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Embedded solitons: solitary waves in resonance with the linear spectrum

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Abstract

It is commonly held that a necessary condition for the existence of solitons in nonlinear-wave systems is that the soliton's frequency (spatial or temporal) must not fall into the continuous spectrum of radiation modes. However, this is not always true. We present a new class of *codimension-one* solitons (i.e., those existing at isolated frequency values) that are *embedded* into the continuous spectrum. This is possible if the spectrum of the linearized system has (at least) two branches, one corresponding to exponentially localized solutions, and the other to radiation modes. An embedded soliton (ES) is obtained when the latter component exactly vanishes in the solitary-wave's tail. The paper contains both a survey of recent results obtained by the authors and some new results, the aim being to draw together several different mechanism underlying the existence of ESs. We also consider the distinctive properties of *semi-stability* of ESs, and *moving* ESs. Results are presented for four different physical models, including an extended fifth-order KdV equation describing surface waves in inviscid fluids, and three models from nonlinear optics. One of them pertains to a resonant Bragg grating in an optical fiber with a cubic nonlinearity, while two others describe second-harmonic generation (SHG) in the temporal or spatial domain (i.e., respectively, propagating pulses in nonlinear-optical fibers, or stationary patterns in nonlinear planar waveguides). Special attention is paid to the SHG model in the temporal domain for a case of competing quadratic and cubic nonlinearities. In particular, a new result is that when both harmonics have anomalous dispersion, an ES can exist which is, virtually, completely stable.

Keywords: Embedded solitons; Solitary waves; Linear spectrum

1. Introduction

Recent studies have revealed a novel type of solitary waves (which we will loosely call "solitons", without assuming integrability of the underlying models) that are *embedded* into the continuous spectrum, i.e., the soliton's internal frequency is in resonance with linear (radiation) waves. Generally, such a soliton should not exist, one finding instead a quasi-soliton (a delocalized one) with non-vanishing oscillatory tails (radiation component) [1]. Nevertheless, *bona fide* (exponentially decaying) solitons can exist as *codimension-one* solutions if, at discrete values of the (quasi-)soliton's internal frequency, the amplitude of the tail exactly vanishes, while the soliton remains embedded into the continuous spectrum of the radiation modes. This requires the spectrum of the corresponding

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linearized system to consist of (at least) two branches, one corresponding to exponentially localized solutions, and the other to oscillatory radiation modes. In terms of the corresponding ordinary differential equations (ODEs) for the traveling-wave solutions, the origin must be a *saddle-center*, i.e. its linearization gives rise to both real and pure imaginary eigenvalues.

Examples of such embedded solitons (ESs) were found in water-wave models [2–4] and in several nonlinear-optical ones, including a Bragg-grating model with forward and backward waves [5], and a model of second-harmonic generation (SHG) in the presence of a Kerr nonlinearity [6]. The term "ES" was proposed in the latter work.

ESs are interesting for several reasons, firstly because they frequently appear when higher-order (singular) perturbations are added to the system, which may completely change its soliton spectrum (see, e.g., [5]). Secondly, optical ESs have considerable potential for applications, just because they are isolated solitons, rather than members of continuous families. Finally, and most crucial for physical applications, ESs are *semi*-stable objects. That is, as argued in [6] heuristically, in a general form applicable to ESs in any system, and checked numerically for the SHG model with the additional defocusing Kerr nonlinearity, ESs are stable in the linear approximation, but do have a slowly growing (sub-exponential) *one-sided* nonlinear instability (see Section 4). This semi-stability property of ESs was further rigorously proved in [4] for long surface waves described by an extended Korteweg–de Vries (KdV) equation.

In the next section, we discuss the existence and stability of ESs in four different nonlinear partial differential equation (PDE) models. Mathematically speaking, in each case the reduced traveling-wave or steady-state ODEs have the structure of a fourth-order, reversible Hamiltonian system. In accord with what was said above, we focus on a parameter region where the origin (trivial fixed point) in these ODE systems is a saddle-center. That is, after diagonalizing the system, one two-dimensional (2D) component of the dynamical system gives rise to imaginary eigenvalues $\pm i\omega$ (corresponding to a continuous radiation branch in the linear spectrum of the PDE system), and the other to real eigenvalues $\pm\lambda$ (corresponding to a gap in a section of the radiation spectrum).

All four models considered in this paper share the feature that ESs exist as non-generic, codimension-one solutions. Section 3 then presents three different views as to why this should be, which together provide general insight into the existence and multiplicity of ESs. Section 4 goes on to discuss their stability, arguing that ESs, in general, are neutrally stable linearly, but suffer from a one-sided sub-exponential instability. This situation has been termed *semi-stability* in [6] (see also [4]). Section 5 treats "moving" ESs and, finally, Section 6 draws conclusions and briefly discusses potential physical applications of ESs in optical memory devices.

2. Physical examples

2.1. An extended fifth-order KdV equation

One of the first systems in which ESs were found (although without being given that name) is an extended fifth-order KdV equation [2–4],

$$u_t = [\tfrac{2}{15}u_{xxxx} - bu_{xx} + au + \tfrac{3}{2}u^2 + \mu(\tfrac{1}{2}(u_x)^2 + (uu_x)_x)]_x, \tag{1}$$

which with $\mu = 0$ reduces to the usual fifth-order KdV equation studied by a number of authors (see, e.g., [7,8]). The extended form (1) may be derived via a regular Hamiltonian perturbation theory from an exact Euler equation formulation for water-waves with surface tension [9]. Looking for traveling-wave solutions $u(x - ct)$, integrating once, setting the constant of integration to be 0, and absorbing the linear term $\sim u_x$ by redefining a, one arrives at the following ODE (the prime stands for $\mathrm{d}/\mathrm{d}(x - ct)$),

$$\tfrac{2}{15}u'''' - bu'' + au + \tfrac{3}{2}u^2 + \mu[\tfrac{1}{2}(u')^2 + (uu')'] = 0. \tag{2}$$

When $a < 0$, Eq. (2) is in the ES regime, since the linearization around the origin, $u = 0$, yields both real and imaginary eigenvalues. Note that, for the particular case $\mu = 0$, it has been proved that there are no dynamical orbits homoclinic to the origin [10]. Nevertheless, at least in the limit $a \to -0$, the system does possess a large family of homoclinic connections to periodic orbits, rather than to fixed points [4,11]. The latter generic solutions correspond to the above-mentioned delocalized solitary waves [1].

However, ESs *do* exist in the case $\mu = 1$ [2], where one can find the *explicit* family of homoclinic-to-zero solutions,

$$u(t) = 3\left(b + \frac{1}{2}\right)\operatorname{sech}^2\left(\sqrt{\frac{3(2b+1)}{4}}\,t\right), \qquad a = \frac{3}{5}(2b+1)(b-2), \quad b \geq -\frac{1}{2}. \tag{3}$$

Moreover, this curve (family) of ESs was found in [2] to be only the first in a countable set of curves that appear to bifurcate from $a = 0$ at a discrete set of negative values of b, see Fig. 1. Note that there are numerical difficulties in computing up to the limit point $a = 0$ for $b < 0$; this is because, as will be motivated in Section 3.3, the bifurcation of these solutions from $a = 0$ is a "beyond-all-orders" effect.

We remark that Grimshaw and Cook [12] found a similar bifurcation-type phenomenon in a system of two coupled KdV equations, although they did not explicitly identify the vanishing of the tail amplitude of the delocalized solitary waves as what we now call ESs. Also, Fujioka and Espinosa [13] found a single *explicit* ES in a higher-order NLS equation with a quintic nonlinearity.

Note also the following feature of this family of ES states. The first member of the family, viz., the explicit solution (3) is a fundamental "ground state", while all other solution branches represent "excited states", having the form of the ground-state soliton with small-amplitude "ripples" superimposed on it. As yet unpublished numerical results suggest that the ground-state soliton appears to be dynamically stable (in the sense described in Section 4).

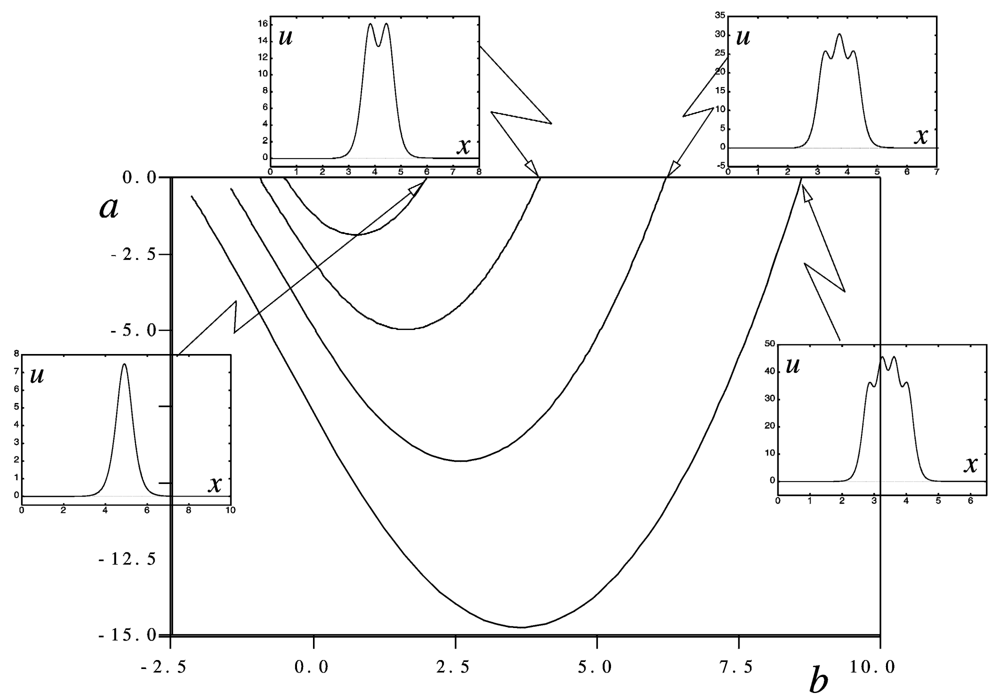

Fig. 1. The region of existence of the ESs for Eq. (2).

2.2. A generalized massive Thirring model

The first example of ESs in nonlinear optics was found in a generalized Thirring model (GTM), introduced long ago in [14,15]. ESs appear in this model when additional wave-propagation (second-derivative) terms are included (k being the carrier wavenumber), so that the model takes the form

$$\begin{aligned} &\mathrm{i}u_t + \mathrm{i}u_x + (2k)^{-1}(u_{xx} - u_{tt}) + [\tfrac{1}{2}|u|^2 + |v|^2]u + v = 0, \\ &\mathrm{i}v_t - \mathrm{i}v_x + (2k)^{-1}(v_{xx} - v_{tt}) + [\tfrac{1}{2}|v|^2 + |u|^2]v + u = 0. \end{aligned} \tag{4}$$

Here $u(x,t)$ and $v(x,t)$ are right-traveling and left-traveling waves coupled by resonant reflections on the grating. Soliton solutions are sought for as $u(x,t) = \exp(-\mathrm{i}\,\Delta\omega t)\,U(\xi)$, $v(x,t) = \exp(-\mathrm{i}\,\Delta\omega t)\,V(\xi)$, where $\xi \equiv x - ct$, c and $\Delta\omega$ being velocity and frequency shifts. The functions $U(\xi)$ and $V(\xi)$ satisfy ODEs

$$\begin{aligned} &\chi U + \mathrm{i}(1-C)U' + DU'' + [\tfrac{1}{2}|U|^2 + |V|^2]U + V = 0, \\ &\chi V - \mathrm{i}(1+C)V' + DV'' + [\tfrac{1}{2}|V|^2 + |U|^2]V + U = 0, \end{aligned} \tag{5}$$

where $\chi \equiv \Delta\omega + (\Delta\omega)^2/2k$, the effective velocity is $C \equiv (1 + \Delta\omega/k)c$, and an effective *dispersion coefficient* is $D \equiv (1 - c^2)/2k$. The same equations were derived in [5] for: (i) temporal-soliton propagation in nonlinear fiber gratings, including spatial-dispersion effects, and (ii) spatial solitons in a planar waveguide with a Bragg grating in the form of parallel scores, taking diffraction into regard, with t replaced by the propagation coordinate z, while x is the transverse coordinate. If we set $C = 0$ then Eq. (5) admit the further invariant reduction $U = V^*$, leading to a single ODE [5],

$$DU'' + \mathrm{i}U' + \chi U + \tfrac{3}{2}|U|^2 U + U^* = 0, \tag{6}$$

which is equivalent to a (Hamiltonian and reversible) system of four first-order equations for the real and imaginary parts of U and U'. In this system, the origin is a saddle-center, provided that $D > 0$ and $|\chi| < 1$. In [5], it was found numerically that Eq. (6) admits exactly *three* branches of the fundamental ESs (in contrast to Eq. (1) where countably many ES states have been found). Also *moving* ($c \neq 0$) ESs, satisfying the eigth-order system (5), have also recently been found in [16], see Section 5 for more details.

2.3. A three-wave interaction model

ESs can be found in far greater abundance in a model for *spatial solitons*, assuming a planar waveguide with a quadratic ($\chi^{(2)}$) nonlinearity [17], where two fundamental-harmonic (FH) waves $v_{1,2}$ are coupled by the Bragg reflections from a set of parallel scores. These two waves then interact nonlinearly, and generate a third wave, the second-harmonic (SH), with its wave-vector equal to the sum of those of the two FH components. The set of equations are

$$\mathrm{i}(v_{1,2})_z \pm \mathrm{i}(v_{1,2})_x + v_{2,1} + v_3 v_{2,1}^* = 0, \qquad 2\mathrm{i}(v_3)_z - qv_3 + D(v_3)_{xx} + v_1 v_2 = 0. \tag{7}$$

Here v_3 is the SH field, x the normalized transverse coordinate, q is a mismatch parameter, and D is an effective diffraction coefficient. Solutions to Eq. (7) are sought in the form $v_{1,2}(x,z) = \exp(\mathrm{i}kz)\,u_{1,2}(\xi)$, $v_3(x,z) = \exp(2\mathrm{i}kz)\,u_3$, with $\xi \equiv x - cz$, c being the slope of the soliton's axis relative to the propagation direction z. In [18], many ESs of the zero-walkoff type ($c = 0$) were found (summarized in Fig. 2), in which case the ODEs reduce, as in Eq. (5), to a fourth-order real system. When $c \neq 0$, one cannot assume all the amplitudes to be real. In this case, one finds "moving" ESs as solutions to an eighth-order real ODE system (see Section 5).

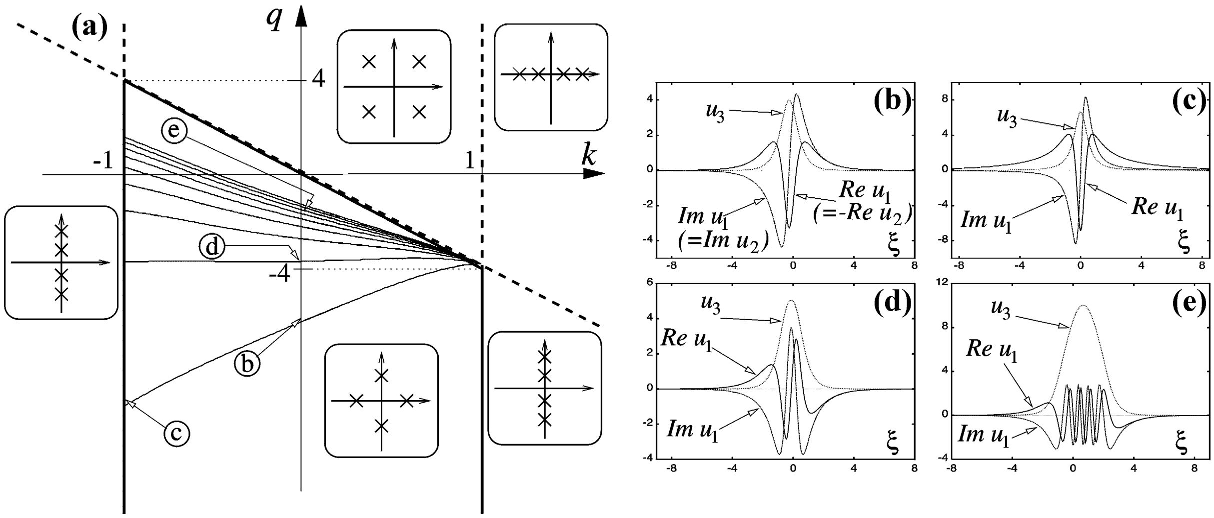

Fig. 2. (a) The (k, q) parameter plane of the three-wave model (7). The linear analysis (summarized in the inset boxes) shows that ESs can occur only in the region between the bold lines. The bundle of curves emanating from the point ($k = 1, q = -4$) are branches of embedded-soliton solutions with $c = 0$. The panels (b)–(e) depict solutions at the labeled points.

2.4. An SH-generation system

The model we shall study in most detail is that for which the term "ES" was first proposed in [6], viz., a nonlinear-optical medium with competing quadratic and cubic nonlinearities [19,20]

$$\mathrm{i}u_z + \tfrac{1}{2}u_{tt} + u^*v + \gamma_1(|u|^2 + 2|v|^2)u = 0, \qquad \mathrm{i}v_z - \tfrac{1}{2}\delta v_{tt} + qv + \tfrac{1}{2}u^2 + 2\gamma_2(|v|^2 + 2|u|^2)v = 0. \tag{8}$$

Here, u and v are FH and SH amplitudes, $-\delta$ is a relative SH/FH dispersion coefficient, q is a phase-velocity mismatch, and $\gamma_{1,2}$ are cubic (Kerr) nonlinear coefficients. In the absence of the Kerr nonlinearities, these equations are the same as those used by Karamzin and Sukhorukov in 1974 [21] to obtain their famous $\chi^{(2)}$ soliton solution, in which both the FH and SH fields are proportional to sech^2. A detailed analysis of the higher-order soliton solutions in that model with purely quadratic nonlinearity can be found in [22]. However, the solutions that we will consider here are in a different class, in that the FH field will be more like a sech than a sech^2.

The particular case of Eq. (8) with $\delta = -\frac{1}{2}$ is specially important, as it corresponds, with t replaced by the transverse coordinate x, to an SH-generation model in the spatial domain (in fact, in a nonlinear planar optical waveguide). In this special case, the model (8) is Galilean invariant, which allows one to generate a whole family of "moving" solitons from the single zero-walkoff one [23]. At all other values of δ, construction of a "moving" (non-zero-walkoff) soliton is a non-trivial problem.

Stationary solutions to Eq. (8) are sought for in the form $u = U(t)\exp(\mathrm{i}kz)$, $v = V(t)\exp(2\mathrm{i}kz)$, where k is real, and U, V satisfy ODEs

$$\begin{aligned} &\tfrac{1}{2}U'' - kU + U^*V + \gamma_1(|U|^2 + 2|V|^2)U = 0, \\ &-\tfrac{1}{2}\delta V'' + (q - 2k)V + \tfrac{1}{2}U^2 + 2\gamma_2(|V|^2 + 2|U|^2)V = 0. \end{aligned} \tag{9}$$

In [6], ES solutions to these equations were found for $\delta > 0$ and $\gamma_{1,2} < 0$ (which implies anomalous and normal dispersions, respectively, at FH and SH, and self-defocusing Kerr nonlinearity. Alternatively, the same case may be

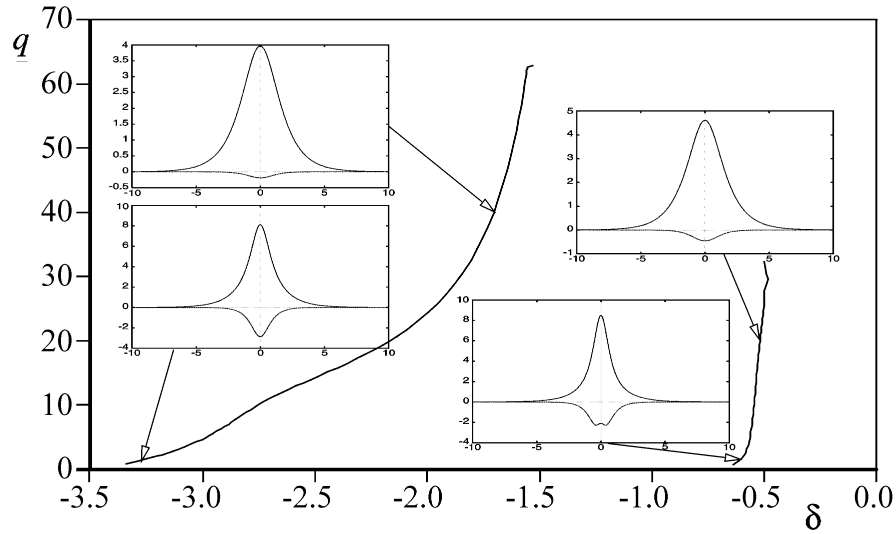

Fig. 3. Two branches, in the (δ, q) plane, of ES solutions to the SHG model (8) with $k = 0.3$ and $\gamma_1 = \gamma_2 = 0.05$. The insets depict the profiles of $U(t)$ (positive component) and $V(t)$ (negative one).

physically realized as the normal and anomalous dispersions at FH and SH and self-focusing Kerr nonlinearity). In the same work, stability of these ESs was studied in detail.

Here, we shall present new results for $\delta < 0$, which corresponds to a more common case where the dispersion has the same sign at both harmonics. The results can be naturally displayed in the form of curves of ESs in the (δ, q) or (q, k) parameter planes, see Figs. 3 and 4.

3. Existence

We now give three distinct explanations of why and how an ES may exist.

3.1. Nonlinearizablity

Consider first the ODE system (9). It is important to note that the system gets fully decoupled in the linear approximation, the linearization of its second equation immediately telling one that no true soliton (with exponentially decaying tails) can exist inside the continuous (radiation-mode) SH spectrum. However, it may happen that the tail of the soliton's SH component decays at the same rate as the *square* of the tail of the FH component. In that case, the second equation of the system (9) is *nonlinearizable*, which opens the way for the existence of truly localized solitons inside the continuous spectrum. Note from the profiles of the solutions in Figs. 3 and 4, that the V-component appears to decay to zero much faster than U, in accordance with this nonlinearizability property.

Let us remark on some qualitative features of the ES branches presented in Fig. 3. First, the computed branches appear to end in "mid air". At the low q end, we have a boundary. The condition for the origin of the ODE system (9) to be a saddle-center is $k < \frac{1}{2}q$. For q below this limit, one may linearize (9), and find regular (non-embedded) solitons, for any q, provided $k > 0$. These latter solitons continuously match into the ESs at $q = \frac{1}{2}k$. Thus, since we took $k = 0.3$, we have that both branches of ESs will end at $q = 0.6$. At the high q end, there were numerical difficulties in continuing the branches, using the software AUTO [24], for precisely the same reasons why the

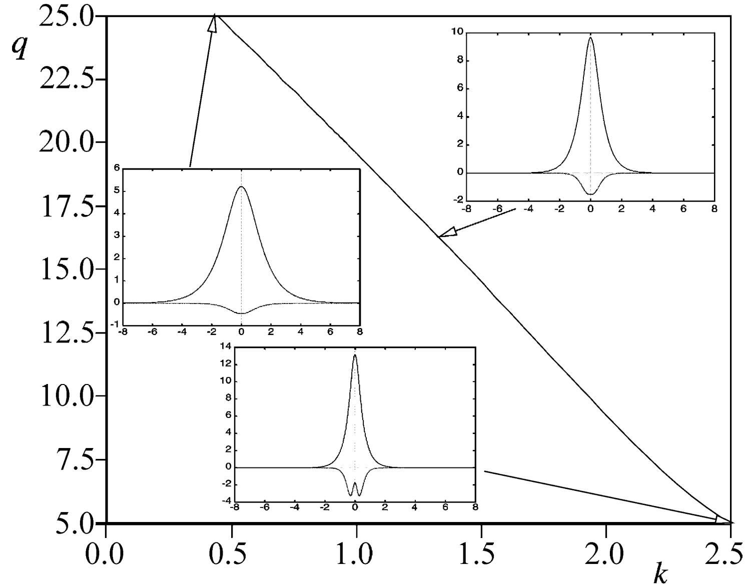

Fig. 4. The branch of $\delta = -0.5$ ESs in the (k, q)-plane corresponding to the right-hand branch in Fig. 3 (with $\gamma_1 = \gamma_2 = 0.05$), the insets similarly showing the soliton's profile.

computations presented in Fig. 1 were difficult at $a = 0^-$ for $b < 0$. As one can see in the figure, one does have the amplitudes of both U and V tending to zero (note also that the V-component is much smaller, in accordance with the nonlinearizability principle).

Second, note that for high q, both ES solutions appear to be single-humped and fundamental. But upon going to smaller q, where the amplitudes of both U and V become larger, it is apparent that the right-hand branch is a higher-order state, with a superimposed ripple in the V component (akin to the second branch in Figs. 1 and 2.

Finally, we remark that only this second branch passes through the physically significant value of $\delta = -\frac{1}{2}$. For the spatial ESs, corresponding to $\delta = -\frac{1}{2}$, one is interested in how the solutions will vary as a function of k as well. That is shown in Fig. 4. Again, the low-q limit corresponds to the boundary of the saddle-center region is $k = \frac{1}{2}q$.

3.2. Homoclinic orbits to saddle-centers

Next, let us try to understand why ESs should be of codimension-one. We start from a general fourth-order Hamiltonian and reversible (invariant with respect to reflections of time and "velocity" variables) ODE system. Linearizing about the zero solution (the origin), we assume the system to have a saddle-center equilibrium. Thus, we have

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^4, \qquad \exists R, \qquad R^2 = \mathrm{id}, \qquad Rf(Rx) = -f(x), \qquad \exists H(x) = \text{const. along solutions}, \\ \mathrm{eigs}(Df(0)) = \{\lambda, -\lambda, \mathrm{i}\omega, -\mathrm{i}\omega\}. \tag{10}$$

For example, for the ODE (2) the reversibility operator R is given by

$$(u, u', u'', u''') \mapsto (u, -u', u'', -u'''),$$

and for the system (9) by

$$R:(U,U',V,V')\mapsto(U,-U',V,-V').$$

A homoclinic orbit connecting such a saddle-center equilibrium to itself is formed by a trajectory simultaneously belonging to the one-dimensional unstable and stable manifolds of the origin. Both of these manifolds lie in a three-dimensional phase space $H(0)$. Therefore, were it not for the reversibility, such homoclinic orbits would be of codimension-two in general [25], since we require the coincidence of two lines in the three-dimensional space. But, *reversible* homoclinic solutions (i.e., solutions that somewhere intersect the fixed-point set of the reversibility) are of codimension-one. This is because the unstable manifold and $\mathrm{fix}(R)\cap H(0)$ are both one-dimensional, and we only require a point intersection between them. Hence, varying two parameters, we should expect to see ESs occurring along lines in the corresponding 2D parameter plane. Moreover, the solutions themselves must be reversible; asymmetric ESs would be of a higher codimension still.

Mielke et al. [26] proved a general theorem valid in the neighborhood of such a curve in the parameter plane of reversible saddle-center homoclinic orbits: under a sign condition, essentially governing how reversibility and the Hamiltonian interact, they showed that there will be an accumulation of infinitely many curves of N-pulse "bound states" of the primary homoclinic orbit for each $N>1$. Note that, for systems that are reversible and also have odd symmetry (such as (6)), the sign condition is always satisfied by virtue of the system's admitting both reversibilities R and $-R$ (i.e., the model (6) is symmetric too under $U\to-U$, and hence is also invariant under the reversibilities $(U,U')\to(U^*,-U'^*)$ and $(U,U')\to(-U^*,U')$). Here, the sign condition determines whether there are "up–up" or "up–down" bound states. In [5], a large number of the bound states of the "up–down" type were found for the generalized massive Thirring model (6) in agreement with this theory. We also remark that Buryak (see [27]) found a similar discrete sequence of bound states of "non-existent" dark solitons in an SHG model and a higher-order NLS equation, however none of these solutions were linearly stable.

So far, in every case that we have investigated, we have found the higher-order ESs to be unstable against linear perturbations. A distinction should be stressed between what one would call "bound-states" — which are like several copies of a fundamental soliton placed end to end — and the higher-order solitons, such as those displayed in Figs. 1 and 2, which are like a fundamental with internal ripples. At the moment, there seems to be no connection between these states, although it is still may happen that, as some parameter is varied, a continuous branch may connect solutions of the two different types.

3.3. A singular limit

We now look at a mechanism which explains how fundamental ESs (possibly with ripples) may appear from the singular limit $\lambda\to0^+$. Such a limit for the general class of systems (10) has been studied using the normal-form theory by Lombardi [28–30], incorporating careful estimation of various exponentially small terms (cf. related results obtained using exponential asymptotics, e.g., [11,12]). A crucial additional ingredient we shall add to Lombardi's work is that λ and ω are assumed to play the role of two independent parameters. We provide here only an oversimplified sketch, more details will appear elsewhere [31].

The appropriate normal form is [28]

$$\dot{x}_1=x_2,\qquad \dot{x}_2=\lambda^2x_1-\tfrac{3}{2}x_1^2-b_1(x_3^2+x_4^2)+\rho N_2(x;\lambda),\qquad \dot{x}_3=-x_4(\omega+b_2x_1)+\rho N_3(x;\lambda),$$
$$\dot{x}_4=x_3(\omega+b_2x_1)+\rho N_4(x;\lambda),\tag{11}$$

where $b_{1,2}$ are ω-dependent constants to be determined for a particular system, and N_i are higher-order (remainder) terms that break (for non-zero ρ) the completely-integrable structure of the truncated normal form. It is not difficult to see that the truncated system possess a sech^2-like homoclinic connection to the origin, whose amplitude is $O(\lambda)$. The key question is whether this homoclinic orbit persists under the inclusion of the remainder terms. This question can be posed in terms of the vanishing of a certain Melnikov integral (which measures the splitting distance between the stable and unstable manifolds), which, after a lengthy calculation [29], can be written as

$$I = \frac{\rho}{\lambda^2}\exp\left(-\frac{\omega\pi}{\lambda}\right)(\Lambda(N_3, N_4, \omega)(1 + O(\lambda) + O(\rho)),$$

where $\Lambda(N_3, N_4, \omega)$ can be computed explicitly for each monomial which is pure in x_1 and x_2 in the Taylor-series expansion of either N_3 or N_4.

Now, something beautiful happens because, for each such monomial, Λ turns out to be [29] a (single-signed ω-dependent) constant multiple of either a Bessel function, $\Lambda \sim J_n(4\sqrt{\omega b_2})$ for $b_2 > 0$, or modified Bessel function $\Lambda \sim I_n(4\sqrt{\omega|b_2|})$ for $b_2 < 0$, for some integer n. Recall the basic properties of the Bessel functions, according to which $J_n(x)$ has infinitely many zeros for $x > 0$, whereas the modified Bessel function $I_n(x) = \mathrm{i}^{-n}J_n(\mathrm{i}x)$ has no zeros. Hence, notice the crucial role played by the coefficient b_2 (in the truncated, scaled normal form (11) in the case when the remainder N consists of a single monomial in (x_1, x_2). If $b_2 > 0$, there will be an infinite number of ω values corresponding to zeros of Λ and hence homoclinic solutions will exist for small ρ and λ. However, if $b_2 < 0$, then Λ is strictly non-zero and hence there are no homoclinic solutions to the origin.

The coefficient b_2 is easy to calculate in the examples with the quadratic nonlinearities, such as the extended fifth-order KdV model (2) with $\mu = 1$ (see [31] for details). There it is found $b_2 > 0$ and this entirely explains the approximately periodic sequence of points on the negative b-axis at which the homoclinic solutions bifurcate from $a = 0$ with the zero amplitude, as shown in Fig. 1. In contrast, for $\mu = -1$, b_2 is negative and no ES bifurcates from $a = 0$. For models with purely cubic nonlinearity, such as the GTM (6), it can be shown that all quadratic coefficients in the normal form (11) vanish, hence a new, odd-symmetric normal form would need to be studied. This is left for future work. We mention also that for the SHG model (9), although there are quadratic terms in it, the coefficient b_2 is identically zero, so this analytic technique gives no information.

4. Stability

The stability of the ESs in the SHG model (8) with $\delta > 0$ has been studied both numerically and analytically in [6]. It was shown that the fundamental (single-hump) ES is linearly stable, but nonlinearly semi-unstable, while all the multi-humped ES are linearly unstable. In showing semi-stability of the fundamental ES, a crucial role is played by the energy, as ESs are *isolated* solutions with uniquely determined values of the energy, with the adjacent delocalized soliton states, on either side, having an infinite energy. It was argued in [6] (see also a similar argument given by Buryak [27]) that perturbations which slightly increase the fundamental ES's energy can be safe, while perturbations which decrease the energy inevitably trigger a slow (sub-exponential) decay of the soliton into radiation. Thus, the weak instability of an ES is *one-sided*. This fact also follows from a simple argument that the usual exponential instability is always two-sided. Equivalently, the usual instability is linear, while the weak one-sided instability of ESs must be *nonlinear*.

The situation is the same in the present case where $\delta < 0$ in the SHG model (8). A study of the linearized equation around the ES shows that the fundamental ES branch (the left one in Fig. 3) is linearly stable, while the branch of multi-humped ES (the right one in Fig. 3) is linearly unstable. The semi-stability argument for the fundamental ES branch also applies here. Positive energy perturbations can be safe, while negative energy perturbations trigger decay of ES. However, as we shall see below, for $\delta < 0$, this decay seems to be significantly slower than that found in [6] for $\delta > 0$. As was done there, we numerically simulated the system (8), with the initial data

$$u(0, t) = U(t) + \alpha_1 \,\mathrm{sech}\, 2t, \qquad v(0, t) = V(t) + \alpha_2 \,\mathrm{sech}\, 2t, \tag{12}$$

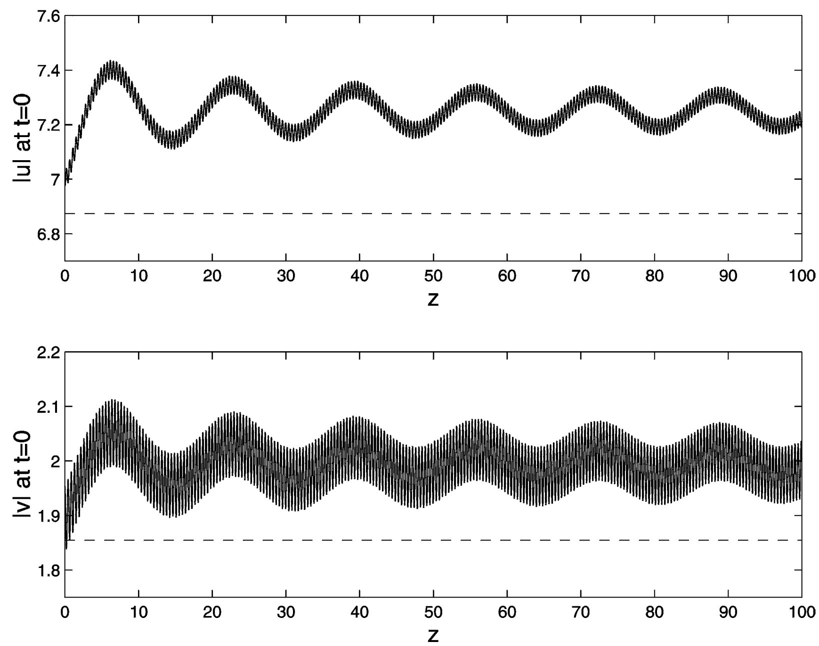

Fig. 5. Evolution of a perturbed soliton with positive energy perturbation ($\alpha_1 = 0.1, \alpha_2 = -0.1$ in (12)) for (8) from the left-hand fundamental branch of Fig. 3.

where $U(t)$ and $V(t)$ is an ES solution on the left-hand branch of Fig. 3 (the values are $\delta = -2.9292$, $q = 6.0556$, $k = 0.2936251$ and $\alpha_1 = \alpha_2 = 0.05$). Figs. 5 and 6 depict, respectively, the effects of the positive-energy and negative-energy perturbations. In both cases, we observe fast oscillations on top of slow ones in the evolution of $|u|$ and $|v|$ at $t = 0$. These oscillations are very similar to those reported in [23] for perturbed solitons in the standard SHG system with no cubic nonlinearity (note that those solitons are ordinary ones, rather than ES). In that case, a superposition of fast and slow oscillations was observed in the numerical simulations of a perturbed soliton when the propagation constant was close to the instability threshold. The slow oscillations were attributed to a genuine (localized) internal mode of the soliton, while the fast oscillations were attributed to a quasi-mode, which belongs to the continuous spectrum of the SH component, but is out of the continuous spectrum in the FH component. Both oscillations could last for a very long time, even though they were expected to eventually decay due to a very weak radiation damping.

The similarities in oscillational dynamics between the SHG model in [23] and in our SHG model (8) suggest that the underlying mechanisms may also be similar, i.e., the slow oscillations in Figs. 5 and 6 are due to an internal mode, and the fast oscillations are due to a quasi-mode. However, the “internal” mode in the present case cannot be the internal (localized) modes as known in the regular gap solitons [23]. Indeed, in order for an internal localized mode to exist, there must be enough free parameters in the tails of the modes, so that by adjusting these parameters, one can get an internal localized mode. But in the linearized equation around any ES, there are just not enough free parameters in the tails. Nevertheless, one does see very persistent oscillations in Figs. 5 and 6.

Although according to the semi-stability argument, a negative perturbation of an ES, as shown in Fig. 6 would eventually decay, it is a very slow decay. Detailed examination of the numerical solutions shows that the central pulse (the v component) in Fig. 6 keeps shedding oscillating tails into the far field. However, the tail amplitudes are extremely small (about 0.001 or smaller). This is why the expected decay of the perturbed ES is not obvious in that

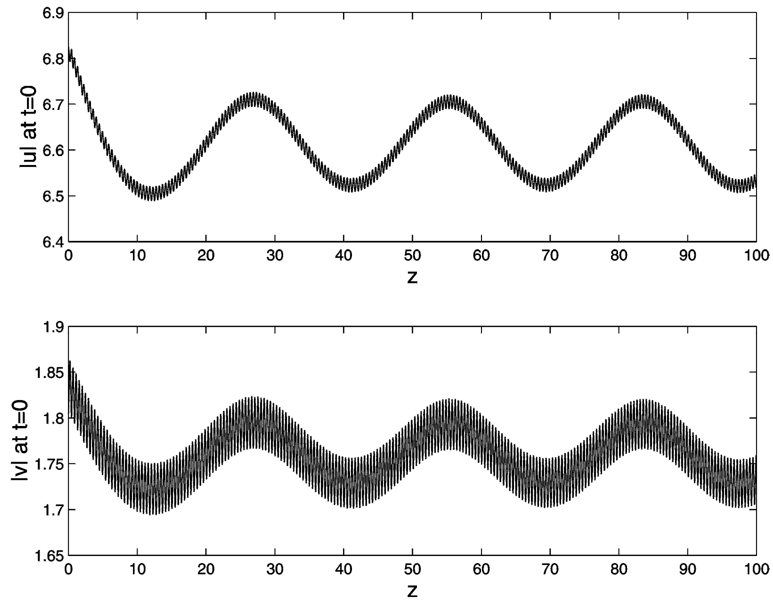

Fig. 6. Evolution of a perturbed soliton with a negative energy perturbation ($\alpha_1 = -0.05$, $\alpha_2 = 0.05$ in Eq. (12)) in the model (8). The soliton belongs to the left-hand fundamental branch in Fig. 3.

figure. Because of this, the actually observed evolution is dominated by a superposition of slow and fast oscillations, just as in Fig. 5 with a positive perturbation, and as in [23]. It will take an extremely long time for the pulse in Fig. 6 to show considerable decay. In fact, it is clear that the ESs in the present model with $\delta < 0$ are *virtually stable*, when compared to the previously considered case [6], $\delta > 0$, where the semi-instability was a really observed feature. The relative stability of an ES for $\delta < 0$ is a new result reported in the present paper, and its importance is quite obvious.

If the ES were linearly unstable, the semi-stability argument would not apply. This is the case for the solutions investigated in [27], and also for the right-hand branch depicted in Fig. 3, which corresponds to the case of a nonlinear planar optical waveguide in the spatial domain. To verify this, we have performed a time integration of the PDE, the results of which are presented in Fig. 7. One can see the onset of a violent exponential instability. We have

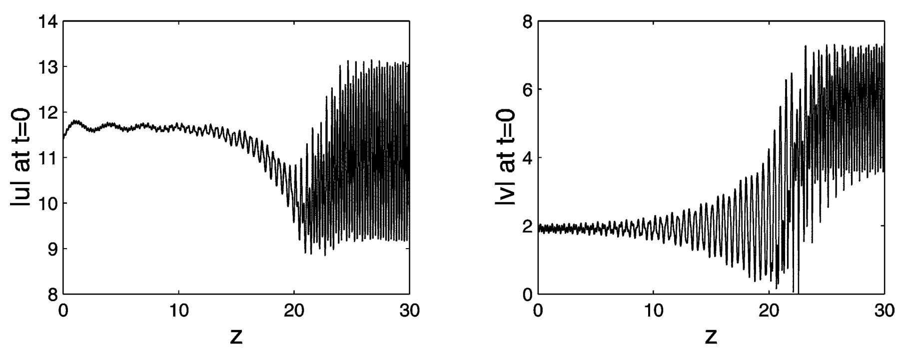

Fig. 7. The linear instability of a perturbed soliton belonging to the right-hand branch in Fig. 3 at $\delta = -0.5$, $\gamma_1 = \gamma_2 = 0.05$, $q = 10$ and $k = 1.9298$.

verified that this solution does have exponentially unstable eigenmodes, from a numerical study of the linearization of Eq. (8) about this solution.

In view of these results, we conjecture the following. Fundamental ESs are, in general, linearly neutrally stable but semi-stable nonlinearly. The higher-order ESs (which usually have internal ripples in their profiles) are, generally, linearly unstable. Preliminary results for the extended fifth-order KdV (1) indicate qualitatively the same properties. This would also be in accord with previous numerical results for higher-order NLS equations, in that an isolated fundamental ES [13] is semi-stable whereas multi-humped bound-states [27] are exponentially unstable. We want to point out that the semi-stability of fundamental ESs in the extended KdV equations has been rigorously proved in [4] by a soliton perturbation method. In addition, dynamics of ESs was also shown there.

To conclude this section, we notice that multi-humped solitons (which are *not* a subject of the present paper) of the ordinary (non-embedded) type were found in many models, see, e.g., early works [32,33] dealing with the Langmuir waves in plasmas, and a recent work on the resonant three-wave interactions [17].

5. Moving ESs

We will now discuss a possibility that ESs in the generalized massive Thirring model (5) may be moving at a non-zero velocity c (this discussion applies also to ESs in the three-wave system (7) with a non-zero walkoff). In this case, the reduction of the eighth-order ODE to a fourth-order one is no longer possible. Moreover, since the eighth-order system is obtained by separating the real and imaginary parts of a fourth-order complex system, the spectrum will have a double degeneracy when $c = 0$. This means that, in the parameter region of interest in the (χ, c, D)-space, the linearization yields four pure imaginary eigenvalues, plus two with positive real parts and two with negative ones. A similar counting argument, as in Section 3.2, shows that reversible homoclinic orbits to such equilibria are of *codimension-two*. Hence, ESs lie on one-dimensional curves in the three-parameter space. Alternatively, this property can be explained as follows: in addition to the energy E, the full system also preserves the momentum P, so we can view a moving ES as being isolated in *both* invariants, i.e., the ES solution family is described by curves $E(D)$ and $P(D)$. Finally, we find that such curves can be found naturally as bifurcation points at $c = 0$ from curves of the zero-velocity ESs.

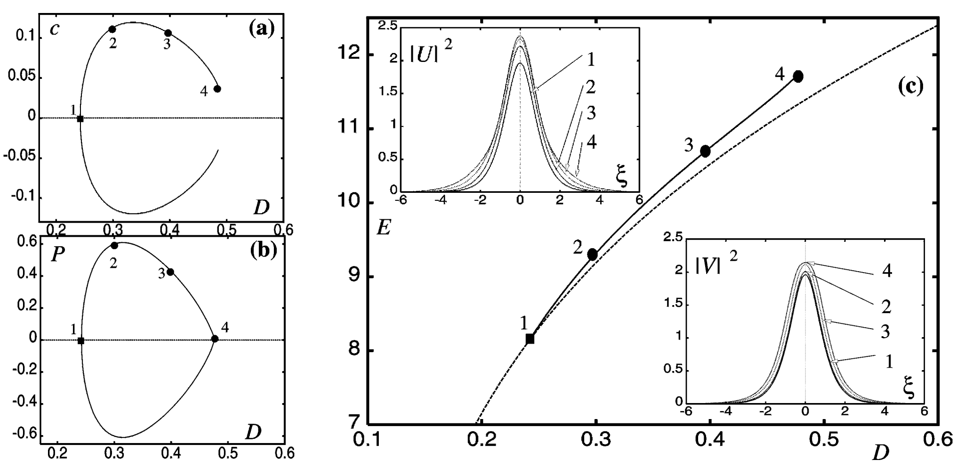

Fig. 8. A branch of moving ESs bifurcating from one of the three simplest branches of the fundamental quiescent ESs in Eq. (5) with $\sigma = \frac{1}{2}$: the velocity (a), momentum (b), and energy (c) vs. D, with insets showing the shape of the moving solitons at labeled points.

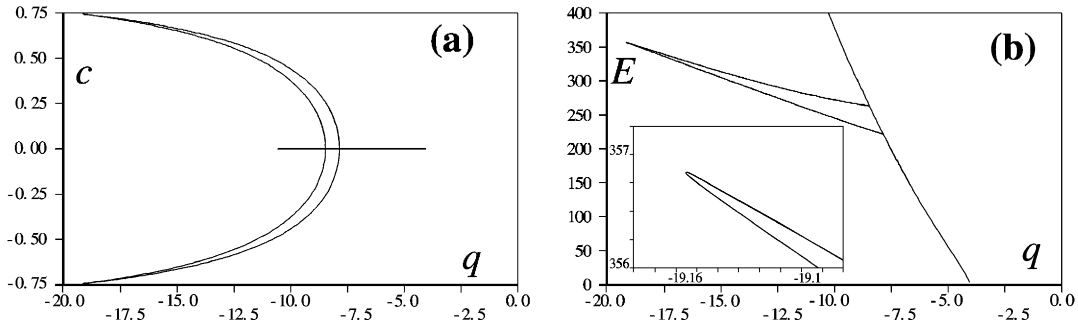

Fig. 9. Two branches of "walking" ($c \neq 0$) ESs bifurcating from the ground-state $c = 0$ branch (solid horizonal line) for the three-wave model (7): (a) the walkoff c, and (b) the energy flux E vs. the mismatch q. The inset in (b) shows that the two branches meet on the left and merge in a typical tangent bifurcation.

Fig. 8 shows one such bifurcation occurring from one of the three branches of fundamental ESs for the Thirring model. The ES branches terminate (as at point "4") and beyond that, ES become ordinary (non-embedded) solitons (i.e., where the saddle-center equilibrium becomes a pure saddle with only real eigenvalues).

Similarly, Fig. 9 shows two branches of moving ESs bifurcating from the ground-state ES (the nearly vertical line) in the three-wave model. Note that the two bifurcating curves become globally connected on the left, through a regular tangent (fold) bifurcation.

Finally, let us turn to the SH model (8). As it was mentioned above, precisely at the value $\delta = -\frac{1}{2}$, this model gives rise to moving solitons in a trivial way, via the Galilean transformation. Search for moving ES in this model with $\delta \neq -\frac{1}{2}$ is the subject of ongoing work.

6. Conclusion

In this paper, we have given a brief overview of recent results that establish the existence of isolated (codimension-one) solitons embedded into the continuous spectrum of radiation modes. A necessary condition for the existence of the ESs is the presence of (at least) two different branches in the spectrum of the corresponding linearized system, so that one branch can correspond to purely imaginary eigenvalues, and another to purely real ones. The fundamental (single-humped) ESs are always stable in the linear approximation, being subject to a weak sub-exponential one-sided instability. Moving ESs may also exist as codimension-two solutions. Moreover, bound states in the form of multi-humped ESs exist too, but they are linearly unstable.

We should remark that the work presented and reviewed here does not necessarily represent the very first time that the existence of ESs have been established in any physical model. For example, as stated earlier, multi-humped dark ESs were already observed in an SHG model and for higher-order NLS equations [27] where they were found to be unstable. Fundamental ESs were also found for the latter with competing cubic and quintic nonlinearities [13] and their existence was also suggested for a coupled KdV system [12], without evidence to suggest their stability. Note also that works [34,35] have shown the existence of codimension-one front solutions to higher-order Frenkel–Kontorova models which are in resonance with the linear spectrum and may therefore be naturally termed "embedded kinks". Nevertheless, we believe the results presented in this paper — the new stability results for fundamental ESs in Section 4, together with the three distinct views given in Section 3 as to why ESs should exist, and the similarities found in Section 2 between their existence properties for four distinct models — provide a new theory for ESs as a phenomenon in their own right.

Moreover, the very fact that ESs are isolated states suggest that they may find potential application in photonics, such as in all-optical switching. For example, taking the examples in Sections 2.2 and 2.3, switching from one ES state to a neighboring one with a smaller energy might be easily initiated by a small perturbation, in view of the semi-stability inherent to ESs. Switching between two branches of moving ESs with $c \neq 0$ might be quite easy to realize too, due to the small energy-flux and walkoff differences between them. There remains much work to be done in investigating these potential aplications further.

Many issues concerning the ESs remain open and are a subject of ongoing investigations. In particular, immediate questions arise concerning moving ESs, and interactions caused by collisions between them.

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