2,7], as well as Bose–Einstein condensates with a non-$\mathcal{PT}$-symmetric trap and gain-loss distribution [31]. In this NLS model, we show that no soliton families can bifurcate out from localized linear modes of a non-periodic potential or Bloch-band edges of a periodic potential. This means that no soliton families can bifurcate out from linear guided modes (if such modes exist). This result suggests that 1D non-$\mathcal{PT}$-symmetric potentials do not support continuous families of solitons. In other words, $\mathcal{PT}$-symmetry of a 1D complex potential is a necessary condition for the existence

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of soliton families (although it is not necessary for all-real linear spectra).

2. Preliminaries

The model equation we consider is the following 1D NLS equation with a linear non-$\mathcal{PT}$-symmetric complex potential

$$iU_t + U_{xx} - V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \tag{2.1}$$

where $V(x)$ is complex-valued and non-$\mathcal{PT}$-symmetric, i.e.,

$$V^*(x) \neq V(-x), \tag{2.2}$$

the asterisk represents complex conjugation, and $\sigma = \pm 1$ is the sign of nonlinearity. This equation governs paraxial light transmission as well as Bose–Einstein condensates in non-$\mathcal{PT}$-symmetric media. In this model, the nonlinearity is cubic. But extension of our analysis to an arbitrary form of nonlinearity is straightforward without much more effort [32].

Regarding the non-$\mathcal{PT}$-symmetric potential $V(x)$, a remark is in order. If this $V(x)$ is non-$\mathcal{PT}$-symmetric, but becomes $\mathcal{PT}$-symmetric after a certain spatial translation x_0, i.e., $V(x - x_0)$ is $\mathcal{PT}$-symmetric, then wave dynamics in this non-$\mathcal{PT}$-symmetric potential $V(x)$ is equivalent to that in the $\mathcal{PT}$-symmetric potential $V(x - x_0)$ and is thus not the subject of our study. Hence, in this article we require that the non-$\mathcal{PT}$-symmetric potential $V(x)$ in Eq. (2.1) remains non-$\mathcal{PT}$-symmetric under any spatial translation.

For non-$\mathcal{PT}$-symmetric complex potentials, their linear spectra may or may not contain real eigenvalues. In this article, we will consider those potentials that admit real eigenvalues in their linear spectra. Non-$\mathcal{PT}$ potentials with all-real spectra are special but important examples of such potentials.

We seek solitons in Eq. (2.1) of the form

$$U(x, t) = e^{i\mu t}u(x), \tag{2.3}$$

where $u(x)$ is a localized function satisfying the equation

$$u_{xx} - V(x)u - \mu u + \sigma|u|^2 u = 0, \tag{2.4}$$

and μ is a real-valued propagation constant. The question we will investigate is, does this equation admit soliton families for a continuous range of propagation-constant values when the potential $V(x)$ is non-$\mathcal{PT}$-symmetric?

It is noted that Eq. (2.4) is phase-invariant. That is, if $u(x)$ is a solitary wave, then so is $u(x)e^{i\alpha}$, where α is any real constant. In this article, solitons that are related by this phase invariance will be considered as equivalent.

3. Non-existence of soliton families bifurcating from localized linear modes

In this section, we consider non-$\mathcal{PT}$-symmetric potentials that are not periodic (for instance, localized potentials). Such potentials can admit discrete real eigenvalues, i.e., linear guided modes [4, 26–30]. If this potential were real or $\mathcal{PT}$-symmetric, soliton families would always bifurcate out from those linear guided modes (see the last section of this article). However, when the potential is non-$\mathcal{PT}$-symmetric, we will show that such soliton-family bifurcations are forbidden.

Suppose $V(x)$ is a non-$\mathcal{PT}$-symmetric potential which admits a simple discrete real eigenvalue μ_0, with the corresponding localized eigenfunction $\psi(x)$, i.e.,

$$L\psi = 0, \tag{3.1}$$

where

$$L \equiv \frac{d^2}{dx^2} - V(x) - \mu_0. \tag{3.2}$$

Since μ_0 is a simple eigenvalue, the equation $L\psi_g = \psi$ for the generalized eigenfunction ψ_g should not admit any solution. This means that the solvability condition of this ψ_g equation should not be satisfied, i.e., its inhomogeneous term ψ should not be orthogonal to the adjoint homogeneous solution ψ^*, or

$$\langle \psi^*, \psi \rangle \neq 0, \tag{3.3}$$

where

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} f^*(x)g(x)\,dx \tag{3.4}$$

is the standard inner product.

If a soliton family in Eq. (2.4) bifurcates out from this localized linear eigenmode, then we can expand these solitons into a perturbation series. We will show that this perturbation series requires an infinite number of nontrivial conditions to be satisfied simultaneously, which is impossible in practice due to lack of spatial symmetries in the 1D potential $V(x)$.

To proceed, let us expand these solitons into a perturbation series

$$u(x; \mu) = \epsilon^{1/2}\big[u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots\big], \tag{3.5}$$

where $\epsilon \equiv \mu - \mu_0$ is small. Substituting this expansion into Eq. (2.4), at $O(\epsilon^{1/2})$ we get

$$Lu_0 = 0, \tag{3.6}$$

hence

$$u_0 = c_0\psi, \tag{3.7}$$

where c_0 is a certain non-zero constant.

At $O(\epsilon^{3/2})$, we get the equation for u_1 as

$$Lu_1 = c_0\big(\psi - \sigma|c_0|^2|\psi|^2\psi\big). \tag{3.8}$$

Here the u_0 solution (3.7) has been utilized. The solvability condition of this u_1 equation is that its right hand side be orthogonal to the adjoint homogeneous solution ψ^*. This condition yields an equation for c_0 as

$$|c_0|^2 = \frac{\langle \psi^*, \psi \rangle}{\sigma\langle \psi^*, |\psi|^2\psi \rangle}. \tag{3.9}$$

Here we have assumed that the denominator $\langle \psi^*, |\psi|^2\psi \rangle \neq 0$. If it is zero, perturbation expansions different from (3.5) would be needed, but the qualitative result would remain the same as that given below.

Since $|c_0|$ is real and $\sigma = \pm 1$, Eq. (3.9) then requires that

$$Q_1 \equiv \frac{\langle \psi^*, \psi \rangle}{\langle \psi^*, |\psi|^2\psi \rangle} \quad \text{must be real.} \tag{3.10}$$

In a non-$\mathcal{PT}$-symmetric complex potential, Q_1 is generically complex, thus this condition is generically not satisfied.

It turns out that Eq. (3.10) is only the first condition for soliton-family bifurcations. As we pursue the perturbation expansion (3.5) to higher orders, infinitely more conditions will also appear. This will be demonstrated below.

If condition (3.10) is met, then the u_1 equation (3.8) is solvable. Its solution is

$$u_1 = \hat{u}_1 + c_1\psi, \tag{3.11}$$

where $\hat{u}_1$ is a particular solution to Eq. (3.8), and c_1 is a constant coefficient of the homogeneous solution ψ.

At $O(\epsilon^{5/2})$, the u_2 equation is

$$Lu_2 = u_1 - \sigma\left(u_0^2 u_1^* + 2|u_0|^2 u_1\right). \tag{3.12}$$

Substituting the above u_1 solution into this equation, we get

$$Lu_2 = c_1\left(1 - 2\sigma|u_0|^2\right)\psi - c_1^* \sigma u_0^2 \psi^* + h_2, \tag{3.13}$$

where

$$h_2 \equiv \left(1 - 2\sigma|u_0|^2\right)\hat{u}_1 - \sigma u_0^2 \hat{u}_1^*.$$

The solvability condition of this u_2 equation is that its right hand side be orthogonal to the adjoint homogeneous solution ψ^*. Recalling the u_0 solution (3.7) and utilizing the solvability condition of the u_1 equation (3.8), the solvability condition of the above u_2 equation then reduces to

$$c_1 + c_1^* = \frac{\langle \psi^*, h_2 \rangle}{\langle \psi^*, \psi \rangle}. \tag{3.14}$$

In order for this equation to admit c_1 solutions, we need to require that

$$Q_2 \equiv \frac{\langle \psi^*, h_2 \rangle}{\langle \psi^*, \psi \rangle} \quad \text{must be real.} \tag{3.15}$$

This is the second condition that must be satisfied in order for the perturbation series solution (3.5) of $u(x; \mu)$ to exist. In a non-$\mathcal{PT}$-symmetric complex potential, this condition is generically not satisfied either.

Carrying out this perturbative calculation to higher orders, we can show that infinitely more conditions of the type (3.15) will appear. Due to lack of symmetry of the involved functions, it is practically impossible for these infinite conditions to be met simultaneously. Thus soliton families cannot bifurcate out from a localized linear eigenmode in a non-$\mathcal{PT}$-symmetric potential.

4. Non-existence of soliton families bifurcating from Bloch-band edges

In this section, we consider periodic non-$\mathcal{PT}$-symmetric potentials. According to the Bloch–Floquet theory, these potentials do not admit discrete eigenvalues, but they possess Bloch bands which can be partially-real or all-real [7,12]. In periodic real or $\mathcal{PT}$-symmetric potentials, soliton families can bifurcate out from edges of Bloch bands [2,12]. However, when the periodic potential is non-$\mathcal{PT}$-symmetric, we will show that these soliton-family bifurcations from band edges are also forbidden.

Suppose $V(x)$ is a periodic non-$\mathcal{PT}$-symmetric complex potential that possesses a real segment of Bloch bands, and μ_0 is a real-valued edge of this Bloch band with the corresponding Bloch mode $p(x)$, i.e.,

$$Lp = 0, \tag{4.1}$$

where L is as defined in Eq. (3.2). According to the Bloch–Floquet theory, the Bloch mode $p(x)$ at edge μ_0 is either T- or $2T$-periodic, where T is the period of the potential $V(x)$. In addition, at the band edge, the eigenvalue μ_0 is simple, i.e., $Lp_g = p$ does not admit generalized eigenfunctions p_g. This means that the inhomogeneous term p should not be orthogonal to the adjoint homogeneous solution p^*, i.e.,

$$\langle p^*, p \rangle \neq 0, \tag{4.2}$$

where the inner product here (and throughout this section) is defined as

$$\langle f, g \rangle \equiv \int_0^T f^*(x) g(x)\, dx. \tag{4.3}$$

Now we consider bifurcations of soliton families from this real band edge μ_0. If the potential $V(x)$ is real, this soliton-family bifurcation has been studied in great detail in [2,33–35], and it was shown that two soliton families could bifurcate out from each Bloch-band edge. In a non-$\mathcal{PT}$-symmetric complex potential, however, we will show below that for this soliton-family bifurcation to occur, an infinite number of nontrivial conditions would have to be satisfied simultaneously, which is impossible in practice.

Suppose a soliton family bifurcates out from the band edge μ_0. Then near this edge, we can expand this soliton family and its propagation constant μ into perturbation series

$$u(x; \mu) = \epsilon\left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots\right), \tag{4.4}$$

$$\mu = \mu_0 + \mu_2 \epsilon^2 + \mu_4 \epsilon^4 + \cdots, \tag{4.5}$$

where ϵ is a small real parameter,

$$u_0 = A(X)p(x) \tag{4.6}$$

is a Bloch-wave packet, $X = \epsilon x$ is the slow spatial variable of the packet envelope $A(X)$, and $\mu_2, \mu_4, \ldots$ are real constants.

Substituting expansions (4.4)–(4.5) into Eq. (2.4), the $O(\epsilon)$ equation is satisfied automatically due to Eq. (4.1). At $O(\epsilon^2)$, we get the equation for u_1 as

$$Lu_1 = -2A_X p_x. \tag{4.7}$$

The solvability condition of this u_1 equation is that its right hand side be orthogonal to the adjoint homogeneous solution $p^*(x)$, which is satisfied automatically. Thus this u_1 equation is solvable. Its solution can be written as

$$u_1 = A_X \nu, \tag{4.8}$$

where $\nu(x)$ is a periodic solution to the equation

$$L\nu = -2p_x. \tag{4.9}$$

At $O(\epsilon^3)$, we get the equation for u_2 as

$$Lu_2 = -A_{XX}(p + 2\nu_x) + \mu_2 A p - \sigma|A|^2 A|p|^2 p. \tag{4.10}$$

Its solvability condition is that its right hand side be orthogonal to $p^*(x)$. This condition yields the following equation for the envelope function $A(X)$,

$$DA_{XX} + \mu_2 A - \alpha|A|^2 A = 0, \tag{4.11}$$

where

$$D \equiv -\frac{\langle p^*, p + 2\nu_x \rangle}{\langle p^*, p \rangle},$$

$$\alpha \equiv \sigma \frac{\langle p^*, |p|^2 p \rangle}{\langle p^*, p \rangle}. \tag{4.12}$$

Under the previous assumption of $V(x)$ possessing a real segment of Bloch bands with μ_0 as its edge, we can show by analyzing the linear Bloch-wave solution of Eq. (2.4) through perturbation expansions near the band edge μ_0 that, the constant D in the above equation (4.12) is related to the dispersion relation $\mu = \mu(k)$ as [2,34]

$$D = \frac{1}{2}\frac{d^2\mu}{dk^2}\bigg|_{\mu=\mu_0}, \tag{4.13}$$

hence D is real. Then in order for the envelope equation (4.11) to admit a localized solution, the coefficient α must be real. Thus, bifurcation of soliton families from a band edge μ_0 requires that

$$R_1 \equiv \frac{\langle p^*, |p|^2 p \rangle}{\langle p^*, p \rangle} \quad \text{must be real.} \tag{4.14}$$

In a non-$\mathcal{PT}$-symmetric periodic potential, R_1 is generically complex, thus this condition is generically not satisfied.

Carrying this perturbation calculation to higher orders, we will find that infinitely more non-trivial conditions also need to be satisfied in order for soliton-family bifurcations from band edges to occur, similar to the case of soliton bifurcations from localized linear modes in the previous section. For instance, the next condition, which comes from the solvability condition of the u_3 equation, is that

$$R_2 \equiv i \frac{\langle p^*, p^2 v^* - |p|^2 v \rangle}{\langle p^*, |p|^2 p \rangle} \quad \text{must be real.} \tag{4.15}$$

Due to lack of symmetry in the complex potential and its Bloch modes, each of these infinite conditions is nontrivial and is generically not satisfied. The requirement of them all satisfied simultaneously is practically impossible. Thus we conclude that in a non-$\mathcal{PT}$-symmetric periodic potential, no soliton families can bifurcate out from Bloch-band edges either.

5. Examples

In this section, we corroborate the general analytical conclusions of the previous two sections by three examples.

In these examples, non-$\mathcal{PT}$-symmetric complex potentials are obtained by the supersymmetry method so that they have all-real spectra [26–28]. This supersymmetry method is briefly summarized below.

5.1. Non-$\mathcal{PT}$-symmetric potentials with all-real spectra

Suppose $V_1(x)$ is a potential with all-real spectrum, and $\mu^{(1)}$ is an eigenvalue of this potential with eigenfunction $\psi^{(1)}$, i.e.,

$$\left[\frac{d^2}{dx^2} - V_1(x) - \mu^{(1)} \right] \psi^{(1)} = 0. \tag{5.1}$$

We first factorize the linear operator in this equation as

$$-\frac{d^2}{dx^2} + V_1(x) + \mu^{(1)} = \left[-\frac{d}{dx} + W(x) \right] \left[\frac{d}{dx} + W(x) \right]. \tag{5.2}$$

The function $W(x)$ in this factorization can be obtained by requiring $\psi^{(1)}$ to annihilate $d/dx + W(x)$, and this gives $W(x)$ as

$$W(x) = -\frac{d}{dx} \ln\left(\psi^{(1)} \right). \tag{5.3}$$

It is easy to directly verify that this $W(x)$ does satisfy the factorization equation (5.2).

Now we switch the two operators on the right side of the above factorization, and this leads to a new potential $V_2(x)$,

$$-\frac{d^2}{dx^2} + V_2(x) + \mu^{(1)} = \left[\frac{d}{dx} + W(x) \right] \left[-\frac{d}{dx} + W(x) \right], \tag{5.4}$$

where

$$V_2 = V_1 + 2W_x. \tag{5.5}$$

This V_2 potential is referred to as the partner potential of V_1, and it has the same spectrum as V_1, since operators AB and BA share the same spectrum. The only possible exception is the eigenvalue

$\mu^{(1)}$. Indeed, using the V_2-factorization (5.4) we can show that $\mu^{(1)}$ is not in the spectrum of V_2 (unless $\mu^{(1)}$ is a degenerate eigenvalue of V_1, i.e., its algebraic multiplicity is higher than its geometric multiplicity in the V_1 potential).

The new potential V_2, however, is only real or $\mathcal{PT}$-symmetric if V_1 is so. In order to derive non-$\mathcal{PT}$-symmetric potentials, we build a new factorization for the V_2 potential,

$$-\frac{d^2}{dx^2} + V_2(x) + \mu^{(1)} = \left[\frac{d}{dx} + \widetilde{W}(x) \right] \left[-\frac{d}{dx} + \widetilde{W}(x) \right]. \tag{5.6}$$

Using the previous V_2 factorization (5.4), the function $\widetilde{W}$ in this new factorization can be derived as [28]

$$\widetilde{W}(x) = -\frac{d}{dx} \ln\left(\widetilde{\psi}^{(1)} \right), \tag{5.7}$$

where

$$\widetilde{\psi}^{(1)}(x) = \frac{\psi^{(1)}(x)}{c + \int_0^x [\psi^{(1)}(\xi)]^2 \, d\xi}, \tag{5.8}$$

and c is an arbitrary complex constant. For this new V_2 factorization, its partner potential, defined through

$$-\frac{d^2}{dx^2} + \widetilde{V}_1(x) + \mu^{(1)} = \left[-\frac{d}{dx} + \widetilde{W}(x) \right] \left[\frac{d}{dx} + \widetilde{W}(x) \right], \tag{5.9}$$

is then

$$\widetilde{V}_1 = V_2 - 2\widetilde{W}_x. \tag{5.10}$$

Utilizing the V_2 and $\widetilde{W}$ formulae (5.5) and (5.7), this $\widetilde{V}_1$ potential is then found to be

$$\widetilde{V}_1(x) = V_1(x) - 2\frac{d^2}{dx^2} \ln\left[c + \int_0^x \left[\psi^{(1)}(\xi) \right]^2 d\xi \right]. \tag{5.11}$$

For generic values of the complex constant c, this $\widetilde{V}_1$ potential is complex and non-$\mathcal{PT}$-symmetric. In addition, its spectrum is identical to that of V_1. Indeed, even though $\mu^{(1)}$ may not lie in the spectrum of V_2, it is in the spectrum of $\widetilde{V}_1$ with eigenfunction $\widetilde{\psi}^{(1)}$. Hence if V_1 has an all-real spectrum, so does $\widetilde{V}_1$. Notice that this $\widetilde{V}_1$ potential, referred to as superpotential below, is actually a family of potentials due to the free complex constant c.

If the original potential V_1 is localized, taking $\psi^{(1)}$ as any of its discrete eigenmodes would lead to a localized superpotential. However, if we want to construct a periodic superpotential from a periodic original potential V_1, then it is easy to see from Eq. (5.11) and the Bloch–Floquet theory that the Bloch mode $\psi^{(1)}$ must be T- or $2T$-periodic, and

$$\int_0^T \left[\psi^{(1)}(x) \right]^2 dx = 0, \tag{5.12}$$

where T is the period of the V_1 potential. The former requirement means that the Bloch mode $\psi^{(1)}$ is located at the center or edge of the Brillouin zone. The latter requirement (5.12) means that the Bloch mode $\psi^{(1)}$ must be complex, hence so is the V_1 potential. In addition, $\langle \psi^{(1)*}, \psi^{(1)} \rangle = 0$, thus this Bloch mode is degenerate. At such a degenerate point, the local dispersion curve is two lines crossing each other like '×'. Due to this degeneracy, when the V_1 potential is perturbed, complex eigenvalues will bifurcate out from $\mu^{(1)}$ [12]. Thus if the V_1 potential is $\mathcal{PT}$-symmetric, then it must be at the phase-transition point (also known as $\mathcal{PT}$-symmetry-breaking point) [6,7,12].
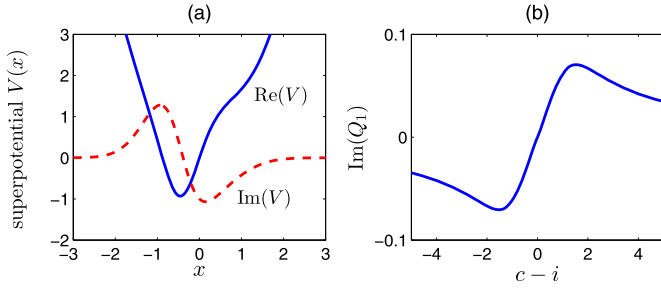
Fig. 1. (Color online.) (a) Superpotential (5.15) with $c = 1 + i$; (b) imaginary part of Q_1 in Eq. (3.10) for various complex values of c with $\text{Im}(c) = 1$.

5.2. Three examples

Now we consider three examples of non-$\mathcal{PT}$-symmetric superpotentials with all-real spectra, and show that the conditions for soliton-family bifurcations in them are not satisfied. Of these three examples, the first one pertains to an unbounded superpotential, the second one to a localized superpotential, and the third to a periodic superpotential.

Example 1. In our first example, the superpotential (5.11) is created from the harmonic potential

$$V_1(x) = x^2 \tag{5.13}$$

and its first eigenmode of $\mu^{(1)} = -1$ with

$$\psi^{(1)} = e^{-x^2/2}. \tag{5.14}$$

In other words, the superpotential (5.11) is

$$V(x) = x^2 - 2\frac{d^2}{dx^2} \ln\left[c + \int_0^x e^{-\xi^2}\, d\xi\right], \tag{5.15}$$

where c is a complex constant. When c is real, so is $V(x)$. When c is purely imaginary, $V(x)$ is complex and $\mathcal{PT}$-symmetric. For all other c values, the superpotential (5.15) is complex and non-$\mathcal{PT}$-symmetric. An example of this non-$\mathcal{PT}$-symmetric superpotential with $c = 1 + i$ is illustrated in Fig. 1(a). The spectrum of this superpotential (for all c values) is $\{-1, -3, -5, \dots\}$, which is the same as that of the harmonic potential (5.13).

For this superpotential (5.15), we consider the bifurcation of soliton families from its first eigenmode of $\mu_0 = -1$, whose eigenfunction is that given in Eq. (5.8), i.e.,

$$\psi = \frac{e^{-x^2/2}}{c + \int_0^x e^{-\xi^2}\, d\xi}. \tag{5.16}$$

Substituting this eigenmode into the Q_1 condition (3.10), we find that this condition is never satisfied for any complex c value that is not real or purely imaginary. For instance, if the imaginary part of c is fixed as one, then the imaginary part of Q_1 versus the real part of c is plotted in Fig. 1(b). One can see that $\text{Im}(Q_1) \neq 0$ when $\text{Re}(c) \neq 0$, indicating that Q_1 is never real when the superpotential (5.15) is non-$\mathcal{PT}$-symmetric; thus condition (3.10) is not satisfied. As a consequence, bifurcation of soliton families from the first eigenmode of the non-$\mathcal{PT}$-symmetric superpotential (5.15) cannot take place.

Example 2. In our second example, the superpotential (5.11) is created from a $\mathcal{PT}$-symmetric double-well potential

$$V_1(x) = -3\left[\text{sech}^2(x+1) + \text{sech}^2(x-1)\right]$$
$$\quad + 0.5i\left[\text{sech}^2(x+1) - \text{sech}^2(x-1)\right]. \tag{5.17}$$
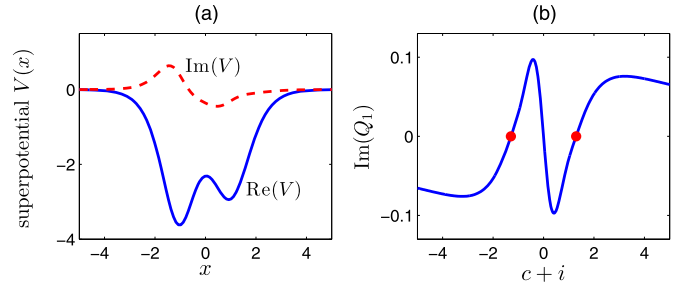

Fig. 2. (Color online.) (a) Superpotential (5.11) built from a double-well potential (5.17) and its first discrete eigenmode with $c = 4 - i$; (b) imaginary part of Q_1 in Eq. (3.10) for various complex values of c with $\text{Im}(c) = -1$.

This V_1 potential has an all-real spectrum that contains three positive discrete eigenvalues and a continuous spectrum of $(-\infty, 0]$. Its first discrete eigenvalue is $\mu^{(1)} \approx 2.3687$, and the eigenfunction $\psi^{(1)}$ of this first eigenvalue will be used to build the superpotential (5.11).

This superpotential is always complex, and is non-$\mathcal{PT}$-symmetric if c is not purely imaginary. When $c = 4 - i$, this superpotential is illustrated in Fig. 2(a).

For this superpotential (with arbitrary c), we also consider the bifurcation of soliton families from its first eigenmode $\widetilde{\psi}^{(1)}$, whose eigenvalue $\mu^{(1)}$ is as given above. In the notations of our analysis in Section 3, we choose

$$\mu_0 = \mu^{(1)}, \qquad \psi = \widetilde{\psi}^{(1)}. \tag{5.18}$$

Here the formula for $\widetilde{\psi}^{(1)}$ is provided by Eq. (5.8), where $\psi^{(1)}$ is the first eigenmode of the original double-well potential V_1, which can be obtained numerically.

Substituting eigenmode ψ of (5.18) into the Q_1 formula (3.10), we find that in the complex c-plane, this Q_1 is non-real everywhere except on the imaginary axis and on a certain quasi-ellipse. The c values on the imaginary axis only yield $\mathcal{PT}$-symmetric superpotentials and are not our concern. For c values on that quasi-ellipse, the superpotential is non-$\mathcal{PT}$-symmetric and Q_1 is real, thus the first condition (3.10) for soliton-family bifurcations is satisfied. However, we have found that on that c-ellipse, the second condition (3.15) is not met, thus this soliton-family bifurcation cannot occur.

To illustrate, we fix $\text{Im}(c) = -1$. Then $\text{Im}(Q_1)$ versus $\text{Re}(c)$ is plotted in Fig. 2(b). For non-$\mathcal{PT}$-symmetric superpotentials, $\text{Re}(c) \neq 0$. Then we see that at $\text{Re}(c) \approx \pm 1.2918$ (marked by red dots in that figure), $\text{Im}(Q_1) = 0$, i.e., Q_1 is real. These two c values, $\pm 1.2918 - i$, are on that c-ellipse mentioned above. But at these two c values, we have checked that Q_2 is not real, thus the second condition (3.15) for soliton-family bifurcations is not met.

Example 3. Our third example pertains to a periodic superpotential. In view of the discussions in the end of the previous subsection, this periodic superpotential (5.11) can be built from the original $\mathcal{PT}$-symmetric periodic potential

$$V_1(x) = -V_0^2 e^{2ix}, \tag{5.19}$$

and its Bloch mode

$$\psi^{(1)} = I_1\left(V_0 e^{ix}\right) \tag{5.20}$$

with eigenvalue $\mu^{(1)} = -1$. Here V_0 is a real constant, and I_n is the modified Bessel function. It is known that this V_1 potential is at the phase transition point [7,12], and its Bloch mode $\psi^{(1)}$ is located at the edge of the first Bloch band with a '×'-shaped local dispersion curve [36,37]. The resulting periodic superpotential (5.11) is
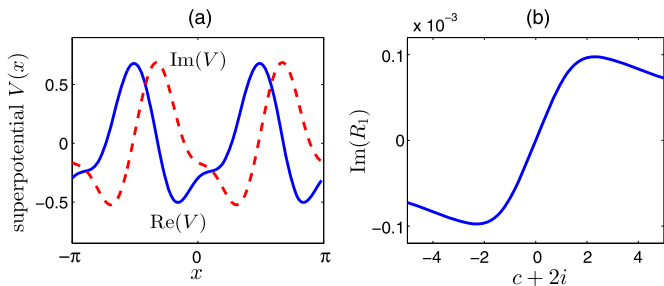
Fig. 3. (Color online.) (a) Periodic superpotential (5.21) with $c = 0.5 - 2i$ and $V_0 = 1$; (b) imaginary part of R_1 in Eq. (4.14) for various complex values of c with $\text{Im}(c) = -2$ and $V_0 = 1$.

$$V(x) = -V_0^2 e^{2ix} - 2\frac{d^2}{dx^2}\ln\left[c + \int_0^x I_1^2\left(V_0 e^{i\xi}\right) d\xi\right], \qquad (5.21)$$

where c is a complex constant.

This superpotential (5.21) is π-periodic, and is non-$\mathcal{PT}$-symmetric as long as c is not purely imaginary. When $c = 0.5 - 2i$ and $V_0 = 1$, this superpotential is illustrated in Fig. 3(a).

The dispersion relation of this superpotential (for all c values) is the same as that of the original potential (5.19), i.e.,

$$\mu = -(k + 2m)^2, \qquad (5.22)$$

where k is in the Brillouin zone $[-1, 1]$, and m is any non-negative integer [37]. From this dispersion relation, we see that Bloch bands of this superpotential cover the entire interval of $-\infty < \mu < 0$. Thus the only possible band edge for soliton bifurcations is $\mu_0 = 0$ (upper edge of the first Bloch band with $k = 0$). At this band edge, the Bloch mode in the original V_1 potential is

$$p^{(1)}(x) = I_0\left(V_0 e^{ix}\right). \qquad (5.23)$$

Then the corresponding Bloch mode in the superpotential (5.21) can be derived from Eqs. (5.2), (5.4), (5.6) and (5.9) as [28]

$$p = \left(-\frac{d}{dx} + \widetilde{W}\right)\left(\frac{d}{dx} + W\right)p^{(1)}, \qquad (5.24)$$

where W and $\widetilde{W}$ are given by Eqs. (5.3) and (5.7).

Substituting this Bloch mode $p(x)$ into the R_1 formula (4.14), we find that this R_1 is non-real everywhere in the complex c-plane, except for the imaginary axis and a certain quasi-circle. The c values on the imaginary axis lead to $\mathcal{PT}$-symmetric superpotentials which are irrelevant for our study. For c values on that quasi-circle, R_1 is real, but R_2 in Eq. (4.15) is non-real, thus the second condition (4.15) is not met. As a consequence, soliton-family bifurcations from this Bloch-band edge $\mu_0 = 0$ cannot occur. This situation is similar to that in Example 2.

For demonstration purpose, we fix $\text{Im}(c) = -2$ and $V_0 = 1$. Then $\text{Im}(R_1)$ versus $\text{Re}(c)$ is plotted in Fig. 3(b). We see that on this line of c values, $\text{Im}(R_1) \neq 0$ when $\text{Re}(c) \neq 0$, indicating that R_1 is non-real when the superpotential (5.21) is non-$\mathcal{PT}$-symmetric, hence the first condition (4.14) for soliton-family bifurcations is not met.

6. Summary and discussion

In this article, we have shown that for the 1D NLS equation with a non-$\mathcal{PT}$-symmetric periodic or non-periodic potential, no continuous families of solitons can bifurcate out from linear modes of the potential, even if this potential has an all-real spectrum. This analytical finding is also corroborated by several specific examples containing complex superpotentials with all-real spectra. This result suggests that $\mathcal{PT}$-symmetry of a 1D complex potential is a

necessary condition for the existence of soliton families. This conclusion highlights the importance of $\mathcal{PT}$-symmetry for the study of nonlinear soliton states, even though it is not necessary for all-real linear spectrum.

If a complex potential is $\mathcal{PT}$-symmetric, then repeating the perturbative calculations in Sections 3 and 4 of this article, we will find that those infinite conditions, such as (3.10), (3.15), (4.14) and (4.15), are all automatically satisfied due to $\mathcal{PT}$-symmetry of the potential and other involved functions. For instance, for $\mathcal{PT}$-symmetric non-periodic potentials, the linear eigenmode ψ and solutions u_0, $\hat{u}_1$ in Section 3 can be made $\mathcal{PT}$-symmetric through phase invariance. Thus quantities Q_1, Q_2 in Eqs. (3.10), (3.15) are automatically real, making conditions (3.10) and (3.15) automatically fulfilled. As a consequence, soliton families *can* be successfully constructed from perturbation expansions. This analytical result is consistent with earlier numerical reports of soliton families in various $\mathcal{PT}$-symmetric potentials [8,9,12,13]. Combining this result with the finding of this article, we argue that in the 1D NLS equation with a complex potential, $\mathcal{PT}$-symmetry of the potential is both necessary and sufficient for the existence of soliton families (assuming that this potential admits real discrete eigenvalues or real Bloch bands). Soliton families that exist in a 1D $\mathcal{PT}$-symmetric potential are always $\mathcal{PT}$-symmetric, as was shown recently in [38].

References

[1] Y.S. Kivshar, G.P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, San Diego, 2003.
[2] J. Yang, Nonlinear Waves in Integrable and Nonintegrable Systems, SIAM, Philadelphia, 2010.
[3] N. Akhmediev, A. Ankiewicz (Eds.), Dissipative Solitons, Springer, Berlin, 2005.
[4] D.A. Zezyulin, Y.V. Kartashov, V.V. Konotop, Opt. Lett. 36 (2011) 1200–1202.
[5] B.A. Malomed, H.G. Winful, Phys. Rev. E 53 (1996) 5365–5368.
[6] C.M. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243–5246.
[7] Z.H. Musslimani, K.G. Makris, R. El-Ganainy, D.N. Christodoulides, Phys. Rev. Lett. 100 (2008) 030402.
[8] H. Wang, J. Wang, Opt. Express 19 (2011) 4030–4035.
[9] Z. Lu, Z. Zhang, Opt. Express 19 (2011) 11457–11462.
[10] F.K. Abdullaev, Y.V. Kartashov, V.V. Konotop, D.A. Zezyulin, Phys. Rev. A 83 (2011) 041805.
[11] Y. He, X. Zhu, D. Mihalache, J. Liu, Z. Chen, Phys. Rev. A 85 (2012) 013831.
[12] S. Nixon, L. Ge, J. Yang, Phys. Rev. A 85 (2012) 023822.
[13] D.A. Zezyulin, V.V. Konotop, Phys. Rev. A 85 (2012) 043840.
[14] C. Huang, C. Li, L. Dong, Opt. Express 21 (2013) 3917–3925.
[15] Y.V. Kartashov, Opt. Lett. 38 (2013) 2600–2603.
[16] R. Driben, B.A. Malomed, Opt. Lett. 36 (2011) 4323–4325.
[17] N.V. Alexeeva, I.V. Barashenkov, A.A. Sukhorukov, Yu.S. Kivshar, Phys. Rev. A 85 (2012) 063837.
[18] F.C. Moreira, F.Kh. Abdullaev, V.V. Konotop, A.V. Yulin, Phys. Rev. A 86 (2012) 053815.
[19] V.V. Konotop, D.E. Pelinovsky, D.A. Zezyulin, Europhys. Lett. 100 (2012) 56006.
[20] I.V. Barashenkov, L. Baker, N.V. Alexeeva, Phys. Rev. A 87 (2013) 033819.
[21] P.G. Kevrekidis, D.E. Pelinovsky, D.Y. Tyugin, SIAM J. Appl. Dyn. Syst. 12 (2013) 1210–1236.
[22] K. Li, P.G. Kevrekidis, Phys. Rev. E 83 (2011) 066608.
[23] D.A. Zezyulin, V.V. Konotop, Phys. Rev. Lett. 108 (2012) 213906.
[24] D.A. Zezyulin, V.V. Konotop, J. Phys. A 46 (2013) 415301.
[25] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205–214.
[26] F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. 251 (1995) 267–385.
[27] F. Cannata, G. Junker, J. Trost, Phys. Lett. A 246 (1998) 219–226.
[28] M. Miri, M. Heinrich, D.N. Christodoulides, Phys. Rev. A 87 (2013) 043819.
[29] A.E. Siegman, J. Opt. Soc. Am. A 20 (2003) 1617–1628.
[30] Y. Chen, V. Sudesh, T. McComb, M.C. Richardson, M. Bass, J. Ballato, J. Opt. Soc. Am. B 24 (2007) 1683–1688.
[31] L.P. Pitaevskii, S. Stringari, Bose–Einstein Condensation, Oxford University Press, Oxford, 2003.

[32] J. Yang, Stud. Appl. Math. 129 (2012) 133–162.
[33] B.B. Baizakov, V.V. Konotop, M. Salerno, J. Phys. B 35 (2002) 5105.
[34] D.E. Pelinovsky, A.A. Sukhorukov, Y.S. Kivshar, Phys. Rev. E 70 (2004) 036618.
[35] G. Hwang, T.R. Akylas, J. Yang, Physica D 240 (2011) 1055–1068.

[36] C.M. Bender, G.V. Dunne, P.N. Meisinger, Phys. Lett. A 252 (1999) 272.
[37] S. Nixon, Y. Zhu, J. Yang, Opt. Lett. 37 (2012) 4874–4876.
[38] J. Yang, Stud. Appl. Math. (2013), http://dx.doi.org/10.1111/sapm.12032, see also arXiv:1309.1652 [nlin.PS].