Can Parity-Time-Symmetric Potentials Support Families of Non-Parity-Time-Symmetric Solitons?

By Jianke Yang

For the one-dimensional nonlinear Schrödinger equations with parity-time (PT) symmetric potentials, it is shown that when a real symmetric potential is perturbed by weak PT-symmetric perturbations, continuous families of asymmetric solitary waves in the real potential are destroyed. It is also shown that in the same model with a general PT-symmetric potential, symmetry breaking of PT-symmetric solitary waves does not occur. Based on these findings, it is conjectured that one-dimensional PT-symmetric potentials cannot support continuous families of non-PT-symmetric solitary waves.

1. Introduction

Solitary waves play an important role in the dynamics of nonlinear wave equations [1–3]. In conservative systems, solitary waves generally exist as continuous families. Familiar examples include the nonlinear Schrödinger (NLS) equations with or without external real potentials. In dissipative systems, solitary waves generally exist as isolated solutions, with the Ginzburg–Landau equation as one of the best known examples [4]. However, a recent discovery is that, in dissipative but parity-time (PT) symmetric systems [5], solitary waves can still exist as continuous families, parameterized by their propagation constants. This exact balance of continually deformed wave profiles in the presence of gain and loss is very remarkable. So far, various PT-symmetric wave systems have been investigated. Examples include the NLS equations

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with linear and/or nonlinear PT-symmetric potentials [6–14], vector NLS equations with PT-symmetric potentials [15], PT-symmetric couplers [16, 17], the $\chi^{(2)}$ system with PT-symmetric potentials [18], discrete NLS-PT lattices [19–21], and so on. In these systems, continuous families of PT-symmetric solitary waves (or solitons in short) have been reported. Other PT systems, such as finite-dimensional PT systems, have also been explored [22–24]. In certain finite-dimensional PT systems (such as the quadrimer model), it was found that continuous families of PT-symmetric solutions could coexist with isolated solutions. Experimentally, PT-symmetric potentials have been fabricated in optical settings [25, 26].

Because solitons in PT systems could exist for continuous ranges of propagation constants, bifurcations of such solitons become an important issue. In conservative wave systems, various archetypical bifurcations of solitons have been reported, including fold bifurcations (also known as saddle-node or saddle-center bifurcations), symmetry-breaking bifurcations (also known as pitchfork bifurcations), and transcritical bifurcations (see [27] and the references therein). In PT systems, fold bifurcations have been found [12, 13, 19, 20], but the other types of bifurcations are still unknown.

Of all bifurcations, symmetry-breaking bifurcations are particularly interesting, because such bifurcations create solitary waves that do not obey the symmetry of the original system. One of the most familiar symmetry-breaking bifurcations is in the NLS equation with a real symmetric potential, where families of asymmetric solitons bifurcate out from symmetric solitons at certain propagation-constant values [28–34]. These asymmetric solitons are often more stable than their symmetric counterparts. PT-symmetric systems have been shown to admit families of PT-symmetric solitons. Then a natural question is, can symmetry breaking occur for these PT-symmetric solitons? If so, continuous families of non-PT-symmetric solitons would appear in a PT-symmetric system.

Symmetry breaking is a dominant way for the creation of families of asymmetric solitons, but it may not be the only way. Thus, a more general question is, can PT-symmetric systems admit continuous families of non-PT-symmetric solitons?

In this article, we investigate the existence of families of non-PT-symmetric solitons in a familiar PT system—the one-dimensional NLS (1D NLS) equation with a linear PT-symmetric complex potential. This PT system governs paraxial nonlinear light propagation in a medium with symmetric refractive index and antisymmetric gain and loss [3, 6], as well as Bose–Einstein condensates in a symmetric trap with spatially balanced gain and loss [35]. For this PT system, we show that continuous families of asymmetric solitons in a real symmetric potential are destroyed when this real potential is perturbed by weak PT-symmetric perturbations. We further show that in a general 1D PT-symmetric potential, symmetry breaking of PT-symmetric solitons cannot occur. Based on these findings, we conjecture that one-dimensional PT-symmetric potentials cannot support continuous families of non-PT-symmetric solitons. In other words, continuous families of solitons in 1D PT-symmetric potentials must be PT-symmetric.

2. Preliminaries

Our study of solitary waves in PT-symmetric systems is based on the following 1D NLS equation with a linear PT-symmetric potential

$$iU_t + U_{xx} - V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \quad (1)$$

where $V(x)$ is a complex-valued (nonreal) PT-symmetric potential

$$V^*(x) = V(-x), \quad (2)$$

with the asterisk representing complex conjugation, and $\sigma = \pm 1$ is the sign of nonlinearity ($\sigma = 1$ for self-focusing and $\sigma = -1$ for self-defocusing). Here, the PT-symmetry (2) means that the real part of the potential $V(x)$ is symmetric in x, and the imaginary part of $V(x)$ is antisymmetric in x. The Equation (1) is the appropriate mathematical model for paraxial light transmission in PT-symmetric media (where the refractive index is symmetric and gain–loss profile antisymmetric). It also governs the dynamics of Bose–Einstein condensates in a symmetric potential with spatially balanced gain and loss (in this community, Equation (1) is called the Gross–Pitaevskii equation). In the model (1), the nonlinearity is only cubic. But extension of our analysis to an arbitrary form of nonlinearity is straightforward without much more effort (see [27]).

Solitary waves in Equation (1) are sought of the form

$$U(x,t) = e^{i\mu t}u(x), \quad (3)$$

where $u(x)$ is a complex-valued localized function, which satisfies the equation

$$u_{xx} - \mu u - V(x)u + \sigma|u|^2u = 0, \quad (4)$$

and μ is a real-valued propagation constant. Even though Equation (1) is dissipative due to the complex potential $V(x)$, a remarkable phenomenon is that it can support continuous families of solitary waves (3), parameterized by the propagation constant μ—just like in real potentials [7, 8, 10–14]. Then under certain conditions, these solitary waves may undergo bifurcations at special values of μ.

If the potential $V(x)$ were strictly real, then the PT symmetry condition (2) would become $V(-x) = V(x)$, i.e., this potential would be symmetric. It is well known that in real symmetric potentials, symmetry breaking of solitary waves often occurs. Specifically, in addition to continuous families of symmetric and antisymmetric solitons, continuous families of asymmetric solitons can also bifurcate out from those symmetric and antisymmetric soliton branches. This symmetry breaking is most familiar in double-well potentials [28, 30], but it can occur in other symmetric potentials (such as periodic potentials) as well [29, 32]. Because of this symmetry breaking, continuous families of asymmetric solitons appear in a real symmetric potential.

When the potential $V(x)$ is complex but PT-symmetric, there exist continuous families of solitary waves $u(x;\mu)$ with the same PT-symmetry

$$u^*(x;\mu) = u(-x;\mu), \tag{5}$$

see [7, 8, 11, 13]. Then, the question is, can PT-symmetry breaking occur for these PT-symmetric solitons? More generally, can continuous families of non-PT-symmetric solitons exist in a 1D PT-symmetric potential? These are the questions we will address in this article.

Remark 1. It is noted that Equation (4) is phase invariant. That is, if $u(x)$ is a solitary wave, then so is $u(x)e^{i\alpha}$, where α is any real constant. For a solitary wave $u(x)$, if there exists a real constant α so that $u(x)e^{i\alpha}$ is PT-symmetric, then we say $u(x)$ is reducible to PT-symmetric. For instance, a complex solitary wave $u(x)$ with anti-PT-symmetry $u^*(x) = -u(-x)$, i.e., with an antisymmetric real part and symmetric imaginary part, is reducible to PT-symmetric by multiplying it by i. In general, a simple way to determine whether a complex-valued solitary wave $u(x)$ is reducible to PT-symmetric or not is to examine the function $u(x)e^{-i\theta}$, where θ is the phase of $u(0)$. If $u(x)e^{-i\theta}$ is PT-symmetric, then $u(x)$ is reducible to PT-symmetric and vise versa. Graphically, a simple way to decide whether a solitary wave $u(x)$ is reducible to PT-symmetric is to plot the amplitude $|u(x)|$ of the function. If this amplitude is not symmetric in x, then $u(x)$ is not reducible to PT-symmetric. In this article, when we say non-PT-symmetric solitary waves or solitons, we mean solitary waves that are not reducible to PT-symmetric.

3. Disappearance of families of asymmetric solitons under weak PT-potential perturbations

To explore the existence of continuous families of non-PT-symmetric solitons in Equation (1) with a PT-symmetric potential, we first investigate what happens to families of asymmetric solitons of a real symmetric potential when this real potential is weakly perturbed by an imaginary antisymmetric term (which makes the perturbed potential nonreal but PT-symmetric). Can these families of asymmetric solitons survive and turn into families of non-PT-symmetric solitons in the resulting PT potential? The answer is negative. We will show that these families of asymmetric solitons of real potentials disappear under weak PT-potential perturbations.

When a real symmetric potential is perturbed by an imaginary antisymmetric term, the model Equation (4) can be written as

$$u_{xx} - \mu u - V_r(x)u + \sigma|u|^2 u = i\epsilon W(x)u, \tag{6}$$

where $V_r(x)$ is a real symmetric potential, $W(x)$ is a real antisymmetric function,

$$V_r(x) = V_r(-x), \quad W(x) = -W(-x), \tag{7}$$

and ϵ is a small real parameter. Note that the combined potential in Equation (6), $V = V_r + i\epsilon W$, is complex and PT-symmetric.

Suppose we seek a continuous family of solitons in the perturbed potential (with $0 < \epsilon \ll 1$), near a continuous family of asymmetric solitons in the unperturbed real potential (with $\epsilon = 0$). Because the soliton family in the perturbed potential should exist for a continuous range of μ values, each soliton in that family with a fixed μ value should converge to the asymmetric soliton of $\epsilon = 0$ with the same μ value when $\epsilon \to 0$. Because of this, in our perturbation expansion we can fix μ and expand the soliton at this μ value into a perturbation series of ϵ,

$$u(x;\epsilon) = u_r(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots, \tag{8}$$

where $u_r(x)$ is an asymmetric (i.e., neither symmetric nor antisymmetric) real soliton in the real potential V_r, i.e.,

$$\frac{d^2 u_r}{dx^2} - \mu u_r - V_r(x)u_r + \sigma u_r^3 = 0, \tag{9}$$

and

$$u_r(-x) \neq \pm u_r(x). \tag{10}$$

We will show below that, in order for this perturbation series to be constructed, an infinite number of nontrivial conditions would have to be satisfied simultaneously, which is impossible in practice. This conclusion will also be corroborated by several specific examples.

We start by substituting the expansion (8) into Equation (6). The $O(1)$ equation is satisfied automatically due to Equation (9). At $O(\epsilon)$, the equation for u_1 is

$$L_r \begin{bmatrix} u_1 \\ u_1^* \end{bmatrix} = \begin{bmatrix} iWu_r \\ -iWu_r \end{bmatrix}, \tag{11}$$

where

$$L_r = \begin{bmatrix} \frac{d^2}{dx^2} - V_r(x) - \mu + 2\sigma u_r^2 & \sigma u_r^2 \\ \sigma u_r^2 & \frac{d^2}{dx^2} - V_r(x) - \mu + 2\sigma u_r^2 \end{bmatrix} \tag{12}$$

is a real and self-adjoint operator. The kernel of L_r contains an eigenfunction $[u_r, -u_r]^{\mathrm{T}}$, where the superscript "T" represents the transpose of a vector, due to Equation (9). Thus,

$$L_r \begin{bmatrix} u_r \\ -u_r \end{bmatrix} = 0. \tag{13}$$

Let us assume that the kernel of L_r does not contain any additional eigenfunctions, which is true for generic values of μ. Then the solvability condition for Equation (11) is that its right-hand side be orthogonal to $[u_r, -u_r]^{\mathrm{T}}$, which reduces to

$$Q_1(\mu) \equiv \langle u_r, W u_r \rangle = 0, \tag{14}$$

where the inner product is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^{*\mathrm{T}}(x) g(x)\, dx. \tag{15}$$

Because W is antisymmetric and u_r asymmetric (see Equation (10)), Equation (14) is then a nontrivial condition which must be satisfied in order for the asymmetric soliton $u_r(x)$ to persist under weak PT-potential perturbations.

It turns out that Equation (14) is only the first of infinitely many conditions which must be satisfied in order for the perturbation-series solution (8) to be constructed. Indeed, at each higher odd order, a new condition would appear. For instance, if condition (14) is met, then the u_1 equation (11) can be solved. This solution can be written as

$$u_1 = i\hat{u}_1 + i d_1 u_r, \tag{16}$$

where $\hat{u}_1$ is a real and localized function solving the equation

$$L_r \begin{bmatrix} \hat{u}_1 \\ -\hat{u}_1 \end{bmatrix} = \begin{bmatrix} W u_r \\ -W u_r \end{bmatrix}, \tag{17}$$

and d_1 is a real constant. This $i d_1 u_r$ term in u_1, when combined with the first term u_r in the expansion (8), only amounts to a constant phase shift to the solution $u(x; \epsilon)$, thus it can be set to be zero without any loss of generality (see Remark 1). Thus,

$$u_1 = i\hat{u}_1. \tag{18}$$

At $O(\epsilon^2)$, the u_2 equation is

$$L_r \begin{bmatrix} u_2 \\ u_2^* \end{bmatrix} = \begin{bmatrix} h_2 \\ h_2 \end{bmatrix}, \tag{19}$$

where

$$h_2 = -\sigma u_r \hat{u}_1^2 - W\hat{u}_1 \tag{20}$$

is a real function. The solvability condition for this equation is satisfied automatically, thus u_2 has a solution

$$u_2 = \hat{u}_2, \tag{21}$$

where $\hat{u}_2$ is a real and localized function. As before, we have excluded the homogeneous term (proportional to iu_r) in the u_2 solution without loss of generality.

Now we proceed to $O(\epsilon^3)$, where the u_3 equation is

$$L_r \begin{bmatrix} u_3 \\ u_3^* \end{bmatrix} = \begin{bmatrix} ih_3 \\ -ih_3 \end{bmatrix}, \tag{22}$$

and

$$h_3 = -\sigma(\hat{u}_1^3 + 2u_r\hat{u}_1\hat{u}_2) + W\hat{u}_2 \tag{23}$$

is a real function. The solvability condition of this u_3 equation is that

$$Q_2(\mu) \equiv \langle u_r, h_3 \rangle = 0. \tag{24}$$

Because $u_r, \hat{u}_1$, and $\hat{u}_2$ are all asymmetric functions, so is h_3. Then, Equation (24) is the second nontrivial condition, which has to be met. Following similar calculations to higher orders, infinitely more conditions will appear.

The fundamental reason for this infinite number of conditions for the perturbation series solution (8) is that, due to phase invariance of the solitons in Remark 1, each u_n solution does not contain any nonreducible free constants. But for each odd n, the u_n equation is of the form $L_r[u_n, u_n^*]^{\mathrm{T}} = [ih_n, -ih_n]^{\mathrm{T}}$, where h_n is a certain real function. In order for this u_n equation to be solvable, the solvability condition $\langle u_r, h_n \rangle = 0$ must be satisfied. All these solvability conditions then constitute an infinite number of conditions for the perturbation series solution (8).

In one spatial dimension, neither the perturbed PT-symmetric potential nor the underlying asymmetric solitons possesses additional spatial symmetries. Because of that, each of these infinitely many conditions is nontrivial and is generally not satisfied for generic values of μ. The requirement of them all satisfied simultaneously is practically impossible. This means that continuous families of asymmetric solitons in the real potential would disappear under weak PT-potential perturbations.

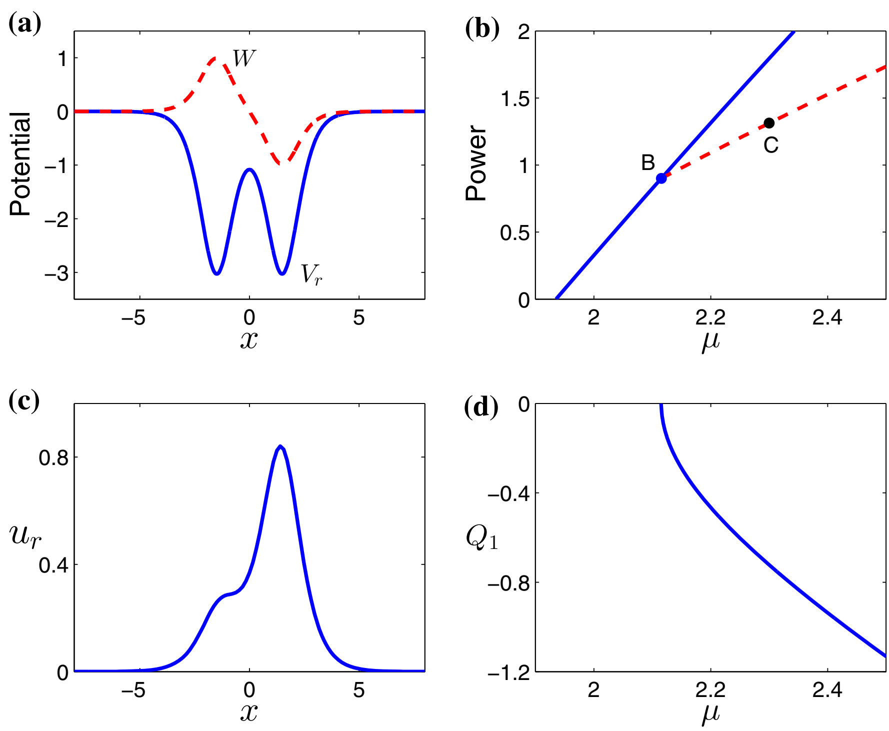

Figure 1. Disappearance of the continuous family of asymmetric solitons under PT-potential perturbations in Example 1. (a) Real potential (25) and PT perturbation (26); (b) power curves of symmetric (blue solid) and asymmetric (red dashed) solitons; (c) asymmetric soliton $u_r(x)$ at point "C" of panel (b), where $\mu = 2.3$; and (d) function $Q_1(\mu)$ in condition (14). This function being nonzero indicates that the continuous family of asymmetric solitons (red dashed line in panel b) are destroyed under PT-potential perturbations.

Now we use three specific examples to corroborate the above statement.

EXAMPLE 1. *In this first example, we take V_r to be a symmetric double-well potential*

$$V_r(x) = -3\left[\operatorname{sech}^2(x+1.5) + \operatorname{sech}^2(x-1.5)\right], \tag{25}$$

and W to be an antisymmetric function

$$W(x) = \operatorname{sech}^2(x+1.5) - \operatorname{sech}^2(x-1.5). \tag{26}$$

Both functions are displayed in Figure 1(a). In addition, we take $\sigma = 1$, i.e., self-focusing nonlinearity.

In this double-well potential V_r, a branch of real symmetric solitons exist, whose power curve is shown in Figure 1(b) (solid blue line). In addition, symmetry breaking occurs at $\mu_0 \approx 2.1153$, where a branch of asymmetric solitons appear for $\mu > \mu_0$. An example of such asymmetric solitons (with

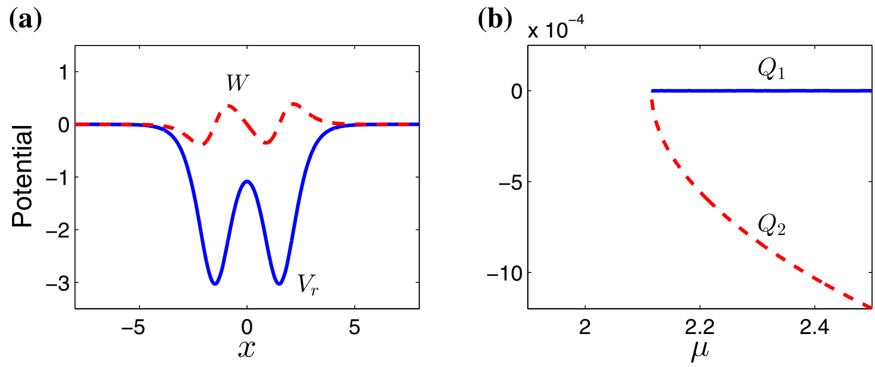

Figure 2. Disappearance of the continuous family of asymmetric solitons under PT-potential perturbations in Example 2. (a) Real potential (25) and PT perturbation (27); (b) functions $Q_1(\mu)$ and $Q_2(\mu)$ in conditions (14) and (24). Here $Q_2 \neq 0$, thus the continuous family of asymmetric solitons are destroyed under PT-potential perturbations.

$\mu = 2.3$*) is illustrated in Figure 1(c). For these asymmetric solitons, we have numerically calculated the function* $Q_1(\mu)$ *as defined in Equation* (14), *and this function is plotted in Figure 1(d). We can see that this function is nonzero for all* $\mu > \mu_0$, *thus the first condition* (14) *is never satisfied, let alone all the other conditions such as* (24). *Thus, we conclude that in this example, the continuous family of asymmetric solitons in the real symmetric potential* (25) *are destroyed under weak PT-potential perturbations* (26).

EXAMPLE 2. *In the second example, we keep the real potential* V_r *of Example 1, but choose a different anti-symmetric function for* W *as*

$$W(x) = \text{sech}^2(x + 1.5)\tanh(x + 1.5) + \text{sech}^2(x - 1.5)\tanh(x - 1.5). \tag{27}$$

This new function W *is displayed in Figure 2 (a). The significance of this new* W *function is that it is proportional to* $V_r'(x)$. *In this case, multiplying Equation* (9) *by* $u_r'(x)$ *and integrating from* $-\infty$ *to* $+\infty$, *we find that*

$$\langle u_r, V_r' u_r \rangle = 0.$$

Because $W \propto V_r'$, $Q_1(\mu)$ *is then always zero, thus the first condition* (14) *is satisfied automatically for all* $\mu > \mu_0$. *However, for this* W *function, the second condition* (24) *is never satisfied. Indeed, we have numerically computed the function* Q_2 *in this condition and plotted it in Figure 2(b); one can see that it is never zero for* $\mu > \mu_0$. *Because this second condition is not met, this family of asymmetric solitons cannot persist and have to disappear under PT-potential perturbations* (27) *as well.*

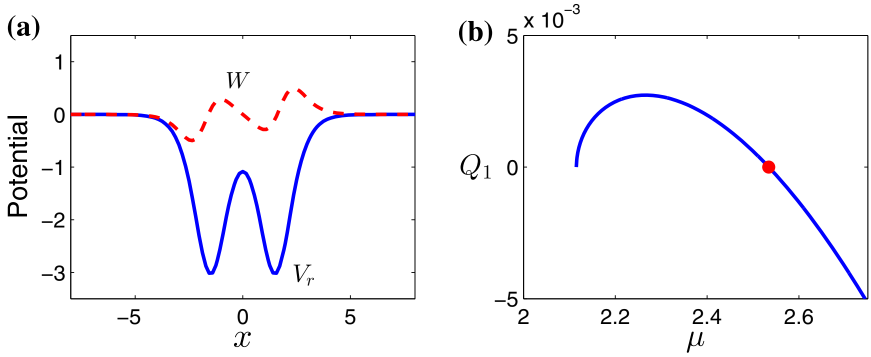

Figure 3. Disappearance of the continuous family of asymmetric solitons under PT-potential perturbations in Example 3. (a) Real potential (25) and PT perturbation (28); (b) function $Q_1(\mu)$ in condition (14); this function is zero at the red-dot point.

EXAMPLE 3. *In the third example, we keep the real potential V_r of Examples 1 and 2, but choose yet another antisymmetric function for W as*

$$W(x) = \text{sech}^2(x+1.5) - \text{sech}^2(x-1.5) - 1.15\left[\text{sech}^2(x+2) - \text{sech}^2(x-2)\right]. \tag{28}$$

This W function is plotted in Figure 3 (a). For this choice of W, the function $Q_1(\mu)$ in condition (14) *is displayed in Figure 3(b). We see that this Q_1 is zero only at a special μ value of $\mu_c \approx 2.5343$, which is marked as a red dot in Figure 3(b). Because of that, under this $W(x)$ perturbation, the continuous family of asymmetric solitons (with $\mu \neq \mu_c$) in the real potential* (25) *are all destroyed.*

What about families of symmetric and antisymmetric solitons of real potentials under weak PT perturbations? In that case, repeating the above perturbation calculations, we can easily show that all conditions, such as (14) and (24), are automatically satisfied due to symmetries of the involved functions. As a consequence, perturbation series (8) for solitary waves $u(x;\epsilon)$ can be constructed to all orders. In addition, the constructed solutions $u(x;\epsilon)$ are PT-symmetric or reducible to PT-symmetric. This indicates that, families of symmetric and antisymmetric solitons of real potentials, under PT perturbations, persist and turn into families of PT-symmetric solitons.

4. No symmetry breaking of solitons in PT-symmetric potentials

In this section, we turn our attention to general PT-symmetric potentials whose imaginary parts are not necessarily small. In such a general PT potential, if Equation (4) admits a branch of PT-symmetric solitons (which is often the case), we ask whether a PT-symmetry-breaking bifurcation can occur, where new branches of non-PT-symmetric solitons bifurcate out from this PT-symmetric branch.

If the potential were real symmetric, symmetry breaking of solitary waves would occur frequently [28–32]. However, when the potential becomes complex and PT-symmetric, we will show that PT-symmetry breaking cannot occur in Equation (4).

Before the analysis, we first introduce some notations and make some basic observations.

4.1. Notations and simple observations

The linearization operator of the solitary-wave equation (4) plays an important role in the bifurcation analysis. Because the solitary wave $u(x)$ is complex-valued due to the complex potential, this linearization operator is vector rather than scalar and can be written as

$$L = \begin{bmatrix} \dfrac{d^2}{dx^2} - V(x) - \mu + 2\sigma|u|^2 & \sigma u^2 \\ \sigma u^{*2} & \dfrac{d^2}{dx^2} - V^*(x) - \mu + 2\sigma|u|^2 \end{bmatrix}. \tag{29}$$

This operator is non-Hermitian. Under the standard inner product (15), the adjoint operator of L is then

$$L^A = \begin{bmatrix} \dfrac{d^2}{dx^2} - V^*(x) - \mu + 2\sigma|u|^2 & \sigma u^2 \\ \sigma u^{*2} & \dfrac{d^2}{dx^2} - V(x) - \mu + 2\sigma|u|^2 \end{bmatrix}. \tag{30}$$

The kernel of the linearization operator L is clearly not empty. Indeed, it is easy to see that

$$L \begin{bmatrix} u \\ -u^* \end{bmatrix} = 0 \tag{31}$$

for all μ values in view of Equation (4), thus the dimension of the kernel of L is at least one.

For any eigenfunction $[f, g]^{\mathrm{T}}$ in the kernel of L, it is easy to see that $[g^*, f^*]^{\mathrm{T}}$ is also in this kernel. By adding these two eigenfunctions, we get an eigenfunction in the form of $[w, w^*]^{\mathrm{T}}$, or equivalently $[\hat{w}, -\hat{w}^*]^{\mathrm{T}}$ if one sets $w = i\hat{w}$. Eigenfunctions in these special forms will be chosen as the basis to span the kernel of L. Similar statements go to the kernel of the adjoint operator L^A as well.

Our basic observation on solitary-wave bifurcations in Equation (4) is that, if a bifurcation occurs at $\mu = \mu_0$, by denoting the corresponding solitary wave and the linearization operator as

$$u_0(x) \equiv u(x; \mu_0), \qquad L_0 \equiv L|_{\mu=\mu_0,\ u=u_0}, \tag{32}$$

then dimension of the kernel of L_0 should be at least two, i.e., $\dim[\ker(L_0)] \geq 2$. This means that the kernel of L_0 should contain at least another localized eigenfunction in addition to $[u_0, -u_0^*]^{\mathrm{T}}$. Using the language of multiplicity of eigenvalues, this means that zero should be a discrete eigenvalue of L_0 with geometric multiplicity at least two. This is a necessary (but not sufficient) condition for bifurcations.

The above necessary condition for bifurcations can be made even more precise. Because L_0 is a fourth-order ordinary differential operator, and the Wronskian of fundamental solutions in its kernel is a nonzero constant (see Remark 2 below), this kernel then cannot contain more than two linearly independent localized eigenfunctions. In other words, $\dim[\ker(L_0)] \leq 2$. Then, combined with the above observation, we see that *a necessary condition for solitary-wave bifurcations in Equation* (4) *is that*

$$\dim[\ker(L_0)]=2. \tag{33}$$

Because of this condition, if a bifurcation occurs at $\mu = \mu_0$, then the kernel of L_0 would contain exactly one additional eigenfunction, which can be denoted as $[\psi, \psi^*]^{\mathrm{T}}$. Thus,

$$L_0 \begin{bmatrix} u_0 \\ -u_0^* \end{bmatrix} = L_0 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} = 0. \tag{34}$$

Because operator L_0 for PT potentials is non-self-adjoint, the kernel of the adjoint operator L_0^A will also play an important role in the bifurcation analysis. When $\dim[\ker(L_0)]=2$, we will show in Remark 2 below that

$$\dim[\ker(L_0^A)]=2 \tag{35}$$

as well. Thus, the kernel of L_0^A also contains exactly two linearly independent localized eigenfunctions: one can be denoted as $[\phi_1, -\phi_1^*]^{\mathrm{T}}$, and the other denoted as $[\phi_2, \phi_2^*]^{\mathrm{T}}$, i.e.,

$$L_0^A \begin{bmatrix} \phi_1 \\ -\phi_1^* \end{bmatrix} = L_0^A \begin{bmatrix} \phi_2 \\ \phi_2^* \end{bmatrix} = 0. \tag{36}$$

Remark 2. Here we show that if $\dim[\ker(L_0)]=2$, then $\dim[\ker(L_0^A)]=2$. To show this, notice that both ordinary differential operators L_0 and L_0^A are four-dimensional. Let us rewrite the operator equations $L_0 Y = 0$ and $L_0^A Y^A = 0$ as systems of four first-order equations, $Z_x = QZ$ and $-Z_x^A = Q^\dagger Z^A$, where $Z = [Y_1, Y_{1x}, Y_2, Y_{2x}]^{\mathrm{T}}$, $Z^A = [Y_1^A, Y_{1x}^A, Y_2^A, Y_{2x}^A]^{\mathrm{T}}$, and the superscript "†" represents Hermitian (i.e., transpose conjugation). Then it is easy to see that $\mathrm{tr}(Q) = 0$, thus Wronskians of fundamental matrices for these two first-order systems are both nonzero constants. It is also known that if the fundamental matrix of the system $Z_x = QZ$ is $Z = M$, then the fundamental matrix of the adjoint system $-Z_x^A = Q^\dagger Z^A$ would be $Z^A = (M^{-1})^\dagger$. Under the assumption of $\dim[\ker(L_0)]=2$, two columns of the fundamental matrix M are localized functions. Because the determinant of M is a nonzero constant, using the M^{-1} formula in terms of cofactors, the other two columns in the adjoint fundamental matrix $(M^{-1})^\dagger$ then are localized functions. Hence, dimension of the kernel of L_0^A is also two.

Remark 3. If the solitary wave $u_0(x)$ is PT-symmetric, then eigenfunctions in Equations (34) and (36) can always be chosen so that they are either PT-symmetric or anti-PT-symmetric, i.e.,

$$\psi^*(x) = \pm\psi(-x), \quad \phi_1^*(x) = \phi_1(-x), \quad \phi_2^*(x) = \pm\phi_2(-x). \tag{37}$$

To prove this, we notice that because $V(x)$ and $u_0(x)$ are both PT-symmetric, by taking the complex conjugate of the second equation in (34) and switching x to $-x$, then L_0 is invariant, and $[\psi^*(-x), \psi(-x)]^T$ is also in the kernel of L_0. Because the kernel of L_0 has dimension two, $[\psi^*(-x), \psi(-x)]^T$ then should be a linear combination of $[u_0(x), -u_0^*(x)]^T$ and $[\psi(x), \psi^*(x)]^T$, i.e.,

$$\psi^*(-x) = c_1\psi(x) + c_2 u_0(x), \tag{38}$$

and

$$\psi(-x) = c_1\psi^*(x) - c_2 u_0^*(x), \tag{39}$$

where c_1, c_2 are certain complex constants. Switching x to $-x$ in (39) and using the PT symmetry of u_0, we get

$$\psi(x) = c_1\psi^*(-x) - c_2 u_0(x). \tag{40}$$

Then adding (38) and (40), we get

$$(1 - c_1)\,[\psi(x) + \psi^*(-x)] = 0. \tag{41}$$

Thus, if $c_1 \neq 1$, then $\psi^*(x) = -\psi(-x)$, i.e., $\psi(x)$ is anti-PT-symmetric. If $c_1 = 1$, by taking the complex conjugate of Equation (39) and then subtracting it from Equation (38), we get $c_2^* = -c_2$, i.e., c_2 is purely imaginary. Denoting $c_2 = i\beta$, where β is a real parameter, Equation (40) can be rewritten as

$$\hat{\psi}^*(x) = \hat{\psi}(-x), \tag{42}$$

where $\hat{\psi} \equiv \psi + \frac{1}{2}i\beta u_0$. It is easy to see that $[\hat{\psi}, \hat{\psi}^*]^T$ is a linear combination of $[\psi(x), \psi^*(x)]^T$ and $[u_0(x), -u_0^*(x)]^T$, hence it is also in the kernel of L_0. Then, instead of $[\psi(x), \psi^*(x)]^T$, we can choose $[\hat{\psi}, \hat{\psi}^*]^T$ in the eigenvalue equation (34); and now $\hat{\psi}$ is PT-symmetric in view of Equation (42). Symmetries (37) for kernels of L_0^A in (36) can be proved in a similar way.

4.2. Nonexistence of PT-symmetry breaking of solitons

The observations and remarks in the previous subsection apply to all bifurcations of solitons in the PT system (4). Now we focus on the particular type of bifurcation: symmetry-breaking bifurcation.

Suppose $u_s(x;\mu)$ is a base branch of PT-symmetric solitons. If a symmetry-breaking bifurcation occurs at $\mu = \mu_0$ of this base branch, with $u_0(x) \equiv u_s(x;\mu_0)$, then eigenfunctions (34) and (36) in the kernels of L_0 and L_0^A should have the following symmetries

$$u_0^*(x) = u_0(-x), \quad \psi^*(x) = -\psi(-x), \tag{43}$$

$$\phi_1^*(x) = \phi_1(-x), \quad \phi_2^*(x) = -\phi_2(-x), \tag{44}$$

i.e., u_0, ϕ_1 are PT-symmetric, and ψ, ϕ_2 anti-PT-symmetric (see Remark 3). In addition, because the two functions in the kernel of L_0 should be linearly independent, $\psi \neq i u_0$. For a similar reason, $\phi_2 \neq i\phi_1$.

In the following, we will show that, in a general PT-symmetric potential, the kernel of L_0 generically cannot contain the second eigenfunction $[\psi, \psi^*]^{\mathrm{T}}$ with anti-PT-symmetry (43). Thus, the necessary condition for symmetry breaking is not met. We will also show that even if such a second eigenfunction $[\psi, \psi^*]^{\mathrm{T}}$ appears in the kernel of L_0, symmetry breaking still cannot occur.

First, we show that in a general PT-symmetric potential, the kernel of L_0 generically cannot contain the second eigenfunction $[\psi, \psi^*]^{\mathrm{T}}$ with anti-PT-symmetry (43). Using the language of multiplicity of eigenvalues, we will show that when the zero eigenvalue of L_0 has algebraic multiplicity higher than one, its geometric multiplicity generically cannot be higher than one with an anti-PT-symmetric second eigenfunction $[\psi, \psi^*]^{\mathrm{T}}$.

Suppose when $\mu = \mu_0$, the zero eigenvalue of L_0 has algebraic multiplicity higher than one and geometric multiplicity two, and the second eigenfunction $[\psi, \psi^*]^{\mathrm{T}}$ of this zero eigenvalue is anti-PT-symmetric. When $\mu \neq \mu_0$, the zero eigenvalue of this $[\psi, \psi^*]^{\mathrm{T}}$ eigenmode would move out of the origin. Let us calculate this eigenvalue of L for $|\mu - \mu_0| \ll 1$ by perturbation methods. The eigenvalue equation is

$$L \begin{bmatrix} w \\ w^* \end{bmatrix} = \lambda \begin{bmatrix} w \\ w^* \end{bmatrix}. \tag{45}$$

When $|\mu - \mu_0| \ll 1$, we can expand the eigenvalue λ and the eigenfunction $[w, w^*]^{\mathrm{T}}$ into a perturbation series,

$$\lambda = \lambda_1(\mu - \mu_0) + \lambda_2(\mu - \mu_0)^2 + \cdots, \tag{46}$$

$$\begin{bmatrix} w \\ w^* \end{bmatrix} = \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} + (\mu - \mu_0)\begin{bmatrix} w_1 \\ w_1^* \end{bmatrix} + (\mu - \mu_0)^2 \begin{bmatrix} w_2 \\ w_2^* \end{bmatrix} + \cdots. \tag{47}$$

Similarly, we also expand the operator L into a perturbation series,

$$L = L_0 + (\mu - \mu_0)L_1 + (\mu - \mu_0)L_2 + \cdots. \tag{48}$$

When this eigenmode $[w, w^*]^{\mathrm{T}}$ moves out of the origin, using similar arguments as in Remark 3, we can show that w can be made PT-symmetric or anti-PT-symmetric. Because $w \to \psi$ as $\mu \to \mu_0$ and ψ is anti-PT-symmetric, w then should be anti-PT-symmetric. As a consequence, the other functions $w_1, w_2, \ldots$ in the w expansion are also anti-PT-symmetric.

Substituting the above expansions into Equation (45), the $O(1)$ equation is satisfied automatically because $[\psi, \psi^*]^{\mathrm{T}}$ is in the kernel of L_0 by assumption. At $O(\mu - \mu_0)$, we get

$$L_0 \begin{bmatrix} w_1 \\ w_1^* \end{bmatrix} = \lambda_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} - \begin{bmatrix} g_1 \\ g_1^* \end{bmatrix}, \tag{49}$$

where

$$\begin{bmatrix} g_1 \\ g_1^* \end{bmatrix} \equiv L_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix}. \tag{50}$$

The function g_1 is anti-PT-symmetric in view that L_1 is PT-symmetric and ψ anti-PT-symmetric.

Because the kernel of L_0 has dimension two under the current assumption, the kernel of L_0^A has dimension two as well (see Remark 2), and the two linearly independent eigenfunctions in the kernel of L_0^A are denoted in Equation (36) with symmetries (44). Then in order for the w_1 equation (49) to be solvable, the solvability condition is that the right side of (49) be orthogonal to the two eigenfunctions in the kernel of L_0^A, i.e.,

$$\left\langle \begin{bmatrix} \phi_1 \\ -\phi_1^* \end{bmatrix}, \quad \lambda_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} - \begin{bmatrix} g_1 \\ g_1^* \end{bmatrix} \right\rangle = 0, \tag{51}$$

$$\left\langle \begin{bmatrix} \phi_2 \\ \phi_2^* \end{bmatrix}, \quad \lambda_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} - \begin{bmatrix} g_1 \\ g_1^* \end{bmatrix} \right\rangle = 0. \tag{52}$$

These two conditions give two different expressions for the same eigenvalue coefficient λ_1. In order for these two formulae to be consistent, the following compatibility condition must be satisfied,

$$\frac{\mathrm{Im}\langle \phi_1, g_1 \rangle}{\mathrm{Im}\langle \phi_1, \psi \rangle} = \frac{\mathrm{Re}\langle \phi_2, g_1 \rangle}{\mathrm{Re}\langle \phi_2, \psi \rangle}. \tag{53}$$

Here "Re" and "Im" represent the real and imaginary parts of a complex number. Recalling the PT symmetry of ϕ_1 and anti-PT-symmetries of ϕ_2, ψ and g_1, $\langle \phi_1, g_1 \rangle$ and $\langle \phi_1, \psi \rangle$ are purely imaginary, and $\langle \phi_2, g_1 \rangle$, $\langle \phi_2, \psi \rangle$ are strictly real. In addition, recalling that $\phi_2 \neq i\phi_1$, Equation (53) then is a nontrivial compatibility condition for the existence of a second eigenfunction $[\psi, \psi^*]^{\mathrm{T}}$ in the kernel of L_0. Because this compatibility condition is not satisfied generically, the necessary condition for symmetry breaking is then not met.

Next, we show that even if such a second eigenfunction $[\psi, \psi^*]^T$ appears in the kernel of L_0, symmetry breaking still cannot occur, because such a bifurcation further requires an infinite number of additional nontrivial conditions to be satisfied simultaneously, which is impossible in practice.

Suppose at a propagation constant $\mu = \mu_0$, the kernels of L_0 and L_0^A have dimension two, and their eigenfunctions are given in Equations (34) and (36) with symmetries (43) and (44). If a symmetry-breaking bifurcation occurs at this point, then two new branches of non-PT-symmetric solitons would bifurcate out from $u_0(x)$ on only one side of $\mu = \mu_0$. Let us seek such non-PT-symmetric solitons near $\mu = \mu_0$ by perturbation methods.

Suppose these new solitons bifurcate to the right side of μ_0, then their perturbation series can be written as

$$u_a(x;\mu) = \sum_{k=0}^{\infty} (\mu - \mu_0)^{k/2} u_k(x). \tag{54}$$

Substituting this perturbation series into Equation (4), the $O(1)$ equation is satisfied automatically because u_0 is a solitary wave at $\mu = \mu_0$. At $O[(\mu - \mu_0)^{1/2}]$, the equation for u_1 is

$$L_0 \begin{bmatrix} u_1 \\ u_1^* \end{bmatrix} = 0. \tag{55}$$

In view of the kernel structure of L_0 in Equation (34), we see that

$$\begin{bmatrix} u_1 \\ u_1^* \end{bmatrix} = c_1 \begin{bmatrix} \psi \\ \psi^* \end{bmatrix} + d_1 \begin{bmatrix} u_0 \\ -u_0^* \end{bmatrix}, \tag{56}$$

where c_1, d_1 are constants. In order for the resulting u_1^* formula to be complex conjugate of the u_1 formula, c_1 must be strictly real, and d_1 purely imaginary. Then the $d_1 u_0$ term in u_1, when combined with the leading-order term u_0 in the expansion (54), only amounts to a phase shift to $u_a(x;\mu)$, which is insignificant in view of Remark 1. Thus, we can set $d_1 = 0$ without loss of generality. Then the u_1 solution becomes

$$u_1 = c_1 \psi, \tag{57}$$

where c_1 is a real constant.

For symmetry-breaking bifurcation to occur, c_1 should be nonzero. In this case, the first-two-term solution of (54),

$$u_0 + (\mu - \mu_0)^{1/2} c_1 \psi,$$

is not PT-symmetric, nor is it reducible to PT-symmetric, because u_0 is PT-symmetric but ψ is anti-PT-symmetric and $\psi \neq i u_0$. This non-PT-symmetry will not be affected by higher-order terms of (54), thus the resulting solution $u_a(x;\mu)$ in (54) would be non-PT-symmetric.

At $O(\mu - \mu_0)$, the equation for u_2 is

$$L_0 \begin{bmatrix} u_2 \\ u_2^* \end{bmatrix} = \begin{bmatrix} g_2 \\ g_2^* \end{bmatrix}, \tag{58}$$

where

$$g_2 = u_0 - \sigma c_1^2 (2u_0|\psi|^2 + u_0^* \psi^2). \tag{59}$$

Here the u_1 solution (57) has been utilized. Because u_0 is PT-symmetric and ψ anti-PT-symmetric, g_2 is PT-symmetric.

The solvability conditions of Equation (58) are that its right-hand side be orthogonal to the kernels of L_0^A in Equation (36), i.e.,

$$\text{Im}\langle \phi_1, g_2 \rangle = \text{Re}\langle \phi_2, g_2 \rangle = 0. \tag{60}$$

Recalling the symmetries of ϕ_1 and ϕ_2 in (44) as well as the PT-symmetry of g_2, we see that $\langle \phi_1, g_2 \rangle$ is strictly real, and $\langle \phi_2, g_2 \rangle$ is purely imaginary, thus both solvability conditions in Equation (60) are automatically satisfied. As a result, a localized particular solution $\hat{u}_2$ can be found. This particular solution can be split into two parts, corresponding to the two terms of g_2 in (59):

$$\hat{u}_2 = \hat{u}_{21} + c_1^2 \hat{u}_{22}. \tag{61}$$

Here, $\hat{u}_{21}$ solves

$$L_0 \begin{bmatrix} \hat{u}_{21} \\ \hat{u}_{21}^* \end{bmatrix} = \begin{bmatrix} u_0 \\ u_0^* \end{bmatrix}, \tag{62}$$

and $\hat{u}_{22}$ solves

$$L_0 \begin{bmatrix} \hat{u}_{22} \\ \hat{u}_{22}^* \end{bmatrix} = \begin{bmatrix} -\sigma(2u_0|\psi|^2 + u_0^* \psi^2) \\ -\sigma(2u_0|\psi|^2 + u_0^* \psi^2)^* \end{bmatrix}. \tag{63}$$

Because both terms of g_2 are PT-symmetric, $\hat{u}_{21}$ and $\hat{u}_{22}$ can be made PT-symmetric as well. The general solution of u_2 is then this particular solution plus the homogeneous solutions. Similar to the u_1 solution case, we can exclude the homogeneous u_0 term and set

$$u_2 = \hat{u}_{21} + c_1^2 \hat{u}_{22} + c_2 \psi \tag{64}$$

without loss of generality. Here c_2 is another real constant to be determined.

The calculations so far have been benign. However, from the next order, we will start to get an infinite number of additional conditions which have to be satisfied in order for the perturbation series (54) to be constructed. Let us begin with the u_3 equation, which is

$$L_0 \begin{bmatrix} u_3 \\ u_3^* \end{bmatrix} = \begin{bmatrix} g_3 \\ g_3^* \end{bmatrix}, \tag{65}$$

where

$$g_3 = u_1 - \sigma\left(2u_0^* u_1 u_2 + 2u_0 u_1^* u_2 + 2u_0 u_1 u_2^* + |u_1|^2 u_1\right). \tag{66}$$

Substituting the u_1 and u_2 solutions (57) and (64) into g_3, we get

$$g_3 = c_1\left(g_{31} + c_1^2 g_{32} + c_2 g_{33}\right), \tag{67}$$

where

$$g_{31} = \psi - 2\sigma\left(u_0^* \psi \hat{u}_{21} + u_0 \psi^* \hat{u}_{21} + u_0 \psi \hat{u}_{21}^*\right),$$

$$g_{32} = -2\sigma\left(u_0^* \psi \hat{u}_{22} + u_0 \psi^* \hat{u}_{22} + u_0 \psi \hat{u}_{22}^*\right) - \sigma|\psi|^2\psi,$$

and

$$g_{33} = -2\sigma(2u_0|\psi|^2 + u_0^* \psi^2).$$

Notice that both g_{31} and g_{32} are anti-PT-symmetric, and g_{33} is PT-symmetric.

The solvability conditions of Equation (65) are

$$\mathrm{Im}\langle \phi_1, g_3\rangle = \mathrm{Re}\langle \phi_2, g_3\rangle = 0. \tag{68}$$

Using the g_3 formula (67) and the symmetry properties of the involved functions, these solvability conditions then yield a condition for symmetry-breaking bifurcations as

$$\frac{\mathrm{Im}\langle \phi_1, g_{31}\rangle}{\mathrm{Im}\langle \phi_1, g_{32}\rangle} = \frac{\mathrm{Re}\langle \phi_2, g_{31}\rangle}{\mathrm{Re}\langle \phi_2, g_{32}\rangle}. \tag{69}$$

In view of the PT symmetry of ϕ_1 and anti-PT-symmetries of ϕ_2, g_{31} and g_{32}, $\langle \phi_1, g_{31}\rangle$ and $\langle \phi_1, g_{32}\rangle$ are purely imaginary, and $\langle \phi_2, g_{31}\rangle$, $\langle \phi_2, g_{32}\rangle$ are strictly real. In addition, $\phi_2 \neq i\phi_1$. Thus, Equation (53) is a nontrivial condition for symmetry-breaking bifurcations.

When we pursue this perturbation expansion to higher orders, infinitely more nontrivial conditions will also appear (because these calculations are straightforward, details are omitted here for brevity). The fundamental reason for this infinite number of conditions is that, due to the phase invariance of solitary waves, when we solve the inhomogeneous u_n equation, we can only introduce one real parameter into the u_n solution, which is the coefficient of the ψ term. But each u_n equation has two solvability conditions (because the kernel of L_0^A has dimension two), and neither solvability condition can be satisfied automatically from symmetry considerations (for $n \geq 3$). This means that we have twice as many solvability conditions as real parameters. Because of this, we have an overdetermined system for real parameters, which results in an infinite number of nontrivial conditions for symmetry-breaking bifurcations. In one spatial dimension, Equation (4) does not admit any additional spatial symmetries (except the PT symmetry). Because of this lack of additional symmetries, it is practically impossible for these infinite conditions to be satisfied simultaneously.

From the above analysis, we see that in a PT-symmetric potential, the necessary condition for symmetry breaking, i.e., dim[ker(L_0)]=2 with eigenfunction symmetries (43), is generically not satisfied. Even if that necessary condition is met, symmetry breaking still requires infinitely more conditions to be satisfied simultaneously, which is practically impossible for the 1D system (4). Thus, we conclude that symmetry breaking cannot occur in the PT-symmetric system (4).

Regarding the three specific examples (25)–(28) in the PT system (6) for various values of ϵ (not necessarily small), we have found numerically that the kernel of L never contains a second eigenfunction $[\psi, \psi^*]^T$ with anti-PT-symmetry at any μ value, thus the necessary condition for symmetry breaking is not satisfied. This numerical finding corroborates our analytical result that this necessary condition for symmetry breaking is generically not met for a PT-symmetric potential.

5. Summary and discussion

In this article, we have investigated the possibility of continuous families of non-PT-symmetric solitons in 1D PT-symmetric potentials. We have shown that families of asymmetric solitons in a real symmetric potential are destroyed when this real potential is perturbed by weak PT-symmetric perturbations. We have also shown that in a general 1D PT-symmetric potential, symmetry breaking of PT-symmetric solitons cannot occur. This contrasts real symmetric potentials where symmetry breaking of solitary waves often takes place.

Based on these findings and Remark 1, we make the following conjecture:

The one-dimensional NLS equation (1) *with a complex PT-symmetric potential cannot admit continuous families of non-PT-symmetric solitary waves.*[1]

Equivalently, this conjecture says that all continuous families of solitary waves in a 1D PT-symmetric potential must be PT-symmetric.

The absence of continuous families of non-PT-symmetric solitons in 1D PT-symmetric potentials is an interesting phenomenon, because it contrasts real symmetric potentials, where families of asymmetric solitons often exist. This means that, even though PT-symmetric potentials can support continuous families of solitons, which makes such dissipative potentials analogous

[1]It is noted that for a PT-symmetric periodic potential $V(x)$, a spatial translation of this potential by half a period $T/2$, i.e., $V(x+T/2)$, is also PT-symmetric since $V^*(x+T/2) = V(-x+T/2)$. Soliton families that are PT-symmetric in the shifted potential $V(x+T/2)$, which clearly could exist, would appear non-PT-symmetric in the original PT potential $V(x)$. To avoid this ambiguity, for a PT-symmetric periodic potential $V(x)$, non-PT-symmetric solitary waves in this conjecture would mean solitons that are non-PT-symmetric with respect to both the original potential $V(x)$ and the shifted potential $V(x+T/2)$.

to conservative real potentials, the types of soliton families allowed by PT-symmetric potentials are nonetheless limited. So the dissipative nature of a PT-symmetric potential does leave its signature on the structure of its solitary waves, and this signature distinguishes PT-symmetric potentials from real symmetric ones.

We would like to point out that the above conjecture does not exclude the possibility of 1D PT-symmetric potentials supporting *isolated* non-PT-symmetric solitons (i.e., non-PT-symmetric solitons existing at isolated propagation-constant values). In a certain finite-dimensional PT-symmetric system (the quadrimer model), isolated non-PT-symmetric solutions have been reported [22]. In the 1D NLS equation (1) with a PT-symmetric potential, such isolated non-PT-symmetric solitons can also exist, as our preliminary numerics has shown. These isolated solitons are reminiscent of dissipative solitons in the Ginzburg–Landau and other dissipative equations [4], and they can coexist with continuous families of solitons in a PT-symmetric potential.

The analytical results in this article can be extended to a large class of higher-dimensional NLS equations with PT-symmetric potentials, but not to all of them. In higher spatial dimensions, the PT symmetry of a potential is compatible with certain other spatial symmetries, such as x-symmetry or y-symmetry. For instance, we can easily construct 2D complex potentials $V(x, y)$ with the following PT-symmetry as well as x-symmetry,

$$V^*(x, y) = V(-x, -y), \quad V(x, y) = V(-x, y),$$

i.e., the real part of the potential is symmetric in both x and y, but the imaginary part of the potential is symmetric in x and antisymmetric in y. Because of this additional x-symmetry, continuous families of non-PT-symmetric solitons *can* exist in this 2D PT-symmetric potential, and symmetry breaking of PT-symmetric solitons *can* occur. Putting this 2D problem in the framework of the earlier analysis in this article, the reason for the existence of PT-symmetry breaking and families of non-PT-symmetric solitons in this 2D PT potential is that, due to the additional x-symmetry of the potential and its ramifications for the symmetries of the underlying solitons and eigenfunctions in the kernels of the linearization operators, those infinite conditions in our earlier analysis can now be all satisfied. Details on this 2D problem will be reported elsewhere. However, if this 2D PT-symmetric potential $V(x, y)$ does not admit those additional spatial symmetries (such as x-symmetry and y-symmetry), then the analysis in this article would still apply, and families of non-PT-symmetric solitons still cannot be expected. Thus, absence of continuous families of non-PT-symmetric solitons in PT-symmetric potentials is not restricted to one spatial dimension, but holds for most higher-dimensional PT-symmetric potentials as well.

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