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Rogue Waves in the Generalized Derivative Nonlinear Schrödinger Equations

Bo Yang[1] · Junchao Chen[2] · Jianke Yang[1]

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Abstract

General rogue waves are derived for the generalized derivative nonlinear Schrödinger (GDNLS) equations by a bilinear Kadomtsev–Petviashvili (KP) reduction method. These GDNLS equations contain the Kaup–Newell equation, the Chen–Lee–Liu equation and the Gerdjikov–Ivanov equation as special cases. In this bilinear framework, it is shown that rogue waves to all members of these equations are expressed by the same bilinear solution. Compared to previous bilinear KP reduction methods for rogue waves in other integrable equations, an important improvement in our current KP reduction procedure is a new parameterization of internal parameters in rogue waves. Under this new parameterization, the rogue wave expressions through elementary Schur polynomials are much simpler. In addition, the rogue wave with the highest peak amplitude at each order can be obtained by setting all those internal parameters to zero, and this maximum peak amplitude at order N turns out to be $2N+1$ times the background amplitude, independent of the individual GDNLS equation and the background wavenumber. It is also reported that these GDNLS equations can be decomposed into two different bilinear systems which require different KP reductions, but the resulting rogue waves remain the same. Dynamics of rogue waves in the GDNLS equations is also analyzed. It is shown that the wavenumber of the constant background strongly affects the orientation and duration of the rogue wave. In addition, some new rogue patterns are presented.

Keywords Rogue waves · Derivative nonlinear Schrödinger equations · Bilinear method

Mathematics Subject Classification 37K10 · 35L05

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✉ Jianke Yang
Jianke.Yang@uvm.edu

1 Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05405, USA

2 Department of Mathematics, Lishui University, Lishui 323000, China

1 Introduction

Rogue waves are large and spontaneous local excitations of nonlinear wave equations that "appear from nowhere and disappear with no trace" (Akhmediev et al. 2009a). More specifically, these local excitations arise from a flat constant-amplitude background, reach a transient high amplitude and then retreat back to the same flat background. Such solutions were first reported for the nonlinear Schrödinger (NLS) equation by Peregrine in 1983 (Peregrine 1983). In recent years, such waves were linked to freak waves in the ocean (Dysthe et al. 2008; Kharif et al. 2009) and extreme events in optics (Solli et al. 2007; Wabnitz 2017), and were observed in water-tank and optical-fiber experiments (Chabchoub et al. 2011, 2012; Kibler et al. 2010; Frisquet et al. 2016; Baronio et al. 2018). Motivated by these physical applications, rogue waves have been derived in a large number of physically relevant integrable nonlinear wave equations, including the NLS equation (Akhmediev et al. 2009b; Ankiewicz et al. 2010a; Dubard et al. 2010; Kedziora et al. 2011; Guo et al. 2012; Ohta and Yang 2012a; Dubard and Matveev 2013), the derivative NLS equations (Xu et al. 2011; Guo et al. 2013; Chan et al. 2014; Zhang et al. 2017), the Manakov equations (Baronio et al. 2012, 2014), the Davey–Stewartson equations (Ohta and Yang 2012b, 2013), and many others (Ohta and Yang 2014; Yang and Yang 2020; Chen et al. 2018a; Zhang and Chen 2018; Ankiewicz et al. 2010b, c; Chow et al. 2013; Mu and Qin 2016; Ling et al. 2016; Clarkson and Dowie 2017). Indeed, rogue waves are caused by baseband modulation instability of the constant-amplitude background (Baronio et al. 2014). Thus, any integrable equation with baseband modulation instability is expected to admit rogue waves, which can be derived by integrable techniques. All known rogue waves in integrable equations are rational solutions of the underlying systems. This fact is related to baseband modulation instability, since rational rogue-wave solutions are associated with long-wave instability of the background. We note by passing that in non-integrable systems, large and spontaneous local excitations can also arise from a constant-amplitude background if such background admits baseband modulation instability (see Solli et al. 2007 for instance). But such excitations do not retreat back to the same background, and are not expected to admit exact analytical expressions, due to the lack of integrability of the underlying nonlinear wave equations (Ankiewicz et al. 2018).

In this paper, we consider rogue waves in the generalized derivative nonlinear Schrödinger (GDNLS) equations (Kundu 1984; Clarkson and Cosgrove 1987)

$$\mathrm{i}\phi_t + \phi_{\xi\xi} + \rho|\phi|^2\phi + \mathrm{i}a\phi\phi^*\phi_\xi + \mathrm{i}b\phi^2\phi_\xi^* + \frac{1}{4}b(2b-a)|\phi|^4\phi = 0, \tag{1}$$

where ρ, a, b are the arbitrary real constants with $a \neq b$, and the superscript '*' represents complex conjugation (the $a = b$ case will be treated in "Appendix"). In fiber optics, these equations model the propagation of short light pulses where, in addition to dispersion and Kerr (cubic) nonlinearity, self-steepening and fifth-order nonlinearity are also accounted for (even though the Raman effect and third-order dispersion are omitted) (Agrawal 2001; Kivshar and Agrawal 2003). When $\rho = 0$ and $a = 2b$, these equations reduce to the Kaup–Newell equation (Kaup and Newell

1978), which governs the propagation of circularly polarized nonlinear Alfvén waves in plasmas (Mio et al. 1976; Mjolhus 1976). When $\rho = b = 0$, these equations reduce to the Chen-Lee-Liu equation (Chen et al. 1979), which models short-pulse propagation in a frequency-doubling crystal through the interplay of quadratic and cubic nonlinearities (Moses et al. 2007). Due to these physical applications, rogue wave formation in these GDNLS equations is a physically significant issue.

There have been a number of studies on rogue waves in these GDNLS equations. For instance, for the Kaup–Newell equation (with $\rho = 0$ and $a = 2b$), special types of rogue waves were derived by Darboux transformation in Xu et al. (2011), Guo et al. (2013). For the Chen–Lee–Liu-type equation, with $b = 0$ in (1), the fundamental rogue wave was derived by the bilinear Hirota method in Chan et al. (2014), and higher-order rogue waves were derived by Darboux transformation in Zhang et al. (2017). For the Gerdjikov–Ivanov equation (Gerdjikov and Ivanov 1983), with $\rho = a = 0$ in (1), fundamental and higher-order rogue waves were derived by Darboux transformation in Xu and He (2012), Guo et al. (2014). Even for the GDNLS equations (1) themselves, general rogue waves were derived by Darboux transformation in Chen et al. (2019), and their chirping phase structure was examined.

In this article, we derive general rogue waves in the GDNLS equations (1) by the bilinear Kadomtsev–Petviashvili (KP) reduction method. The advantage of this bilinear framework is that rogue waves in all GDNLS equations (1) can be expressed explicitly by the same bilinear solution. Compared to previous bilinear KP reduction methods for rogue waves in other integrable equations (Ohta and Yang 2012a, b, 2013, 2014; Yang and Yang 2020; Chen et al. 2018a; Zhang and Chen 2018), an important improvement in our current KP reduction technique is a new parameterization of internal parameters in rogue waves. Under this parameterization, analytical expressions of rogue waves through Schur polynomials are much simpler. More importantly, when all internal parameters are set to zero, we would get a parity-time-symmetric rogue wave which attains the maximum peak amplitude among rogue waves of that order. This allows us to analytically derive this maximum peak amplitude, which turns out to be $2N + 1$ times the background amplitude at order N, independent of the individual GDNLS equation and the background wavenumber. We also find that the GDNLS equations (1) can be decomposed into two different bilinear systems which require different KP reductions, but the resulting rogue waves are the same. After these rogue waves are derived, their dynamics is also analyzed. It is shown that the wavenumber of the background strongly affects the orientation and duration of the rogue wave. In addition, some new rogue patterns are presented. In "Appendix", general rogue waves for the GDNLS equations (1) with $a = b$ (the so-called Kundu–Eckhaus equation) are also given in the bilinear framework. These results deepen our understanding of rogue waves in the physically significant GDNLS equations (1). Meanwhile, they advance the bilinear KP-reduction methodology for the derivation of rogue waves.

2 Preliminaries

Under a simple gauge transformation (Kakei et al. 1995)

$$\phi(\xi, t) = \sqrt{\frac{2}{a-b}}\, u(x,t)\ \exp\left\{ \mathrm{i}\frac{\rho}{a-b}x + \mathrm{i}\frac{\rho^2}{(a-b)^2}t \right\},$$

where $x = \xi - 2\rho t/(a-b)$, the GDNLS equations (1) with $a \neq b$ reduce to

$$\mathrm{i}u_t + u_{xx} + 2\mathrm{i}\gamma|u|^2u_x + 2\mathrm{i}(\gamma-1)u^2u_x^* + (\gamma-1)(\gamma-2)|u|^4u = 0, \tag{2}$$

where $\gamma = a/(a-b)$. We will work with these normalized GDNLS equations (2) in the remainder of this article. These equations become the Kaup–Newell equation when $\gamma = 2$ (Kaup and Newell 1978), the Chen-Lee-Liu equation when $\gamma = 1$ (Chen et al. 1979), and the Gerdjikov–Ivanov equation when $\gamma = 0$ (Gerdjikov and Ivanov 1983).

It is noted that with an additional gauge transformation

$$u(x,t) = v(x,t)\, e^{\mathrm{i}(2-\gamma)\int|v(x,t)|^2dx}, \tag{3}$$

the normalized GDNLS equations (2) further reduce to the Kaup–Newell equation

$$\mathrm{i}v_t + v_{xx} + 2\mathrm{i}(|v|^2v)_x = 0. \tag{4}$$

Thus, from rogue waves of the Kaup–Newell equation, one can derive rogue waves in the GDNLS equations (2) in principle. However, the gauge transformation (3) involves a nontrivial integral, which makes it difficult to derive explicit solutions to the GDNLS equations from those of the Kaup–Newell equation. For this reason, we will not utilize this gauge transformation. Instead, we will use a bilinear method to directly obtain explicit rogue wave solutions in the GDNLS equations (2) for arbitrary γ values.

Rogue waves in the GDNLS equations (2) approach a constant-amplitude continuous wave background at large x and t. By simple variable scalings, this constant amplitude can be normalized to be unity. Then, these rogue waves approach the unit-amplitude continuous wave background $e^{\mathrm{i}\kappa x - \mathrm{i}\omega t}$, where κ is a free wavenumber, and $\omega = \kappa^2 + 2\kappa - (\gamma-1)(\gamma-2)$ is the frequency. In order for rogue waves to arise, these backgrounds must be unstable to baseband modulations (Baronio et al. 2014). Simple modulation instability calculations show that these backgrounds are base-band unstable when $\kappa < 1 - \gamma$. Thus, rogue waves in the GDNLS equations (2) should approach the following background as $x, t \to \pm\infty$:

$$u(x,t) \to e^{\mathrm{i}(1-\gamma-\alpha)x - \mathrm{i}\left[\alpha^2 + 2(\gamma-2)\alpha + 1 - \gamma\right]t}, \tag{5}$$

where $\alpha > 0$ is a wavenumber parameter. Unlike the NLS equation, the GDNLS equations (2) do not admit Galilean-transformation invariance. Thus, α is a non-reducible parameter in its rogue waves.

In view of the above boundary condition, we introduce the variable transformation

$$u = e^{\mathrm{i}(1-\gamma-\alpha)x - \mathrm{i}\left[\alpha^2 + 2(\gamma-2)\alpha + 1 - \gamma\right]t}\, \frac{(f^*)^{\gamma-1}g}{f^{\gamma}}, \tag{6}$$

where f and g are the complex functions. Under this transformation, the GDNLS Eq. (2) can be decomposed into the following system of four bilinear equations:

$$\left(\mathrm{i}D_t + D_x^2 + 2\mathrm{i}(1-\alpha)D_x\right) g \cdot f^* = 0, \tag{7}$$

$$\left(\mathrm{i}D_t + D_x^2 + 2\mathrm{i}D_x\right) f \cdot f^* = 0, \tag{8}$$

$$(\mathrm{i}D_x - 1)\, f \cdot f^* + |g|^2 = 0, \tag{9}$$

$$D_x^2 f \cdot f^* - \mathrm{i}D_x g \cdot g^* + (2\alpha+1)(|f|^2 - |g|^2) = 0, \tag{10}$$

where D is the Hirota's bilinear differential operator. We will use these bilinear equations to derive rogue waves in the GDNLS Eq. (2). It is important to notice that these bilinear equations are independent of the equation parameter γ. This means that rogue waves in the whole family of GDNLS Eq.(2), for different values of γ, are given by the same f and g solutions, and the γ-dependence of the rogue waves only appears through the bilinear transformation (6). This is a big advantage of the bilinear method for solving the GDNLS Eq. (2).

Interestingly, under the same transformation (6), the GDNLS Eq. (2) can also be decomposed into a different bilinear system, where the first bilinear Eq. (7) is replaced by a new equation

$$\left(\mathrm{i}D_t + D_x^2 - 2\mathrm{i}\alpha D_x\right) g \cdot f = 0, \tag{11}$$

while the other three bilinear Eqs. (8)–(10) remain the same. This replacement of the first bilinear Eq. (7) is admitted because the left side of the above new bilinear Eq. (11) can be written as a linear combination of the left sides of the former bilinear Eqs. (7)–(10). Specifically, denoting the left side of each equation by its equation number, we have the identity

$$\begin{aligned} f \times (7) - g \times (8) + 2(\mathrm{i}g_x + \alpha g) \times (9) = f^* \times (11) - g \times (10) \\ + g(\mathrm{i}\partial_x - 1) \times (9). \end{aligned} \tag{12}$$

Thus, if f and g satisfy the former system of bilinear Eqs. (7)–(10), then they would also satisfy the latter bilinear system (8)–(11). Although these two $(1+1)$-dimensional bilinear systems are equivalent, they have to be reduced from different higher-dimensional bilinear systems which admit different bilinear solutions. But these two different KP reductions will lead to the same rogue wave solutions, which we will show in later texts.

In this article, we will present rogue waves of the GDNLS Eq. (2) through elementary Schur polynomials. These Schur polynomials $S_j(\boldsymbol{x})$ are defined by

$$\sum_{j=0}^{\infty} S_j(\boldsymbol{x})\lambda^j = \exp\left(\sum_{j=1}^{\infty} x_j \lambda^j\right),$$

or more explicitly,

$$S_0(\boldsymbol{x}) = 1, \quad S_1(\boldsymbol{x}) = x_1, \quad S_2(\boldsymbol{x}) = \frac{1}{2}x_1^2 + x_2, \quad \cdots,$$
$$S_j(\boldsymbol{x}) = \sum_{l_1+2l_2+\cdots+ml_m=j} \left(\prod_{j=1}^{m} \frac{x_j^{l_j}}{l_j!} \right),$$

where $\boldsymbol{x} = (x_1, x_2, \cdots)$.

3 General Rogue Wave Solutions

Our general rogue wave solutions to the GDNLS Eq. (2) are given by the following theorem.

Theorem 1 *The GDNLS Eq.* (2) *under the boundary condition* (5) *admit rational rogue wave solutions*

$$u_N(x,t) = e^{\mathrm{i}(1-\gamma-\alpha)x - \mathrm{i}\left[\alpha^2+2(\gamma-2)\alpha+1-\gamma\right]t} \, \frac{(f_N^*)^{\gamma-1} g_N}{f_N^{\gamma}}, \tag{13}$$

where the positive integer N represents the order of the rogue wave,

$$f_N(x,t) = \sigma_{0,0}, \quad g_N(x,t) = \sigma_{-1,1},$$
$$\sigma_{n,k} = \det_{1\le i,j\le N} \left(m_{2i-1,2j-1}^{(n,k)} \right),$$

the matrix elements in $\sigma_{n,k}$ are defined by

$$m_{i,j}^{(n,k)} = \sum_{\nu=0}^{\min(i,j)} \frac{1}{4^{\nu}} \, S_{i-\nu}(\boldsymbol{x}^{+}(n,k) + \nu \boldsymbol{s}) \, S_{j-\nu}(\boldsymbol{x}^{-}(n,k) + \nu \boldsymbol{s}), \tag{14}$$

vectors $\boldsymbol{x}^{\pm}(n,k) = \left(x_1^{\pm}, x_2^{\pm}, \cdots\right)$ are defined by

$$x_1^{+} = \ \ k + \left(n + \frac{1}{2}\right)\left(h_1 + \frac{1}{2}\right) + \sqrt{\alpha}x + 2\sqrt{\alpha}\left[(\alpha-1) + \mathrm{i}\sqrt{\alpha}\right]t + a_1,$$
$$x_1^{-} = -k - \left(n + \frac{1}{2}\right)\left(h_1^* + \frac{1}{2}\right) + \sqrt{\alpha}x + 2\sqrt{\alpha}\left[(\alpha-1) - \mathrm{i}\sqrt{\alpha}\right]t + a_1^*,$$
$$x_r^{+} = \ \ (n + \frac{1}{2})h_r + \frac{1}{r!}\left\{\sqrt{\alpha}x + \left[2\sqrt{\alpha}(\alpha-1) + 2^r\mathrm{i}\alpha\right]t\right\} + a_r, \quad r > 1,$$
$$x_r^{-} = -(n + \frac{1}{2})h_r^* + \frac{1}{r!}\left\{\sqrt{\alpha}x + \left[2\sqrt{\alpha}(\alpha-1) - 2^r\mathrm{i}\alpha\right]t\right\} + a_r^*, \quad r > 1,$$

$h_r(\alpha)$, s_r are coefficients from the expansions

$$\sum_{r=1}^{\infty} h_r \lambda^r = \ln\left(\frac{\mathrm{i}e^{\lambda/2} + \sqrt{\alpha}e^{-\lambda/2}}{\mathrm{i} + \sqrt{\alpha}}\right), \quad \sum_{r=1}^{\infty} s_r \lambda^r = \ln\left[\frac{2}{\lambda}\tanh\left(\frac{\lambda}{2}\right)\right], \tag{15}$$

and a_r $(r = 1, 2, \dots)$ are free complex constants.

Note 1 The first few coefficients $h_r(\alpha)$ and s_r in expansions (15) are

$$h_1(\alpha) = \frac{\mathrm{i} - \sqrt{\alpha}}{2\left(\mathrm{i} + \sqrt{\alpha}\right)}, \quad h_2(\alpha) = \frac{\mathrm{i}\sqrt{\alpha}}{2\left(\mathrm{i} + \sqrt{\alpha}\right)^2}, \quad h_3(\alpha) = \frac{\sqrt{\alpha}\left(1 + i\sqrt{\alpha}\right)}{6\left(\mathrm{i} + \sqrt{\alpha}\right)^3}, \tag{16}$$

$$s_1 = s_3 = \cdots = s_{\mathrm{odd}} = 0, \quad s_2 = -\frac{1}{12}, \quad s_4 = \frac{7}{1440}. \tag{17}$$

Theorem 1 will be proved in Sect. 5.

Some remarks on rogue waves in this theorem are in order. First, one can notice that the matrix-element expression in this theorem is significantly simpler than earlier such expressions for other integrable equations (Ohta and Yang 2012a, b, 2013, 2014; Yang and Yang 2020). Indeed, the current expression in (14) involves a single summation, while previous such expressions involved three summations. Second, our current parameterization of rogue waves in Theorem 1 is very different from the previous ones. In our current rogue wave solution, all internal parameters $a_1, a_2, a_3, \dots$ appear inside the $\boldsymbol{x}^{\pm}(n, k)$ vectors, while previous internal parameters all appeared outside such vectors as summation coefficients (Ohta and Yang 2012a, b, 2013, 2014; Yang and Yang 2020). This different parameterization is the key reason for the simpler matrix-element expression in Theorem 1. More significantly, this parameterization facilitates the analysis of rogue waves, especially regarding the maximum peak amplitude for rogue waves of a given order. Indeed, under previous parameterizations for the NLS equation, the rogue wave with maximum peak amplitude occurs at peculiar internal parameter values (Ohta and Yang 2012a), which makes the derivation of maximum peak amplitudes at arbitrary orders intractable. However, in our current parameterization, rogue waves in Theorem 1 admit the following property.

Theorem 2 *When $a_r = 0$ for all $r \geq 1$, the rogue wave in Theorem 1 is parity-time-symmetric, i.e., $u_N^*(-x, -t) = u_N(x, t)$.*

This property will also be proved in Sect. 5. The significance of this property is that this parity-time-symmetric rogue wave happens to possess the maximum peak amplitude among rogue waves of that order (see Chen et al. 2019). In addition, this maximum peak amplitude is located at the center of this parity-time-symmetric rogue wave, i.e., at $x = t = 0$. Thus, to derive the maximum peak amplitude of rogue waves in Theorem 1, we only need to set all its internal parameters a_r as well as (x, t) to zero, which is much easier. Doing so, our explicit calculations for $N = 1, 2, \dots, 6$ show that

$$
\begin{aligned}
|f_N(0,0)|_{a_r=0} &= \frac{\alpha^{N(N+1)/2}}{2^{2N^2}(\alpha+1)^{N(N+1)/2}}, \\
|g_N(0,0)|_{a_r=0} &= \frac{(2N+1)\alpha^{N(N+1)/2}}{2^{2N^2}(\alpha+1)^{N(N+1)/2}},
\end{aligned} \tag{18}
$$

and thus, the maximum peak amplitude is

$$
|u_N(0,0)|_{a_r=0} = \frac{|g_N(0,0)|_{a_r=0}}{|f_N(0,0)|_{a_r=0}} = 2N+1. \tag{19}
$$

Remarkably, this maximum peak amplitude does not depend on the background wavenumber α, although $|f_N|$ and $|g_N|$ in its numerator and denominator do. While these formulae (18)–(19) were obtained for $N \leq 6$, we believe they hold for all $N > 6$ as well.

In Zhang et al. 2017, Guo et al. (2014) and Chen et al. (2019) for the Chen–Lee–Liu equation, the Gerdjikov–Ivanov equation and the GDNLS equations (1), examination of some low-order rogue waves revealed that their maximum peak amplitude was $2N+1$. Our result above is more general. Interestingly, this maximum peak amplitude for rogue waves in the GDNLS Eq. (2) is exactly the same as that for the NLS equation (Akhmediev et al. 2009b; Ankiewicz et al. 2010a; Ohta and Yang 2012a; Wang et al. 2017).

Another remark on rogue waves in Theorem 1 pertains to the number of their irreducible free parameters. These rogue waves of order N contain $2N-1$ complex parameters $a_1, a_2, \ldots, a_{2N-1}$. But we can show that all even-indexed parameters a_{even} are dummy parameters which cancel out automatically from the solution. To prove this, we first rewrite $\sigma_{n,k}$ in Theorem 1 as (Ohta and Yang 2012a)

$$
\begin{aligned}
\sigma_{n,k} = \sum_{0 \leq \nu_1 < \nu_2 < \cdots < \nu_N \leq 2N-1} & \det_{1 \leq i,j \leq N} \left(\frac{1}{2^{\nu_j}} S_{2i-1-\nu_j}(\boldsymbol{x}^+(n,k) + \nu_j \boldsymbol{s}) \right) \\
& \det_{1 \leq i,j \leq N} \left(\frac{1}{2^{\nu_j}} S_{2i-1-\nu_j}(\boldsymbol{x}^-(n,k) + \nu_j \boldsymbol{s}) \right).
\end{aligned} \tag{20}
$$

In addition, denoting ξ_r and η_r as the real and imaginary parts of a_r, we can easily see that

$$
\partial_{\xi_r} S_n(\boldsymbol{x}^\pm + \nu \boldsymbol{s}) = S_{n-r}(\boldsymbol{x}^\pm + \nu \boldsymbol{s}), \quad \partial_{\eta_r} S_n(\boldsymbol{x}^\pm + \nu \boldsymbol{s}) = \pm \mathrm{i} S_{n-r}(\boldsymbol{x}^\pm + \nu \boldsymbol{s}). \tag{21}
$$

Using these two equations, we can show that

$$
\partial_{\xi_{2r}} \sigma_{n,k} = \partial_{\eta_{2r}} \sigma_{n,k} = 0, \tag{22}
$$

which proves that rogue waves in Theorem 1 are independent of parameters a_{even}. Thus, we will simply set $a_2 = a_4 = \cdots = a_{\text{even}} = 0$ throughout this article. Of the remaining parameters, we can normalize $a_1 = 0$ through a shift of x and t. Then, the N-th order rogue waves in the GDNLS equation (2) contain $N-1$ free irreducible complex parameters, $a_3, a_5, \ldots, a_{2N-1}$. This number of irreducible free parameters is the same as that in rogue waves of the NLS equation (Ohta and Yang 2012a).

4 Dynamical Patterns of Rogue Waves

In this section, we analyze the dynamics of rogue waves in Theorem 1 for the GDNLS equations (2).

First of all, we notice from Eq. (13) that the amplitude profile of the rogue wave is

$$|u_N(x,t)| = \frac{|g_N(x,t)|}{|f_N(x,t)|}, \tag{23}$$

which is independent of the equation parameter γ. This means that the intensity patterns of rogue waves are the same for *all* GDNLS equations (2) regardless of the γ value. But the phase structure of rogue waves is influenced by the γ value. Indeed, the gauge transformation (3) tells us that on top of rogue waves $v(x,t)$ of the Kaup–Newell equation, different values of γ introduce an extra phase $\theta(x,t) = (2-\gamma)\int |v(x,t)|^2 dx$, which can be calculated directly from the bilinear solution (13). This phase induces a "chirp" to an optical rogue wave, which was examined in detail in Chen et al. (2019).

Although the rogue wave intensity pattern in the GDNLS equations (2) is independent of γ, it does depend on the wavenumber parameter α of the constant background. We will focus on this α dependence of the rogue-wave intensity pattern next.

First, we consider fundamental rogue waves, where we set $N = 1$ in Theorem 1. In addition, we normalize $a_1 = 0$ (see the remark in the end of the last section). Then, we get

$$|u_1(x,t;\alpha)| = \left| \frac{\alpha(x+2\alpha t)^2 + (x-2t)^2 - \mathrm{i}(x+6\alpha t) - \frac{3}{4}}{\alpha(x+2\alpha t)^2 + (x-2t)^2 + \mathrm{i}(x+2\alpha t) - 4\mathrm{i}t + \frac{1}{4}} \right|. \tag{24}$$

At three values of $\alpha = 0.5$, 1 and 2, this amplitude profile is shown in Fig. 1a–c, respectively. It is seen that α strongly affects the orientation and duration of the rogue wave. Specifically, as the α value increases, the orientation angle also increases, but the duration of the rogue wave decreases. However, the peak amplitudes of these rogue waves for different α values are all equal to 3, which are attained at the center $x = t = 0$, i.e., $|u_1(0,0;\alpha)| = 3$. Physically, the longer duration of rogue waves at smaller α values can be understood, because in this case, the growth rates of baseband modulation instability can be shown to be smaller, which causes the rogue wave to take longer time to rise from the unit-amplitude background to its peak amplitude of 3. The dependence of the orientation angle on α can also be heuristically understood. It is known that the phase gradient of a pulse generally causes the pulse to move at a velocity which is proportional to this phase gradient. In the present case, the phase gradient of the rogue wave can be estimated from Eq. (13) as the wavenumber $1-\gamma-\alpha$. Then, for a fixed equation parameter γ, larger α causes the velocity to be smaller or negative, leading to a larger orientation angle. To put these results in perspective, we

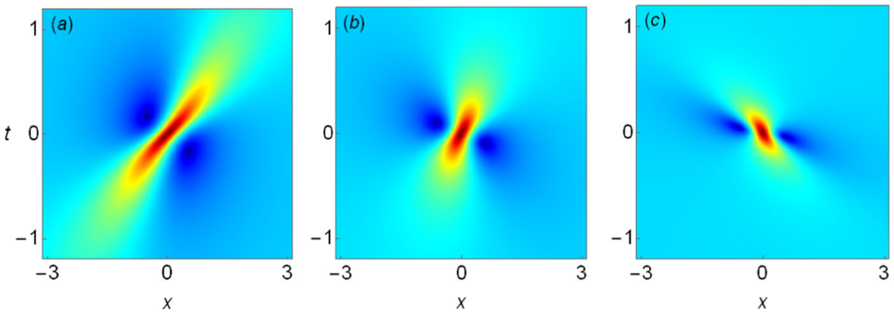

Fig. 1 Amplitude profiles (24) of first-order rogue waves. **a** $\alpha = 0.5$; **b** $\alpha = 1$; **c** $\alpha = 2$

note that for the NLS equation, since the constant-background wavenumber of its rogue waves can be normalized by a Galilean transformation (Ohta and Yang 2012a), the background wavenumber only affects the orientation, but not duration, of its rogue waves.

It is interesting to notice that in the limit of $\alpha \to 0$,

$$|u_1(x, t; \alpha)| \to \left| \frac{(x - 2t)^2 - \mathrm{i}x - \frac{3}{4}}{(x - 2t)^2 + \mathrm{i}x - 4\mathrm{i}t + \frac{1}{4}} \right|, \tag{25}$$

which becomes a quadratic algebraic soliton instead of a rogue wave. This comes about because when $\alpha = 0$, baseband modulation instability disappears in the GDNLS equation (2), and thus, rogue waves no longer exist.

Now we consider second-order rogue waves, where we set $N = 2$ and $a_1 = 0$ in Theorem 1. These solutions contain one complex free parameter a_3. When $a_3 = 0$, the resulting rogue wave is parity-time-symmetric, and it reaches peak amplitude 5 at the center $x = t = 0$ for all α values, i.e., $|u_2(0, 0; \alpha)| = 5$. This peak amplitude 5 is the maximum peak amplitude for all rogue waves of second order, and thus, this parity-time-symmetric rogue wave was called the super rogue wave in Chen et al. (2019). The amplitude profile of this super rogue wave depends on the wavenumber parameter α though. At three α values of 0.5, 1 and 2, these super rogue waves are displayed in Fig. 2. Again, α strongly affects the orientation and duration of these rogue waves.

When $a_3 \neq 0$, the second-order rogue waves generally will split into three separate first-order rogue waves, as has been reported in Guo et al. (2013, 2014), Zhang et al. (2017) and Chen et al. (2019). This phenomenon is similar to second-order rogue waves of the NLS equation (Akhmediev et al. 2009b; Ankiewicz et al. 2010a; Dubard et al. 2010; Kedziora et al. 2011; Guo et al. 2012; Ohta and Yang 2012a; Dubard and Matveev 2013). The orientations and durations of these three separate first-order rogue waves are determined by the wavenumber parameter α.

Having clarified the effect of wavenumber parameter α on rogue waves, at third order, we will fix $\alpha = 1$ and explore new rogue wave patterns. For this purpose, we set $N = 3$ and $a_1 = 0$, and the remaining free complex parameters are a_3 and a_5. When $a_3 = a_5 = 0$, we get a super rogue wave with peak amplitude 7 (see also

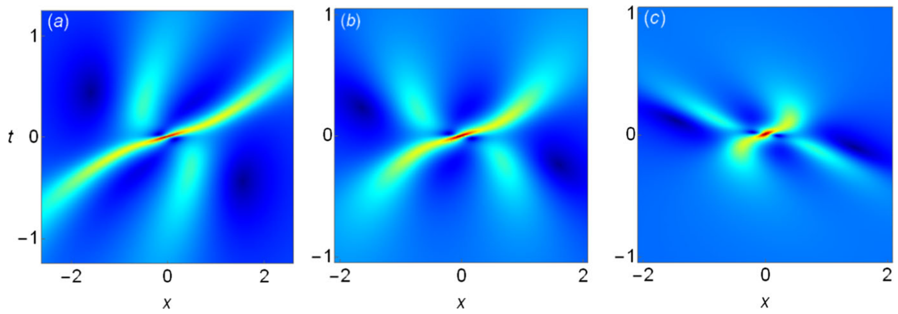

Fig. 2 Amplitude profiles $|u_2(x,t)|$ of second-order super rogue waves (with $a_3 = 0$). **a** $\alpha = 0.5$; **b** $\alpha = 1$; **c** $\alpha = 2$

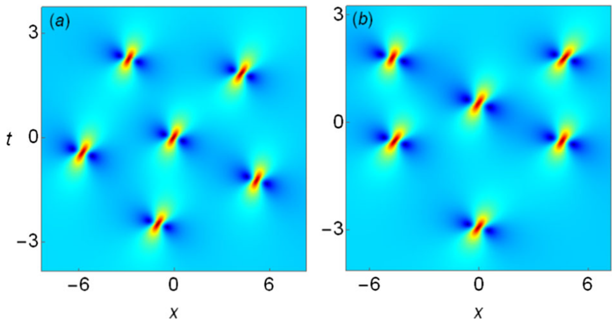

Fig. 3 Third-order rogue waves with $\alpha = 1$. Left: a pentagon pattern, where $a_3 = 0$ and $a_5 = 80 + 80\mathrm{i}$. Right: a mixed pattern, where $a_3 = 10\mathrm{i}$ and $a_5 = 100\mathrm{i}$

Zhang et al. 2017; Guo et al. 2014; Chen et al. 2019). At other a_3 and a_5 values, the third-order rogue wave generally splits into 6 separate first-order rogue waves in various configurations. Two such solutions are displayed in Fig. 3. The left panel shows a pentagon pattern, which has been seen before (Zhang et al. 2017; Guo et al. 2014). But the right panel shows a mix of a first-order rogue wave and a cluster of five first-order rogue waves in square configuration, which is novel to our knowledge. Our results suggest that when a third-order rogue wave splits into 6 separate first-order rogue waves, these 6 first-order rogue waves can appear in arbitrary configurations in the (x,t) plane. The same should hold for higher-order rogue waves too.

5 Derivation of Rogue Waves from the First Bilinear System

As we mentioned in Sect. 2, the GDNLS Eq. (2) can be decomposed into two different bilinear systems. In this section, we will derive rogue waves in Theorem 1 from the first bilinear system (7)–(10). The basic idea of this derivation is similar to that in Ohta and Yang (2012a) for the NLS equation. The main improvement is that we will choose differential operators in the bilinear solutions in a different way, which leads to a more convenient parameterization and simpler expression for rogue waves.

5.1 Gram Determinant Solutions for a Higher-Dimensional Bilinear System

First, we need to derive algebraic solutions to a higher-dimensional bilinear system, which can reduce to the original lower-dimensional bilinear system (7)–(10) under certain reductions.

From Lemma 2 of Chen et al. (2018b), section 3.2 of Feng et al. (2017) and our own calculations, we learn that if functions $m_{i,j}^{(n,k)}$, $\varphi_i^{(n,k)}$ and $\psi_j^{(n,k)}$ of variables (x_{-1}, x_1, x_2) satisfy the following differential and difference relations,

$$\begin{aligned}
&\partial_{x_1} m_{i,j}^{(n,k)} = \varphi_i^{(n,k)} \psi_j^{(n,k)},\\
&\partial_{x_1} \varphi_i^{(n,k)} = \varphi_i^{(n+1,k)},\ \partial_{x_1} \psi_j^{(n,k)} = -\psi_j^{(n-1,k)},\\
&\partial_{x_1} \varphi_i^{(n,k)} = c\varphi_i^{(n,k)} + \varphi_i^{(n,k+1)},\ \partial_{x_1} \psi_j^{(n,k)} = -c\psi_j^{(n,k)} - \psi_j^{(n,k-1)},\\
&\partial_{x_2} \varphi_i^{(n,k)} = \partial_{x_1}^2 \varphi_i^{(n,k)},\ \partial_{x_2} \psi_j^{(n,k)} = -\partial_{x_1}^2 \psi_j^{(n,k)},\\
&\partial_{x_{-1}} \varphi_i^{(n,k)} = \varphi_i^{(n,k-1)},\ \partial_{x_{-1}} \psi_j^{(n,k)} = -\psi_j^{(n,k+1)},
\end{aligned} \tag{26}$$

where c is an arbitrary complex constant, and then, they would also satisfy the following relations:

$$\begin{aligned}
&\partial_{x_2} m_{i,j}^{(n,k)} = \varphi_i^{(n+1,k)} \psi_j^{(n,k)} + \varphi_i^{(n,k)} \psi_j^{(n-1,k)},\\
&\partial_{x_2} m_{i,j}^{(n,k)} = \varphi_i^{(n,k+1)} \psi_j^{(n,k)} + \varphi_i^{(n,k)} \psi_j^{(n,k-1)} + 2c\varphi_i^{(n,k)} \psi_j^{(n,k)},\\
&\partial_{x_{-1}} m_{i,j}^{(n,k)} = -\varphi_i^{(n,k-1)} \psi_j^{(n,k+1)},\\
&m_{i,j}^{(n+1,k)} = m_{i,j}^{(n,k)} + \varphi_i^{(n,k)} \psi_j^{(n+1,k)},\\
&m_{i,j}^{(n,k+1)} = m_{i,j}^{(n,k)} + \varphi_i^{(n,k)} \psi_j^{(n,k+1)}.
\end{aligned} \tag{27}$$

Using these relations, one can show that the determinant

$$\tau_{n,k} = \det_{1\le i,j\le N} \left(m_{i,j}^{(n,k)} \right) \tag{28}$$

would satisfy the following bilinear equations in the extended KP hierarchy

$$\left(D_{x_2} - D_{x_1}^2 - 2cD_{x_1} \right) \tau_{n-1,k+1} \cdot \tau_{n-1,k} = 0, \tag{29}$$

$$\left(D_{x_2} - D_{x_1}^2 \right) \tau_{n,k} \cdot \tau_{n-1,k} = 0, \tag{30}$$

$$\left(cD_{x_{-1}} - 1 \right) \tau_{n,k} \cdot \tau_{n-1,k} + \tau_{n-1,k+1} \tau_{n,k-1} = 0, \tag{31}$$

$$(cD_{x_1} D_{x_{-1}} - D_{x_1} - 2c)\tau_{n,k} \cdot \tau_{n-1,k} + (D_{x_1} + 2c)\tau_{n-1,k+1} \cdot \tau_{n,k-1} = 0. \tag{32}$$

Now, we introduce functions $m^{(n,k)}$, $\varphi^{(n,k)}$ and $\psi^{(n,k)}$ as

$$m^{(n,k)} = \frac{\mathrm{i}p}{p+q} \left(-\frac{p}{q} \right)^n \left(-\frac{p-c}{q+c} \right)^k \mathrm{e}^{\xi+\eta}, \tag{33}$$

$$\varphi^{(n,k)} = (\mathrm{i}p)p^n(p-c)^k e^{\xi}, \tag{34}$$

$$\psi^{(n,k)} = (-q)^{-n}\left[-(q+c)\right]^{-k} e^{\eta}, \tag{35}$$

where

$$\xi = \frac{1}{p-c}x_{-1} + px_1 + p^2x_2 + \xi_0, \tag{36}$$

$$\eta = \frac{1}{q+c}x_{-1} + qx_1 - q^2x_2 + \eta_0, \tag{37}$$

and p, q, ξ_0 and η_0 are the arbitrary complex constants. It is easy to see that these functions satisfy the differential and difference relations (26) with indices i and j ignored. Then, by defining

$$m_{ij}^{(n,k)} = \mathcal{A}_i\mathcal{B}_j m^{(n,k)}, \quad \varphi_i^{(n,k)} = \mathcal{A}_i\varphi^{(n,k)}, \quad \psi_j^{(n,k)} = \mathcal{B}_j\psi^{(n,k)}, \tag{38}$$

where $\mathcal{A}_i$ and $\mathcal{B}_j$ are the differential operators with respect to p and q, respectively, as

$$\mathcal{A}_i = \frac{1}{i!}\left[(p-c)\partial_p\right]^i, \quad \mathcal{B}_j = \frac{1}{j!}\left[(q+c)\partial_q\right]^j, \tag{39}$$

these functions would also satisfy the differential and difference relations (26) since operators $\mathcal{A}_i$ and $\mathcal{B}_j$ commute with differentials ∂_{x_k}. Consequently, for an arbitrary sequence of indices $(i_1, i_2, \cdots, i_N; j_1, j_2, \cdots, j_N)$, the determinant

$$\tau_{n,k} = \det_{1\le\nu,\mu\le N}\left(m_{i_\nu,j_\mu}^{(n,k)}\right) \tag{40}$$

satisfies the higher-dimensional bilinear system (29)–(32).

It is important to notice that the differential operators $\mathcal{A}_i$ and $\mathcal{B}_j$ defined here are simpler than the ones in previous bilinear derivations of rogue waves (Ohta and Yang 2012a, b, 2013, 2014; Yang and Yang 2020; Chen et al. 2018a; Zhang and Chen 2018). Indeed, the current differential operators are single terms, while previous ones were defined as summations. The reason for the previous summation definitions was to introduce internal free parameters in rogue waves. In our current approach, we will introduce free constants through ξ_0 and η_0 in Eqs. (36)–(37), which will be done later in this section.

Next, we will reduce the higher-dimensional bilinear system (29)–(32) to the original bilinear system (7)–(10), so that the higher-dimensional solutions (40) become rogue wave solutions to the GDNLS equations (2). In this reduction, we will need to set

$$c = -\mathrm{i}\alpha, \tag{41}$$

where c is the parameter in the higher-dimensional system (29)–(32), and α is the wavenumber parameter in the original bilinear system (7)–(10).

5.2 Dimensional Reduction

First, we reduce the higher-dimensional bilinear system (29)–(32) to a lower-dimensional one, a process called dimension reduction. This reduction will restrict the indices in the determinant (40), and select the (p, q) values in its matrix element $m_{i_\nu, j_\mu}^{(n,k)}$.

The dimension reduction condition we impose is

$$\left(\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right)\tau_{n,k} = C\tau_{n,k}, \tag{42}$$

where C is some constant. Denoting $\hat{p} \equiv p - c$ and $\hat{q} \equiv q + c$, then $\mathcal{A}_i$ and $\mathcal{B}_j$ in Eq. (39) can be rewritten as

$$\mathcal{A}_i = \frac{1}{i!}\left(\hat{p}\partial_{\hat{p}}\right)^i, \quad \mathcal{B}_j = \frac{1}{j!}\left(\hat{q}\partial_{\hat{q}}\right)^j. \tag{43}$$

In addition,

$$\begin{aligned}\left(\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right) m_{i,j}^{(n,k)} &= \mathcal{A}_i\mathcal{B}_j\left(\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right) m^{(n,k)} \\ &= \mathcal{A}_i\mathcal{B}_j\left[\hat{p} + \frac{\mathrm{i}c}{\hat{p}} + \hat{q} + \frac{\mathrm{i}c}{\hat{q}}\right] m^{(n,k)}.\end{aligned}$$

Using the Leibnitz rule exactly as in Ohta and Yang (2012a), the above equation reduces to

$$\begin{aligned}\left(\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right) m_{i,j}^{(n,k)} &= \sum_{\mu=0}^{i}\frac{1}{\mu!}\left(\hat{p} + (-1)^\mu\frac{\mathrm{i}c}{\hat{p}}\right) m_{i-\mu,j}^{(n,k)} \\ &\quad + \sum_{l=0}^{j}\frac{1}{l!}\left(\hat{q} + (-1)^l\frac{\mathrm{i}c}{\hat{q}}\right) m_{i,j-l}^{(n,k)}.\end{aligned}$$

Recalling $c = -\mathrm{i}\alpha$ from (41), we see that when we set $p = p_0$ and $q = q_0$, where

$$p_0 = \sqrt{\alpha} - \mathrm{i}\alpha, \quad q_0 = \sqrt{\alpha} + \mathrm{i}\alpha, \tag{44}$$

the above equation would further simplify to

$$\begin{aligned}\left(\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right) m_{i,j}^{(n,k)}\Big|_{p=p_0,\ q=q_0} &= 2\sqrt{\alpha}\sum_{\substack{\mu=0,\\ \mu:even}}^{i}\frac{1}{\mu!}\, m_{i-\mu,j}^{(n,k)}\Big|_{p=p_0,\ q=q_0} \\ &\quad + 2\sqrt{\alpha}\sum_{\substack{l=0,\\ l:even}}^{j}\frac{1}{l!}\, m_{i,j-l}^{(n,k)}\Big|_{p=p_0,\ q=q_0}.\end{aligned} \tag{45}$$

Now, we restrict the general determinant (40) to

$$\tau_{n,k} = \det_{1\leq i,j\leq N}\left(m^{(n,k)}_{2i-1,2j-1}\Big|_{p=p_0,\ q=q_0}\right). \tag{46}$$

Then, using the contiguity relation (45) as in Ohta and Yang (2012a), we get

$$\left(\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right)\tau_{n,k} = 4\sqrt{\alpha}\,N\,\tau_{n,k},$$

which shows that the $\tau_{n,k}$ function (46) satisfies the dimension reduction condition (42).

When this dimension reduction equation is used to eliminate x_{-1} from the higher-dimensional bilinear system (29)–(32), and in view of the parameter connection (41), we get

$$\left(D_{x_2} - D^2_{x_1} + 2\mathrm{i}\alpha D_{x_1}\right)\tau_{n-1,k+1}\cdot\tau_{n-1,k} = 0, \tag{47}$$

$$\left(D_{x_2} - D^2_{x_1}\right)\tau_{n,k}\cdot\tau_{n-1,k} = 0, \tag{48}$$

$$\left(\mathrm{i}D_{x_1} - 1\right)\tau_{n,k}\cdot\tau_{n-1,k} + \tau_{n-1,k+1}\tau_{n,k-1} = 0, \tag{49}$$

$$(D^2_{x_1} + \mathrm{i}D_{x_1} + 2\alpha)\tau_{n,k}\cdot\tau_{n-1,k} - (\mathrm{i}D_{x_1} + 2\alpha)\tau_{n-1,k+1}\cdot\tau_{n,k-1} = 0. \tag{50}$$

In addition, using Eq. (49), we can replace the last bilinear equation (50) by

$$\begin{aligned} &D^2_{x_1}\tau_{n,k}\cdot\tau_{n-1,k} - \mathrm{i}D_{x_1}\tau_{n-1,k+1}\cdot\tau_{n,k-1} \\ &\quad +(2\alpha+1)(\tau_{n,k}\cdot\tau_{n-1,k} - \tau_{n-1,k+1}\cdot\tau_{n,k-1}) = 0. \end{aligned} \tag{51}$$

In these reduced bilinear equations, the x_{-1} derivative disappears.

To further reduce the bilinear system (47)–(49) and (51) to the original system (7)–(10), we set

$$x_1 = x - 2t, \quad x_2 = \mathrm{i}t. \tag{52}$$

Under this variable relation, we have

$$\partial_{x_1} = \partial_x, \quad \partial_{x_2} = -\mathrm{i}\partial_t - 2\mathrm{i}\partial_x. \tag{53}$$

Inserting these equations into the bilinear system (47)–(49) and (51), and setting $n = k = 0$, we get

$$\left(\mathrm{i}D_t + D^2_x + 2\mathrm{i}(1-\alpha)D_x\right)g\cdot\bar{f} = 0, \tag{54}$$

$$\left(\mathrm{i}D_t + D^2_x + 2\mathrm{i}D_x\right)f\cdot\bar{f} = 0, \tag{55}$$

$$(\mathrm{i}D_x - 1)f\cdot\bar{f} + g\bar{g} = 0, \tag{56}$$

$$D^2_x f\cdot\bar{f} - \mathrm{i}D_x g\cdot\bar{g} + (2\alpha+1)(|f|^2 - |g|^2) = 0, \tag{57}$$

where f, $\bar{f}$, g and $\bar{g}$ are defined as

$$f=\tau_{0,0},\quad \bar{f}=\tau_{-1,0},\quad g=\tau_{-1,1},\quad \bar{g}=\tau_{0,-1}. \tag{58}$$

5.3 Complex Conjugacy Conditions

Next, we need to impose complex conjugacy conditions $\bar{f}=f^*$ and $\bar{g}=g^*$, i.e.,

$$\tau_{-1,0}=\tau_{0,0}^*,\quad \tau_{0,-1}=\tau_{-1,1}^*, \tag{59}$$

so that the bilinear system (54)–(57) would reduce to the original bilinear system (7)–(10). These conditions can be satisfied by imposing the parameter constraint $\xi_0=\eta_0^*$. Indeed, under this constraint, since $x_1=x-2t$ is real, $x_2=\mathrm{i}t$, $c=-\mathrm{i}\alpha$ are pure imaginary, and $q_0=p_0^*$, we can easily show that

$$\left[m_{i,j}^{(n,k)}\right]^*\Big|_{p=p_0,\ q=q_0}=m_{j,i}^{(-n-1,-k)}\Big|_{p=p_0,\ q=q_0}. \tag{60}$$

Thus, $\tau_{n,k}^*=\tau_{-n-1,-k}$, i.e., the complex conjugacy conditions (59) hold.

5.4 Rogue Wave Solutions in Differential Operator Form

Finally, we need to introduce free parameters into rogue waves. Unlike all previous bilinear approaches (Ohta and Yang 2012a, b, 2013, 2014; Yang and Yang 2020; Chen et al. 2018a; Zhang and Chen 2018), we will introduce free parameters through the arbitrary constant ξ_0 in Eq. (36). Specifically, we choose ξ_0 as

$$\xi_0=\sum_{r=1}^{\infty}\hat{a}_r\ln^r\left(\frac{p-c}{p_0-c}\right)=\sum_{r=1}^{\infty}\hat{a}_r\ln^r\left(\frac{p+\mathrm{i}\alpha}{\sqrt{\alpha}}\right), \tag{61}$$

where $\hat{a}_r$ are the free complex constants. We can show that rogue waves with this new parameterization can be converted to those with the old parameterization through non-trivial parameter connections. But the new parameterization will drastically simplify rogue wave expressions.

Putting all the above results together and setting $x_{-1}=0$, rational solutions to the GDNLS equations (2) are given by the following theorem.

Theorem 3 *The GDNLS equations* (2) *admit rational solutions*

$$u_N(x,t)=e^{\mathrm{i}(1-\gamma-\alpha)x-\mathrm{i}\left[\alpha^2+2(\gamma-2)\alpha+1-\gamma\right]t}\,\frac{(f_N^*)^{\gamma-1}g_N}{f_N^{\gamma}}, \tag{62}$$

where

$$f_N(x,t)=\tau_{0,0},\quad g_N(x,t)=\tau_{-1,1}, \tag{63}$$

$$\tau_{n,k} = \det_{1 \le i,j \le N} \left(m^{(n,k)}_{2i-1,2j-1} \right), \tag{64}$$

the matrix elements in $\tau_{n,k}$ *are defined by*

$$m^{(n,k)}_{i,j} = \frac{\left[(p + \mathrm{i}\alpha)\partial_p\right]^i}{i!} \frac{\left[(q - \mathrm{i}\alpha)\partial_q\right]^j}{j!} \left[\frac{ip}{p+q} \left(-\frac{p}{q}\right)^n \left(-\frac{p+\mathrm{i}\alpha}{q-\mathrm{i}\alpha}\right)^k e^{\Theta(x,t)} \right] \Bigg|_{p=p_0,\ q=q_0}, \tag{65}$$

with

$$\Theta(x,t) = (p+q)(x-2t) + (p^2 - q^2)it + \sum_{r=1}^{\infty} \hat{a}_r \ln^r \left(\frac{p + \mathrm{i}\alpha}{\sqrt{\alpha}} \right) + \sum_{r=1}^{\infty} \hat{a}_r^* \ln^r \left(\frac{q - \mathrm{i}\alpha}{\sqrt{\alpha}} \right), \tag{66}$$

p_0, q_0 *are given in Eq.* (44)*,* $\alpha > 0$*, and* $\hat{a}_r$ $(r = 1, 2, \ldots)$ *are free complex constants.*

5.5 Explicit Rogue Wave Solutions Through Schur Polynomials

The above rational solutions in Theorem 3 involve differential operators, which make them less explicit. More seriously, such forms make analysis of those solutions difficult. For instance, under such forms, it is difficult to prove that they satisfy the boundary conditions (5). In addition, it is difficult to determine the maximum peak amplitudes for rogue waves of each order. Thus, in this subsection, we derive a more explicit form for these solutions, which is the one given in Theorem 1 earlier in the paper.

The technique we use is similar to that in Ohta and Yang (2012a). The differential operators in (65) can be rewritten as (43), where $\hat{p} = p + \mathrm{i}\alpha$ and $\hat{q} = q - \mathrm{i}\alpha$, and the $m^{(n,k)}$ term following the differential operators in (65) can be rewritten as

$$\begin{aligned} m^{(n,k)} = {} & \frac{\mathrm{i}(\hat{p} - \mathrm{i}\alpha)}{\hat{p} + \hat{q}} \left(-\frac{\hat{p} - \mathrm{i}\alpha}{\hat{q} + \mathrm{i}\alpha} \right)^n \left(-\frac{\hat{p}}{\hat{q}} \right)^k \\ & \times \exp\left\{ \left(\hat{p} + \hat{q}\right)(x - 2t) + \left[\hat{p}^2 - \hat{q}^2 - 2\mathrm{i}\alpha(\hat{p} + \hat{q}) \right] \mathrm{i}t \right. \\ & \left. + \sum_{r=1}^{\infty} \hat{a}_r \ln^r \left(\frac{\hat{p}}{\hat{p}_0} \right) + \sum_{r=1}^{\infty} \hat{a}_r^* \ln^r \left(\frac{\hat{q}}{\hat{q}_0} \right) \right\}, \end{aligned}$$

where $\hat{p}_0 = p_0 + \mathrm{i}\alpha$ and $\hat{q}_0 = q_0 - \mathrm{i}\alpha$, i.e., $\hat{p}_0 = \hat{q}_0 = \sqrt{\alpha}$. Then, introducing the generator $\mathcal{G}$ of differential operators $\left(\hat{p}\partial_{\hat{p}}\right)^i \left(\hat{q}\partial_{\hat{q}}\right)^j$ as

$$\mathcal{G} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\zeta^i}{i!}\frac{\lambda^j}{j!}\left[\hat{p}\partial_{\hat{p}}\right]^i \left[\hat{q}\partial_{\hat{q}}\right]^j , \tag{67}$$

and utilizing the formula (Ohta and Yang 2012a)

$$\mathcal{G}F(\hat{p},\hat{q}) = F\left(e^{\zeta}\hat{p}, e^{\lambda}\hat{q}\right), \tag{68}$$

we get

$$\begin{aligned}
&\mathcal{G}m^{(n,k)}\Big|_{\hat{p}=\hat{p}_0,\ \hat{q}=\hat{q}_0} \\
&= \frac{e^{\zeta/2}(\mathrm{i}e^{\zeta/2} + \sqrt{\alpha}e^{-\zeta/2})}{e^{\zeta}+e^{\lambda}}(-1)^k e^{(k+\frac{n}{2})(\zeta-\lambda)}\left(\frac{\mathrm{i}e^{\zeta/2}+\sqrt{\alpha}e^{-\zeta/2}}{-\mathrm{i}e^{\lambda/2}+\sqrt{\alpha}e^{-\lambda/2}}\right)^n \\
&\quad \times \exp\left\{\sqrt{\alpha}\left(e^{\zeta}+e^{\lambda}\right)(x-2t+2\alpha t) + \alpha\left(e^{2\zeta}-e^{2\lambda}\right)\mathrm{i}t \right. \\
&\qquad \left. + \sum_{r=1}^{\infty}\left(a_r\zeta^r + a_r^*\lambda^r\right)\right\}.
\end{aligned}$$

Since

$$m^{(n,k)}\Big|_{\hat{p}=\hat{p}_0,\ \hat{q}=\hat{q}_0} = (-1)^k\,\frac{(\mathrm{i}+\sqrt{\alpha})}{2}\left(\frac{\mathrm{i}+\sqrt{\alpha}}{-\mathrm{i}+\sqrt{\alpha}}\right)^n e^{2\sqrt{\alpha}(x-2t+2\alpha t)},$$

we have

$$\begin{aligned}
&\frac{1}{m^{(n,k)}}\,\mathcal{G}m^{(n,k)}\Big|_{\hat{p}=\hat{p}_0,\ \hat{q}=\hat{q}_0} \\
&= \frac{2}{e^{\zeta}+e^{\lambda}} e^{\zeta/2+(k+\frac{n}{2})(\zeta-\lambda)}\left(\frac{\mathrm{i}e^{\zeta/2}+\sqrt{\alpha}e^{-\zeta/2}}{\mathrm{i}+\sqrt{\alpha}}\right)^{n+1}\left(\frac{-\mathrm{i}+\sqrt{\alpha}}{-\mathrm{i}e^{\lambda/2}+\sqrt{\alpha}e^{-\lambda/2}}\right)^n \\
&\quad \times \exp\left(\sqrt{\alpha}\left(e^{\zeta}+e^{\lambda}-2\right)(x-2t+2\alpha t) + \alpha\left(e^{2\zeta}-e^{2\lambda}\right)\mathrm{i}t \right. \\
&\qquad \left. + \sum_{r=1}^{\infty}(a_r\zeta^r + a_r^*\lambda^r)\right).
\end{aligned} \tag{69}$$

Now, we need to expand the right side of the above equation into power series of ζ and λ. For this purpose, we denote

$$\frac{\mathrm{i}e^{\zeta/2}+\sqrt{\alpha}e^{-\zeta/2}}{\mathrm{i}+\sqrt{\alpha}} = \exp\left[\ln\left(\frac{\mathrm{i}e^{\zeta/2}+\sqrt{\alpha}e^{-\zeta/2}}{\mathrm{i}+\sqrt{\alpha}}\right)\right] = \exp\left(\sum_{r=1}^{\infty} h_r\zeta^r\right),$$

where $h_r(\alpha)$ is as defined in Eq. (15). The exponent in the most right-hand side of Eq. (69) can be rewritten as

$$\exp\left(\sum_{r=1}^{\infty}\frac{\zeta^r}{r!}\left(\sqrt{\alpha}(x-2t+2\alpha t)+2^r\mathrm{i}\alpha t\right)\right.$$
$$\left.+\sum_{r=1}^{\infty}\frac{\lambda^r}{r!}\left(\sqrt{\alpha}(x-2t+2\alpha t)-2^r\mathrm{i}\alpha t\right)+\sum_{r=1}^{\infty}(a_r\zeta^r+a_r^*\lambda^r)\right),$$

and the $2/(e^{\zeta}+e^{\lambda})$ term can be written as Ohta and Yang (2012a)

$$\frac{2}{e^{\zeta}+e^{\lambda}}=\sum_{\nu=0}^{\infty}\left(\frac{\zeta\lambda}{4}\right)^{\nu}\exp\left(\sum_{r=1}^{\infty}(\nu s_r-c_r)\left(\zeta^r+\lambda^r\right)-\frac{\zeta}{2}-\frac{\lambda}{2}\right),$$

where c_r are the Taylor coefficients of λ^r in the expansion of $\ln\cosh(\lambda/2)$, and s_r are given in Eq. (15). Combining the above results, Eq. (69) becomes

$$\frac{1}{m^{(n,k)}}\,\mathcal{G}m^{(n,k)}\Big|_{\hat{p}=\hat{p}_0,\ \hat{q}=\hat{q}_0}$$
$$=\sum_{\nu=0}^{\infty}\left(\frac{\zeta\lambda}{4}\right)^{\nu}\exp\left(\sum_{r=1}^{\infty}\left(x_r^{+}+\nu s_r\right)\zeta^r+\sum_{r=1}^{\infty}\left(x_r^{-}+\nu s_r\right)\lambda^r\right),\tag{70}$$

where $x_r^{+}(n,k)$ and $x_r^{-}(n,k)$ are defined as

$$x_1^{+}(n,k)=\sqrt{\alpha}(x-2t+2\alpha t)+2\mathrm{i}\alpha t+(n+1)h_1+k+\frac{n}{2}-c_1+\hat{a}_1,$$
$$x_1^{-}(n,k)=\sqrt{\alpha}(x-2t+2\alpha t)-2\mathrm{i}\alpha t-nh_1^*-k-\frac{1}{2}(n+1)-c_1+\hat{a}_1^*,$$
$$x_r^{+}(n,k)=\frac{1}{r!}\left[\sqrt{\alpha}(x-2t+2\alpha t)+2^r\mathrm{i}\alpha t\right]+(n+1)h_r-c_r+\hat{a}_r,$$
$$x_r^{-}(n,k)=\frac{1}{r!}\left[\sqrt{\alpha}(x-2t+2\alpha t)-2^r\mathrm{i}\alpha t\right]-nh_r^*-c_r+\hat{a}_r^*.$$

We further define shifted parameters

$$a_1=\hat{a}_1-c_1+\frac{1}{2}h_1-\frac{1}{4},\quad a_r=\hat{a}_r-c_r+\frac{1}{2}h_r.$$

Then, the above x_r^{+} and x_r^{-} reduce to those in Theorem 1. Taking the coefficients of $\zeta^i\lambda^j$ on both sides of Eq. (70), we get

$$\frac{m_{i,j}^{(n,k)}}{m^{(n,k)}\big|_{p=p_0,q=q_0}}=\sum_{\nu=0}^{\min(i,j)}\frac{1}{4^{\nu}}S_{i-\nu}\left(\boldsymbol{x}^{+}+\nu\boldsymbol{s}\right)S_{j-\nu}\left(\boldsymbol{x}^{-}+\nu\boldsymbol{s}\right),$$

where $m_{i,j}^{(n,k)}$ is the matrix element defined in Eq. (65) of Theorem 3. Finally, we define

$$\sigma_{n,k} = \frac{\tau_{n,k}}{\left(m^{(n,k)}\big|_{p=p_0,q=q_0} \right)^N}.$$

Then, the matrix element in $\sigma_{n,k}$ is as given in Theorem 1. Since the bilinear equations (7)–(10) are invariant when f and g are divided by an arbitrary complex constant multiplying an exponential of a linear and real function in x and t, $\sigma_{n,k}$ then is also a solution to the GDNLS Eq. (2).

Regarding boundary conditions of these rational solutions, using the Schur polynomial expressions in Theorem 1 and the same technique as in Ohta and Yang (2012a), we can show that when x or t approaches infinity, $f_N(x, t)$ and $g_N(x, t)$ have the same leading term, which is also real. Thus, the rational solution (13) satisfies the boundary condition (5), and is thus a rogue wave. Theorem 1 is then proved.

5.6 The Parity-Time-Symmetric Rogue Wave

In this subsection, we derive the parity-time-symmetric rogue wave and prove Theorem 2.

When we set all $a_r = 0$ in Theorem 1, x_r^+ and x_r^- satisfy the following relations

$$\widehat{x}_r^{\pm}(x, t) = -x_r^{\mp}(x, t), \quad r \geq 1,$$

where we have defined $\widehat{f}(x, t) \equiv f^*(-x, -t)$ for any function $f(x, t)$. Thus,

$$\widehat{\mathbf{x}}^{\pm}(n, k) + \nu\, \boldsymbol{s} = \mathbf{y}^{\mp}(n, k) + \nu\, \boldsymbol{s} + \mathbf{z}^{\mp}(n),$$

where vectors $\mathbf{y}^{\pm}$ and $\mathbf{z}^{\pm}$ are defined as

$$\mathbf{y}^{\pm} = \left(-x_1^{\pm}, x_2^{\pm}, -x_3^{\pm}, x_4^{\pm}, \cdots\right), \quad \mathbf{z}^{\pm} = \left(0, -2x_2^{\pm}, 0, -2x_4^{\pm}, 0, \cdots\right).$$

Notice that

$$\begin{aligned}
\sum_{j=0}^{\infty} S_j\left(\widehat{\mathbf{x}}^{\mp} + \nu\, \boldsymbol{s}\right)\lambda^j &= \sum_{j=0}^{\infty} S_j\left(\mathbf{y}^{\pm} + \nu\, \boldsymbol{s} + \mathbf{z}^{\pm}\right)\lambda^j \\
&= \exp\left(\sum_{j=1}^{\infty}\left(y_j^{\pm} + \nu\, s_j + z_j^{\pm}\right)\lambda^j\right) \\
&= \exp\left(\sum_{j=1}^{\infty}\left(y_j^{\pm} + \nu\, s_j\right)\lambda^j\right)\exp\left(\sum_{j=1}^{\infty} z_j^{\pm}\lambda^j\right) \\
&= \sum_{j=0}^{\infty} S_j(\mathbf{y}^{\pm} + \nu \boldsymbol{s})\lambda^j \sum_{j=0}^{\infty} S_j(\mathbf{z}^{\pm})\lambda^j
\end{aligned}$$

$$= \sum_{j=0}^{\infty} \sum_{\mu_1+\mu_2=j} S_{\mu_1}(\mathbf{y}^{\pm} + \nu \mathbf{s}) S_{\mu_2}(\mathbf{z}^{\pm}) \lambda^j .$$

Since $s_1 = s_3 = \cdots = s_{\text{odd}} = 0$ in view of Eq. (17), by comparing the coefficient of λ^j on the two sides of this equation and utilizing Lemmas 2 and 3 in Yang and Yang (2020), we get the relation

$$S_j\left(\widehat{\boldsymbol{x}}^{\mp} + \nu\, \boldsymbol{s}\right) = (-1)^j \sum_{\mu=0}^{[j/2]} S_\mu(\boldsymbol{w}^{\pm}) S_{j-2\mu}(\boldsymbol{x}^{\pm} + \nu \boldsymbol{s}), \tag{71}$$

where $\boldsymbol{w}^{\pm} = \left(-2x_2^{\pm}, -2x_4^{\pm}, \cdots\right)$. Recall from Theorem 1 that

$$\sigma_{n,k} = \det_{1\le i,j\le N} \left(\sum_{\nu=0}^{\min(2i-1,\, 2j-1)} \frac{1}{2^\nu} S_{2i-1-\nu}(\boldsymbol{x}^{+}(n,k) + \nu \boldsymbol{s}) \frac{1}{2^\nu} S_{2j-1-\nu}(\boldsymbol{x}^{-}(n,k) + \nu \boldsymbol{s}) \right),$$

and $\widehat{\sigma}_{n,k}$ is equal to the right side of the above equation with $\boldsymbol{x}^{\pm}$ replaced by $\widehat{\boldsymbol{x}}^{\pm}$. By rewriting these two determinants into $3N \times 3N$ determinants as in Ohta and Yang (2012a), utilizing relations (71) and performing simple row manipulations, we can quickly show that $\widehat{\sigma}_{n,k} = \sigma_{n,k}$. Thus, the solution $u_N(x,t)$ in Theorem 1 with all a_r being zero satisfies the parity-time symmetry $\widehat{u}_N = u_N$, i.e., $u_N^*(-x,-t) = u_N(x,t)$. Theorem 2 is then proved.

It turns out that the converse is also true, i.e., if a rogue wave $u_N(x,t)$ in Theorem 1 is parity-time-symmetric, then $a_1 = a_3 = \cdots = a_{\text{odd}} = 0$ [there is no restriction on the a_{even} values because the solution is independent of them, see Eq. (22)]. Our proof is based on calculating the derivatives of the polynomial $\sigma_{n,k}$ with respect to the real part ξ_{2r-1} and imaginary part η_{2r-1} of the parameter a_{2r-1}. Using Eqs. (20)–(21), we can show that each of $\partial_{\xi_{2r-1}}\sigma_{n,k}$ and $\mathrm{i}\partial_{\eta_{2r-1}}\sigma_{n,k}$ contains power terms of (x,t) which are not parity-time-symmetric. Thus, if any a_{odd} is nonzero, the solution $u_N(x,t)$ would not be parity-time-symmetric.

6 Rogue Waves Through a Different KP-Reduction Procedure

As we mentioned in Sect. 2, the GDNLS equations (2) admit two different bilinearizations. The first bilinear system is Eqs. (7)–(10), while the second bilinear system is Eqs. (8)–(10) and (11), i.e.,

$$\left(\mathrm{i}D_t + D_x^2 - 2\mathrm{i}\alpha D_x\right) g \cdot f = 0, \tag{72}$$

$$\left(\mathrm{i}D_t + D_x^2 + 2\mathrm{i}D_x\right) f \cdot f^* = 0, \tag{73}$$

$$(\mathrm{i}D_x - 1)\, f \cdot f^* + |g|^2 = 0 \tag{74}$$

$$D_x^2 f \cdot f^* - \mathrm{i}D_x g \cdot g^* + (2\alpha+1)(|f|^2 - |g|^2) = 0. \tag{75}$$

Rogue waves in the GDNLS equations (2), as given in Theorem 1, can also be derived from this second bilinear system, but the corresponding KP-reduction procedure is different. This will be shown below. This situation is analogous to multi-soliton solutions in the Sasa–Satsuma equation, which also admit two different bilinearizations and two different reduction procedures (Gilson et al. 2003)

6.1 Algebraic Solutions for a Higher-Dimensional Bilinear System

First, we consider the following higher-dimensional bilinear equations in the extended KP hierarchy

$$\left(D_{x_2}-D_{x_1}^2-2dD_{x_1}\right)\tau_{n,k,l+1}\cdot\tau_{n,k,l}=0, \tag{76}$$

$$\left(D_{x_2}-D_{x_1}^2\right)\tau_{n,k,l}\cdot\tau_{n-1,k,l}=0, \tag{77}$$

$$\left(cD_{x_{-1}}+1\right)\tau_{n-1,k,l}\cdot\tau_{n,k,l}=\tau_{n,k-1,l}\tau_{n-1,k+1,l}, \tag{78}$$

$$(cD_{x_1}D_{x_{-1}}-D_{x_1}-2c)\tau_{n,k,l}\cdot\tau_{n-1,k,l}=(D_{x_1}-2c)\tau_{n,k-1,l}\cdot\tau_{n-1,k+1,l}, \tag{79}$$

where c and d are the arbitrary complex constants. The main difference between these bilinear equations and the previous ones (29)–(32) is the introduction of the third index l in the τ function, which is necessary in order to reduce the first bilinear equation (72) to (76). Indeed, the previous two-index τ function (28) is unable to satisfy a higher-dimensional bilinear equation which can be reduced to (72).

We can show that if functions $m_{i,j}^{(n,k,l)}$, $\varphi_i^{(n,k,l)}$ and $\psi_j^{(n,k,l)}$ of variables (x_{-1}, x_1, x_2) satisfy the following differential and difference relations

$$\begin{aligned}
&\partial_{x_1}m_{i,j}^{(n,k,l)}=\varphi_i^{(n,k,l)}\psi_j^{(n,k,l)},\\
&\partial_{x_1}\varphi_i^{(n,k,l)}=\varphi_i^{(n+1,k,l)},\ \partial_{x_1}\psi_j^{(n,k,l)}=-\psi_j^{(n-1,k,l)},\\
&\partial_{x_1}\varphi_i^{(n,k,l)}=c\varphi_i^{(n,k,l)}+\varphi_i^{(n,k+1,l)},\ \partial_{x_1}\psi_j^{(n,k,l)}=-c\psi_j^{(n,k,l)}-\psi_j^{(n,k-1,l)},\\
&\partial_{x_1}\varphi_i^{(n,k,l)}=d\varphi_i^{(n,k,l)}+\varphi_i^{(n,k,l+1)},\ \partial_{x_1}\psi_j^{(n,k,l)}=-d\psi_i^{(n,k,l)}-\psi_j^{(n,k,l-1)},\\
&\partial_{x_2}\varphi_i^{(n,k,l)}=\partial_{x_1}^2\varphi_i^{(n,k,l)},\ \partial_{x_2}\psi_j^{(n,k,l)}=-\partial_{x_1}^2\psi_j^{(n,k,l)},\\
&\partial_{x_{-1}}\varphi_i^{(n,k,l)}=\varphi_i^{(n,k-1,l)},\ \partial_{x_{-1}}\psi_j^{(n,k,l)}=-\psi_j^{(n,k+1,l)},
\end{aligned} \tag{80}$$

then the determinant

$$\tau_{n,k,l}=\det_{1\le i,j\le N}\left(m_{i,j}^{(n,k,l)}\right) \tag{81}$$

would satisfy the new higher-dimensional bilinear system (76)–(79).

Now, we introduce the function $m^{(n,k,l)}$ as

$$m^{(n,k,l)}=\frac{\mathrm{i}p}{p+q}\left(-\frac{p}{q}\right)^n\left(-\frac{p-c}{q+c}\right)^k\left(-\frac{p-d}{q+d}\right)^l \mathrm{e}^{\xi+\eta},$$

where

$$\xi = \frac{1}{p-c}x_{-1} + px_1 + p^2x_2 + \xi_0,$$
$$\eta = \frac{1}{q+c}x_{-1} + qx_1 - q^2x_2 + \eta_0,$$

and ξ_0 and η_0 are the arbitrary complex constants. Then, by defining

$$m_{i,j}^{(n,k,l)} = \mathcal{A}_i\mathcal{B}_j m^{(n,k,l)}, \tag{82}$$

where $\mathcal{A}_i$ and $\mathcal{B}_j$ are the differential operators as defined in Eq. (39); then, this $m_{i,j}^{(n,k,l)}$, together with appropriately chosen $\varphi_i^{(n,k,l)}$ and $\psi_j^{(n,k,l)}$, satisfies those differential-difference equations (80), and thus, the determinant (81) satisfies the bilinear system (76)–(79) for arbitrary sequences of indices $(i_1, i_2, \cdots, i_N; j_1, j_2, \cdots, j_N)$.

To reduce the higher-dimensional bilinear system (76)–(79) to (72)–(75), we will set

$$c = -\mathrm{i}\alpha, \quad d = -\mathrm{i}(1+\alpha). \tag{83}$$

6.2 Dimension Reduction

Our dimension reduction is the same as before, i.e.,

$$\left[\partial_{x_1} + \mathrm{i}c\partial_{x_{-1}}\right]\tau_{n,k,l} = C\tau_{n,k,l}, \tag{84}$$

where C is a certain constant. The same calculations as in Sect. 5.2 show that the determinant

$$\tau_{n,k,l} = \det_{1\le i,j\le N}\left(m_{2i-1,2j-1}^{(n,k,l)}\Big|_{p=p_0,\ q=q_0}\right), \tag{85}$$

with p_0, q_0 given by Eq. (44), would satisfy this dimension reduction condition. Under this reduction, the bilinear equation (78) becomes

$$\left(\mathrm{i}D_{x_1} - 1\right)\tau_{n,k,l}\cdot\tau_{n-1,k,l} + \tau_{n,k-1,l}\tau_{n-1,k+1,l} = 0, \tag{86}$$

and (79), combined with (86), reduces to

$$\begin{aligned} &D_{x_1}^2\tau_{n,k,l}\cdot\tau_{n-1,k,l} + \mathrm{i}D_{x_1}\tau_{n,k-1,l}\cdot\tau_{n-1,k+1,l} \\ &\quad = (2\mathrm{i}c+1)(\tau_{n,k-1,l}\cdot\tau_{n-1,k+1,l} - \tau_{n,k,l}\cdot\tau_{n-1,k,l}). \end{aligned} \tag{87}$$

6.3 The Index Reduction

The key step to reduce the bilinear equation (72) to (76) is the observation that the current three-index τ function (85) admits the following index relation,

$$\tau_{n,k-1,l} = K^N \tau_{n-1,k,l-1}, \quad K = \left(\frac{\sqrt{\alpha}+\mathrm{i}}{\sqrt{\alpha}-\mathrm{i}}\right)^2. \tag{88}$$

Its proof resembles that in Ohta and Yang (2014) for showing a similar index relation but for a different integrable equation. From the definition of $m_{i,j}^{(n,k,l)}$ in Eq. (82), we have

$$m_{i,j}^{(n,k-1,l)} = \mathcal{A}_i \mathcal{B}_j m^{(n,k-1,l)} = \mathcal{A}_i \mathcal{B}_j \left(\frac{p}{q}\right)\left(-\frac{q+c}{p-c}\right)\left(\frac{p-d}{q+d}\right) m^{(n-1,k,l-1)}.$$

Defining

$$H(\hat{p}) = \frac{p(p-d)}{p-c}, \quad \widetilde{H}(\hat{q}) = -\frac{q+c}{q(q+d)},$$

where $\hat{p} = p - c$ and $\hat{q} = q + c$, then

$$m_{i,j}^{(n,k-1,l)} = \mathcal{A}_i \mathcal{B}_j H(\hat{p}) \widetilde{H}(\hat{q}) m^{(n-1,k,l-1)}.$$

From the Leibniz rule, we can rewrite the above equation as

$$m_{i,j}^{(n,k-1,l)} = \sum_{\nu=0}^{i} \sum_{r=0}^{j} \frac{1}{\nu!} \frac{1}{r!} H_\nu(\hat{p}) \widetilde{H}_r(\hat{q})\, m_{i-\nu,j-r}^{(n-1,k,l-1)},$$

where functions $H_\nu(\hat{p})$ and $\widetilde{H}_r(\hat{q})$ are defined as

$$H_\nu(\hat{p}) = \left(\hat{p}\partial_{\hat{p}}\right)^\nu H(\hat{p}), \quad \widetilde{H}_r(\hat{q}) = \left(\hat{q}\partial_{\hat{q}}\right)^r \widetilde{H}(\hat{q}).$$

Introducing two generators

$$\mathcal{G}_1 = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} \left(\hat{p}\partial_{\hat{p}}\right)^\nu, \quad \mathcal{G}_2 = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \left(\hat{q}\partial_{\hat{q}}\right)^r,$$

and using the formula (68), we get

$$\mathcal{G}_1 H(\hat{p}) = H(e^\zeta \hat{p}) = e^\zeta \hat{p} + \frac{c(c-d)}{\hat{p}} e^{-\zeta} + 2c - d,$$
$$\mathcal{G}_2 \widetilde{H}(\hat{q}) = \widetilde{H}(e^\lambda \hat{q}) = \frac{-1}{e^\lambda \hat{q} + \frac{c(c-d)}{\hat{q}} e^{-\lambda} - 2c + d}.$$

For the chosen c, d values (83) and values $\hat{p}_0 = \hat{q}_0 = \sqrt{\alpha}$ from (44), we see that $\mathcal{G}_1 H(\hat{p}_0)$ and $\mathcal{G}_2 \widetilde{H}(\hat{q}_0)$ are even functions of ζ and λ, respectively. Thus, $H_{2\nu-1}(\hat{p}_0) = \widetilde{H}_{2\nu-1}(\hat{q}_0) = 0$ for all $\nu \geq 1$. Utilizing these results, we get the relation

$$m_{i,j}^{(n,k-1,l)}\big|_{p=p_0,q=q_0} = \sum_{\substack{\nu=0,\\ \nu:even}}^{i} \sum_{\substack{r=0,\\ r:even}}^{j} \frac{1}{\nu!}\frac{1}{r!} H_\nu(\hat{p})\widetilde{H}_r(\hat{q})\, m_{i-\nu,j-r}^{(n-1,k,l-1)}\big|_{p=p_0,q=q_0}.$$

Thus,

$$\left(m_{2i-1,2j-1}^{(n,k-1,l)}\Big|_{p=p_0,\ q=q_0} \right)_{1\leq i,j\leq N} = L \left(m_{2i-1,2j-1}^{(n-1,k,l-1)}\Big|_{p=p_0,\ q=q_0} \right)_{1\leq i,j\leq N} U,$$

where L is a certain lower triangular matrix with $H_0(\hat{p}_0)$ on the diagonal, and U is a certain upper triangular matrix with $\widetilde{H}_0(\hat{q}_0)$ on the diagonal. Taking determinants to this equation, we get

$$\tau_{n,k-1,l} = \left[H_0(\hat{p}_0)\widetilde{H}_0(\hat{q}_0)\right]^N \tau_{n-1,k,l-1},$$

which is the same as (88) since $H_0(\hat{p}_0)\ \widetilde{H}_0(\hat{q}_0) = K$.

6.4 Rogue Wave Solutions

Now, we set $x_1 = x - 2t$, $x_2 = \mathrm{i}t$, c, d as in (83), and $n = k = l = 0$ in the above bilinear equations (76), (77), (86) and (87). Since $\tau_{0,0,1} = (K)^N \tau_{-1,1,0}$ due to the index relation (88), we find that when we define

$$f = \tau_{0,0,0}, \quad \bar{f} = \tau_{-1,0,0}, \quad g = \tau_{-1,1,0}, \quad \bar{g} = \tau_{0,-1,0},$$

the above bilinear equations would become

$$\begin{aligned}
&\left(\mathrm{i}D_t + D_x^2 - 2\mathrm{i}\alpha D_x\right) g \cdot f = 0,\\
&\left(\mathrm{i}D_t + D_x^2 + 2\mathrm{i}D_x\right) f \cdot \bar{f} = 0,\\
&(\mathrm{i}D_x - 1) f \cdot \bar{f} + g\bar{g} = 0,\\
&D_x^2 f \cdot \bar{f} - \mathrm{i}D_x g \cdot \bar{g} + (2\alpha + 1)(f\bar{f} - g\bar{g}) = 0.
\end{aligned} \tag{89}$$

Notice that these $(f, \bar{f}, g, \bar{g})$ functions all have index $l = 0$. Thus, these functions are exactly the same as those given in Eq. (58) of the earlier section. Then, following the same complex conjugacy reductions $\bar{f} = f^*$ and $\bar{g} = g^*$ as before, the bilinear system (89) reduces to Eqs. (72)–(75), and its rogue wave solutions are exactly as given in Theorems 1 and 3.

7 Conclusions and Discussions

In this article, we have derived general rogue waves in the GDNLS equations (1) by an improved bilinear KP reduction method. Since these GDNLS equations arise in multiple physical situations and contain the Kaup–Newell equation, the Chen-Lee-Liu equation and others as special cases, these results would be useful for rogue-wave generation in such physical systems. A main benefit of this bilinear framework is that rogue waves to all members of these GDNLS equations can be expressed by the same bilinear solution. Compared to previous bilinear KP reduction methods for rogue waves in other integrable equations, an important improvement in our current KP reduction technique is a new parameterization of internal parameters in rogue waves. Under this new parameterization, the bilinear solution is much simpler than before. In addition, the rogue wave with the highest peak amplitude at each order can be easily obtained by setting all these internal parameters to zero. This way, the maximum peak amplitude at order N is found to be $2N + 1$ times the background amplitude, independent of the individual GDNLS equation and the background wavenumber. We have also found that these GDNLS equations can be decomposed into two different bilinear systems which require different KP reductions, but the resulting rogue waves are the same. Dynamics of rogue waves in the GDNLS equations is also analyzed. It is shown that the wavenumber of the constant background strongly affects the orientation and duration of the rogue wave. In addition, some new rogue patterns are presented.

The GDNLS equations (1) considered in this article have the parameter requirement of $a \neq b$, in which case these equations are gauge equivalent to the derivative NLS equation of Kaup–Newell type (4) (see Sect. 2). If $a = b$, Eq. (1) is called the Kundu–Eckhaus equation in the literature (Kundu 1984). The Kundu–Eckhaus equation is gauge equivalent to the NLS equation rather than the derivative NLS equation, and thus, its rogue waves would be different from those for the GDNLS equations (1) with $a \neq b$. Rogue waves in the Kundu–Eckhaus equation have been studied by Darboux transformation in Zhaqilao (2013), Wang et al. (2014), Qiu et al. (2015). In the bilinear framework, we can derive general rogue waves in the Kundu–Eckhaus equation in a similar way as we did for the GDNLS equations (1) with $a \neq b$. This derivation will be sketched in "Appendix".

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Appendix: Bilinear derivation of rogue waves in the Kundu–Eckhaus equation

When $a = b$, Eq. (1) becomes the Kundu–Eckhaus equation (Kundu 1984)

$$\mathrm{i}\phi_t + \phi_{\xi\xi} + \rho|\phi|^2\phi + \mathrm{i}a(|\phi|^2)_\xi\phi + \frac{1}{4}a^2|\phi|^4\phi = 0. \tag{90}$$

Under a gauge transformation

$$\phi(\xi,t)=w(\xi,t)e^{-\frac{a}{2}\mathrm{i}\int|w(\xi,t)|^2d\xi},$$

this Kundu–Eckhaus equation reduces to the NLS equation

$$\mathrm{i}w_t+w_{\xi\xi}+\rho|w|^2w=0, \tag{91}$$

whose rogue waves have been derived before (Akhmediev et al. 2009b; Ankiewicz et al. 2010a; Dubard et al. 2010; Kedziora et al. 2011; Guo et al. 2012; Ohta and Yang 2012a; Dubard and Matveev 2013). To directly obtain rogue waves in the Kundu–Eckhaus equation (90) without the use of the above gauge transformation, we can apply a similar bilinear approach as we did for the $a\neq b$ case in the main text of this article. Specifically, through a scaling of (ϕ,ξ,t,a) together with a Galilean transformation, we can normalize $\rho=2$ in Eq. (90), and the boundary conditions of its rogue waves can be normalized as

$$\phi(\xi,t)\to e^{\mathrm{i}\left(2t-\frac{1}{2}a\xi\right)},\quad (\xi,t)\to\infty. \tag{92}$$

Then, we employ a bilinear variable transformation

$$\phi(\xi,t)=e^{\mathrm{i}\left[2t-\frac{1}{2}a[\xi+(\ln f)_\xi]\right]}\frac{g}{f}, \tag{93}$$

where f is a real function, and g a complex function. Under this transformation, the Kundu–Eckhaus equation (90) can be split into the following three bilinear equations,

$$\left(\mathrm{i}D_t+D_\xi^2\right)g\cdot f=0, \tag{94}$$

$$(D_\xi^2+2)f\cdot f=2|g|^2, \tag{95}$$

$$D_\xi D_t f\cdot f=2\mathrm{i}D_\xi g\cdot g^*. \tag{96}$$

One can recognize that the first two bilinear equations are the ones for the NLS equation (91) with $\rho=2$ (Ohta and Yang 2012a). It turns out that the (f,g) solutions for rogue waves of the NLS equation also satisfy the third bilinear equation above, and thus, rogue waves for the Kundu–Eckhaus equation (90) are given by (93), where (f,g) are those for the NLS equation (91). The reason for this is that under the same differential and difference relations of τ functions listed in Eq. (3.7) of Ohta and Yang (2012a), the following three multi-dimensional bilinear equations are satisfied simultaneously,

$$(D_{x_1}D_{x_{-1}}-2)\tau_n\cdot\tau_n=-2\tau_{n+1}\tau_{n-1}, \tag{97}$$

$$(D_{x_2}-D_{x_1}^2)\tau_{n+1}\cdot\tau_n=0, \tag{98}$$

$$D_{x_{-1}}D_{x_2}\tau_n\cdot\tau_n=2D_{x_1}\tau_{n-1}\cdot\tau_{n+1}. \tag{99}$$

Thus, with the same dimension reduction and complex conjugacy conditions of the NLS equation (Ohta and Yang 2012a), and setting $x_1 = \xi$, $x_2 = \mathrm{i}t$, these multi-dimensional bilinear equations reduce to (94)–(96), and thus, the (f, g) solutions for rogue waves of the NLS equation (91) are also bilinear solutions for rogue waves of the Kundu–Eckhaus Eq. (90) under the bilinear variable transformation (93).

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