

Bifurcation of Soliton Families from Linear Modes in Non- \mathcal{PT} -Symmetric Complex Potentials

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Continuous families of solitons in the nonlinear Schrödinger equation with non- \mathcal{PT} -symmetric complex potentials and general forms of nonlinearity are studied analytically. Under a weak assumption, it is shown that stationary equations for solitons admit a constant of motion if and only if the complex potential is of a special form $g^2(x) + ig'(x)$, where $g(x)$ is an arbitrary real function. Using this constant of motion, the second-order complex soliton equation is reduced to a new second-order real equation for the amplitude of the soliton. From this real soliton equation, a novel perturbation technique is employed to show that continuous families of solitons bifurcate out from linear discrete modes in these non- \mathcal{PT} -symmetric complex potentials. All analytical results are corroborated by numerical examples.

1. Introduction

Nonlinear wave systems fall into two major categories: conservative and dissipative. Conservative systems are energy conserving, and their solitary waves (solitons) exist as continuous families with continuous ranges of energy values. A typical example is the nonlinear Schrödinger (NLS) equation. Dissipative systems contain gain and loss, and their solitons are generally isolated with certain discrete energy values. A typical example of this type is the Ginzburg–Landau equation. A recent discovery is that, in dissipative but parity-time (\mathcal{PT}) symmetric systems, solitons can still exist

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as continuous families with continuous energy values [1–19]. An example in this category is the NLS equation with a complex but \mathcal{PT} -symmetric potential. These soliton families are allowed because the \mathcal{PT} symmetry assures that the gain and loss of the soliton is perfectly balanced at arbitrary energy levels.

In dissipative and non- \mathcal{PT} -symmetric systems, the expectation is that any solitons will be isolated with discrete energy values, as seen in typical dissipative systems [20, 21]. However, exceptions were reported numerically in [22, 23] for the NLS equation with a non- \mathcal{PT} -symmetric complex potential of special form, where families of solitons with continuous energy values can bifurcate out from the linear modes of the potential. This finding is very surprising in view of the lack of \mathcal{PT} symmetry here. For these special potentials, a constant of motion was discovered in [23] for the stationary soliton equation. Using this constant of motion, soliton families in these special potentials were explained by a numerical shooting argument [23].

In this article, we analytically investigate solitons of the NLS equation with non- \mathcal{PT} -symmetric complex potentials. We focus on three main questions: (1) What types of non- \mathcal{PT} -symmetric complex potentials admit soliton families? (2) How can one analytically explain and calculate soliton families bifurcating from linear modes in such potentials? (3) Do these soliton families exist under other nonlinearities?

Regarding the first question, we recognize that in the absence of \mathcal{PT} symmetry, the existence of a constant of motion in the stationary soliton equation plays a crucial role in the existence of soliton families. Assuming this constant of motion for complex potentials is a continuous deformation of one that exists in the NLS equation without a potential, we show that the only complex potentials which admit a constant of motion are those of the form reported in [22, 23], that is, $V(x) = g^2(x) + ig'(x)$, where $g(x)$ is an arbitrary real function. This strongly suggests that potentials of the above form are the *only* one-dimensional non- \mathcal{PT} -symmetric complex potentials that admit soliton families.

On the second question, through use of the constant of motion, we reduce the second-order complex soliton equation to a new second-order real equation for the square of amplitude of the soliton, which is then solved perturbatively for a continuous range of μ values. This way, the existence of soliton families bifurcating from linear modes in non- \mathcal{PT} -symmetric potentials is analytically explained and explicitly calculated. Interestingly, this perturbation calculation of solitons differs significantly from the method used for real and \mathcal{PT} -symmetric potentials because the linearization operator of the new real equation has a distinctly different kernel structure.

Regarding the third question, we show that these soliton families still exist under a more general class of nonlinearities. Furthermore, the choice of nonlinearity within this class has no effect on the existence of a constant of motion.

These analytical results are compared with numerical examples, and good agreement between them is illustrated.

2. Preliminaries

The mathematical model we consider, in most parts of this article, is the NLS equation with a complex potential,

$$i\Psi_t + \Psi_{xx} + V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \quad (1)$$

where $\sigma = \pm 1$ is the sign of cubic nonlinearity. This model describes paraxial nonlinear light propagation in a waveguide with gain and loss [2, 24], as well as Bose–Einstein condensates with atoms injected into one part of the potential and removed from another part of the potential [25, 26]. Most of the earlier work focused on the case where the complex potential $V(x)$ is \mathcal{PT} -symmetric, that is, $V^*(x) = V(-x)$, with the superscript “*” representing complex conjugation [2–4, 6–9]. In this article, we consider the case where $V(x)$ is not \mathcal{PT} -symmetric, that is,

$$V^*(x) \neq V(-x). \quad (2)$$

Soliton solutions of Eq. (1) take the form

$$\Psi(x, t) = \psi(x)e^{i\mu t}, \quad (3)$$

where $\psi(x)$ is a localized function solving the stationary equation

$$\psi_{xx} - \mu\psi + V(x)\psi + \sigma|\psi|^2\psi = 0, \quad (4)$$

and μ is a real propagation constant. In \mathcal{PT} -symmetric potentials, solitons exist as continuous families parameterized by μ [3, 4, 6–9]. However, in non- \mathcal{PT} -symmetric potentials, soliton families are generically forbidden [21]. Surprisingly, it was reported recently through numerical examples that in complex potentials of the special form

$$V(x) = g^2(x) + ig'(x), \quad (5)$$

where $g(x)$ is an arbitrary real function, soliton families can still bifurcate out from linear modes even when $V(x)$ is non- \mathcal{PT} -symmetric (i.e., when $g(x)$ is not even) [22, 23]. This result is very unintuitive. Indeed, if one performs a regular perturbation calculation of soliton families bifurcating from linear modes in a general complex potential, it will be seen that infinitely many nontrivial conditions would have to be satisfied simultaneously, which

makes such bifurcation almost impossible [21]. However, for the special complex potential (5), all those conditions are met, which is miraculous. Obviously this phenomenon needs better understanding. A step in this direction was made in Ref. [23], where through the discovery of a constant of motion for the soliton equation (4) under the potential (5), soliton families in Eq. (4) were explained through a numerical shooting argument.

Many important questions are currently open regarding soliton families in non- \mathcal{PT} -symmetric complex potentials. For instance, what other non- \mathcal{PT} -symmetric complex potentials admit soliton families? How can one analytically explain and explicitly calculate soliton families bifurcating from linear modes in non- \mathcal{PT} -symmetric potentials? Do these soliton families also exist under other nonlinearities? These questions will be investigated in the remainder of this article.

3. Constant of Motion

A quantity $J(x, \psi)$ is called a constant of motion in the stationary equation (4) if $dJ/dx = 0$. The existence of a constant of motion proves to be important for the existence of soliton families in non- \mathcal{PT} -symmetric potentials. The reason for this can be understood heuristically by a shooting argument. In localized potentials, the soliton equation (4) shows that $\psi(x) \rightarrow c_{\pm} e^{-\sqrt{\mu}|x|}$ as $x \rightarrow \pm\infty$, where c_{\pm} are complex constants. Due to phase invariance of the soliton equation, c_{-} may be normalized to be real. Thus, for a given μ value, there are three real parameters, c_{-} , $\text{Re}(c_{+})$, and $\text{Im}(c_{+})$, in the tails of solitons. Shooting from these tails at $x = \pm\infty$ to any interior point x_0 , the values of $\psi(x_0)$ and $\psi'(x_0)$ from the two sides must match, resulting in four real conditions. This is generically impossible because there are only three tail parameters to adjust. However, if there is a constant of motion, the four matching conditions reduce to three, which makes shooting possible for each given μ , hence a continuous family of solitons result.

Thus, in this section, we study what complex potentials $V(x)$ admit a constant of motion. In this study, solutions, $\psi(x)$, to the stationary equation (4) are allowed to be any solutions, not necessarily solitons. That is, $\psi(x)$ is allowed to be nonlocal.

First, we split the complex potential $V(x)$ into real and imaginary parts,

$$V(x) = v_1(x) + i v_2(x), \quad (6)$$

where $v_1(x), v_2(x)$ are real functions. We also express the complex function $\psi(x)$ in polar forms,

$$\psi(x) = r(x) e^{i \int \theta(x) dx}, \quad (7)$$

where $r(x)$, $\theta(x)$ are real amplitude and phase functions. Substituting these expressions into the soliton equation (4), we get

$$r_{xx} - \mu r + v_1 r + \sigma r^3 - \theta^2 r = 0, \tag{8}$$

$$(r^2 \theta)_x = -v_2 r^2. \tag{9}$$

In the absence of the potential ($v_1 = v_2 = 0$), it is easy to verify that this system admits two constants of motion

$$J_1 = r^2 \theta \tag{10}$$

and

$$J_2 = r_x^2 - \mu r^2 + \frac{\sigma}{2} r^4 + r^2 \theta^2, \tag{11}$$

where $dJ_1/dx = dJ_2/dx = 0$. Because the system is third order, these are the only constants of motion the system can allow (because if more than two constants of motion existed, the solution would have to be a constant). These two constants of motion are associated with the flux terms of the power and momentum conservation laws of the potential-free NLS equation, but this fact is not important to our analysis.

In the presence of the potential, it is reasonable to assume that the corresponding constant of motion J is a continuous deformation of those in the potential-free case. In other words, J approaches constants of motion of the potential-free equation when v_1, v_2 approach zero. Notice that J_1 and J_2 have different ranks [27]. Thus, under the limit $v_1, v_2 \rightarrow 0$, J can only approach one of (J_1, J_2) , not their linear combination. Our strategy then is to calculate dJ_k/dx ($k = 1, 2$) in the presence of the potential and derive conditions on (v_1, v_2) so that dJ_k/dx is a total derivative of x , that is, a constant of motion is admitted.

Now, we calculate dJ_k/dx in the presence of a potential. First, we consider dJ_1/dx . Equation (9) clearly shows that, for dJ_1/dx to be a total derivative, we must have $v_2 = 0$, that is, the potential $V(x)$ is real. This is not what we want because we exclusively consider complex potentials in this paper. Thus, there are no constants of motion in Eqs. (8) and (9) that approach J_1 when the complex potential approaches zero.

Next, we consider dJ_2/dx . Utilizing Eqs. (8) and (9), we readily find that

$$\frac{dJ_2}{dx} = -v_1 (r^2)_x - 2v_2 r^2 \theta. \tag{12}$$

The right side of this equation can be rewritten as

$$\frac{dJ_2}{dx} = W_x + r^2 v_{1x} + 2(r^2 \theta)_x \int v_2 dx, \tag{13}$$

where

$$W = -v_1 r^2 - 2r^2 \theta \int v_2 dx.$$

Then utilizing Eq. (9), the above equation becomes

$$\frac{dJ_2}{dx} = W_x + r^2 \left(v_{1x} - 2v_2 \int v_2 dx \right). \quad (14)$$

For the right side of the above equation to be a total derivative, the necessary and sufficient condition is

$$v_{1x} = 2v_2 \int v_2 dx. \quad (15)$$

This condition can be rewritten as

$$v_{1x} = \left[\left(\int v_2 dx \right)^2 \right]_x, \quad (16)$$

thus

$$v_1 = \left(\int v_2 dx \right)^2 + C, \quad (17)$$

where C is an arbitrary constant. Finally, denoting

$$g = \int v_2 dx, \quad (18)$$

the potential $V(x)$ which admits a constant of motion then is of the form

$$V(x) = g^2(x) + ig'(x) + C. \quad (19)$$

Obviously, the constant C in this potential can be eliminated from Eq. (1) through a simple gauge transformation. The remaining potential is then of the form (5). Thus we conclude that if the constant of motion for the stationary equation (4) with a complex potential is a continuous deformation of J_2 without the potential, then this constant of motion exists if and only if the complex potential $V(x)$ is of the special form (5), and the corresponding motion constant is

$$J = J_2 - W = J_2 + g^2 r^2 + 2gr^2 \theta, \quad (20)$$

or more explicitly,

$$J = r_x^2 - \mu r^2 + \frac{\sigma}{2} r^4 + r^2 (\theta + g)^2, \quad (21)$$

where $dJ/dx = 0$. This constant of motion agrees with that reported in Ref. [23] for these special potentials (5).

It should be pointed out that a constant of motion is necessary only for soliton families in non- \mathcal{PT} -symmetric complex potentials. In \mathcal{PT} -symmetric potentials, soliton families (of \mathcal{PT} -symmetric type) generally exist without constants of motion. The existence of such soliton families in \mathcal{PT} potentials is a consequence of the \mathcal{PT} symmetry, which guarantees that for a continuous range of μ values, all solvability conditions in a perturbation expansion of solitons are satisfied to all orders [21]. From a shooting point of view (see the beginning of this section), \mathcal{PT} -symmetric solitons have tails $\psi(x) \rightarrow c_{\pm}e^{-\sqrt{\mu}|x|}$, where $c_+ = c_-^*$, which contain two real parameters, $\text{Re}(c_-)$ and $\text{Im}(c_-)$ (here imposing \mathcal{PT} symmetry does not allow c_- normalized to be real by phase invariance). Shooting from the tail at $x = -\infty$ to $x = 0$, there are two ending conditions, $\text{Im}[\psi(0)] = \text{Re}[\psi'(0)] = 0$, thus shooting is possible for each given μ . The importance of the \mathcal{PT} -symmetric potential in this process is that, it guarantees when these two ending conditions are met and a soliton $\psi(x)$ is obtained on the left-half axis $-\infty < x \leq 0$, its \mathcal{PT} -symmetric counterpart $\psi^*(-x)$ on the right-half axis $0 \leq x < \infty$ automatically satisfies the soliton equation (4) as well, thus a \mathcal{PT} -symmetric soliton on the whole x -axis results. From this argument, we see that soliton families in \mathcal{PT} -symmetric potentials are due to \mathcal{PT} symmetry, and no constants of motion are needed.

4. Bifurcation of Soliton Families

In this section, we analytically calculate the bifurcation of solitons from linear modes in Eq. (1), with potential of the special form (5), and show that soliton families bifurcate out in such non- \mathcal{PT} -symmetric systems.

For the potential (5), when solitons (3) are expressed in polar forms (7), the equations for r and θ are seen from Eqs. (8) and (9) as

$$r_{xx} - \mu r + g^2 r + \sigma r^3 - \theta^2 r = 0, \tag{22}$$

$$(r^2 \theta)_x = -g_x r^2, \tag{23}$$

and these equations admit a constant of motion (21). For solitons, this constant J can be evaluated at $x = \infty$ as zero, thus

$$r_x^2 - \mu r^2 + \frac{\sigma}{2} r^4 + r^2(\theta + g)^2 = 0. \tag{24}$$

From this equation, we get

$$\theta = -g \pm \sqrt{\mu - \frac{1}{2}\sigma r^2 - \frac{r_x^2}{r^2}}. \tag{25}$$

Inserting it into Eq. (22) and after simple algebra, we get

$$r_{xx} - 2\mu r + \frac{3}{2}\sigma r^3 + \frac{r_x^2}{r} = \mp 2g\sqrt{\mu r^2 - \frac{1}{2}\sigma r^4 - r_x^2}, \quad (26)$$

or equivalently,

$$(r^2)_{xx} - 4\mu r^2 + 3\sigma r^4 \pm 2g\sqrt{4\mu r^4 - 2\sigma r^6 - [(r^2)_x]^2} = 0. \quad (27)$$

Denoting $R = r^2$, we arrive at a single second-order equation for the real amplitude function R as

$$R_{xx} - 4\mu R + 3\sigma R^2 \pm 2g\sqrt{4\mu R^2 - 2\sigma R^3 - R_x^2} = 0, \quad (28)$$

which can also be rewritten as

$$\left(\sqrt{4\mu R^2 - 2\sigma R^3 - R_x^2}\right)_x = \pm 2gR_x. \quad (29)$$

4.1. Perturbation calculations

The sign in Eq. (28) needs to be chosen appropriately according to the function $g(x)$. Indeed, if $g(x)$ switches to $-g(x)$, this sign should switch as well. Without loss of generality, we take the plus sign in Eq. (28),

$$R_{xx} - 4\mu R + 3\sigma R^2 + 2g\sqrt{4\mu R^2 - 2\sigma R^3 - R_x^2} = 0. \quad (30)$$

Note that sometimes the same solution $R(x)$ can lead to mixed signs in Eq. (28) on different x -intervals. This could occur if $4\mu R^2 - 2\sigma R^3 - R_x^2$ is zero somewhere on the x -axis, because the square root is a possible mechanism for inducing a sign change so that the square-rooted quantity remains smooth. We do not consider such mixed cases here. This exclusion will be assured by Assumption 1 in Section 4.3.

For a localized function $g(x)$, it is easy to see that the large- x asymptotics of the soliton solution in Eq. (30) are, to leading order,

$$R(x) \rightarrow a_{\pm}e^{-\sqrt{4\mu}|x|}, \quad x \rightarrow \pm\infty, \quad (31)$$

where a_{\pm} are positive constants. This asymptotics is obtained from the linear terms $R_{xx} - 4\mu R \sim 0$ in Eq. (30), and it simultaneously annihilates the leading-order terms $4\mu R^2 - R_x^2$ under the square root of that same equation.

First, we consider linear modes in Eq. (30), which satisfy the equation

$$\phi_{xx} - 4\mu_0\phi + 2g\sqrt{4\mu_0\phi^2 - \phi_x^2} = 0. \quad (32)$$

This equation can be rewritten as

$$\left(\sqrt{4\mu_0\phi^2 - \phi_x^2}\right)_x = 2g\phi_x. \quad (33)$$

Because g and ϕ are localized functions, we see that

$$\sqrt{4\mu_0\phi^2 - \phi_x^2} = \int_{-\infty}^x 2g\phi_\xi d\xi = - \int_x^\infty 2g\phi_\xi d\xi \tag{34}$$

and

$$\int_{-\infty}^\infty 2g\phi_\xi d\xi = 0. \tag{35}$$

Equation (32) is scaling-invariant and thus an eigenvalue problem, but it is nonlinear in both the eigenvalue μ_0 and eigenfunction ϕ . Thus, this eigenvalue problem differs from most previous ones where the eigenvalue equation is linear in the eigenfunction and the eigenvalue [28,29]. Solving this new eigenvalue problem is equivalent to solving for discrete real eigenvalues in the original eigenvalue problem from Eq. (4), that is,

$$\psi_{xx} + V(x)\psi = \mu_0\psi, \tag{36}$$

and the eigenfunction correspondence is $\phi = |\psi|^2$. Previous results in [22] have shown that for the underlying special potential (5), the linear eigenvalue problem (36) admits discrete real eigenvalues for a large class of functions $g(x)$. The new eigenvalue problem (32) also shows that the existence of such real eigenvalues is plausible for real functions of $g(x)$ because the initial assumption that μ and $\phi(x)$ be real has resulted in a completely real equation.

From such eigenmodes, families of solitons can bifurcate out under variation of μ . We will analytically show this by explicitly calculating this soliton bifurcation from a linear mode (μ_0, ϕ) using perturbation methods.

The perturbation expansion is

$$R = \epsilon (R_0 + \epsilon R_1 + \epsilon^2 R_2 + \dots), \tag{37}$$

$$\mu = \mu_0 + \epsilon, \tag{38}$$

where $\epsilon > 0$ is a small parameter. Here, we have assumed the bifurcation occurs to the right side of μ_0 . As we will see later, this assumption dictates the sign of nonlinearity σ . If the bifurcation occurs to the left side of μ_0 , then only trivial modifications to our analysis are needed, and the bifurcation will occur for the opposite sign of nonlinearity.

Inserting the above expansion into Eq. (30), at order ϵ , we find

$$R_0 = c_0\phi, \tag{39}$$

where c_0 is a positive constant to be determined.

At order ϵ^2 , we get

$$LR_1 = F, \tag{40}$$

where

$$L = \partial_{xx} + p_1 \partial_x + p_2, \tag{41}$$

$$p_1 = -\frac{2g\phi_x}{\sqrt{4\mu_0\phi^2 - \phi_x^2}}, \quad p_2 = 4\mu_0 \left(\frac{2g\phi}{\sqrt{4\mu_0\phi^2 - \phi_x^2}} - 1 \right), \tag{42}$$

$$F = c_0(f_1 - c_0\sigma f_2), \tag{43}$$

and

$$f_1 = 4\phi \left(1 - \frac{g\phi}{\sqrt{4\mu_0\phi^2 - \phi_x^2}} \right), \quad f_2 = \phi^2 \left(3 - \frac{2g\phi}{\sqrt{4\mu_0\phi^2 - \phi_x^2}} \right). \tag{44}$$

Now it is time to analyze the properties of homogeneous solutions and adjoint homogeneous solutions of the operator L and the solvability condition of Eq. (40).

4.2. Kernels of linearization operators L and L^A

First, it is easy to verify that ϕ is a homogeneous solution of L , that is,

$$L\phi = 0. \tag{45}$$

Let us suppose the other homogeneous solution of L is ϕ_2 , then according to Abel's formula, the Wronskian of (ϕ, ϕ_2) is

$$W(\phi, \phi_2) = W_0 e^{-\int p_1 dx} \tag{46}$$

or

$$W(\phi, \phi_2) = W_0 \sqrt{4\mu_0\phi^2 - \phi_x^2} \tag{47}$$

in view of Eqs. (33) and (42). Here, W_0 is a constant. Utilizing Eq. (34), the above Wronskian can be rewritten as

$$W(\phi, \phi_2) = W_0 \int_{-\infty}^x 2g\phi_\xi d\xi = -W_0 \int_x^\infty 2g\phi_\xi d\xi. \tag{48}$$

From this formula we see that, if $g(x)$ is a localized function, then the decay rate of this Wronskian at large $|x|$ is faster than that of ϕ , and thus ϕ_2 is also a localized function.

Using these homogeneous solutions of L , we can build homogeneous solutions of the adjoint operator L^A , where

$$L^A = \partial_{xx} - \partial_x p_1 + p_2, \tag{49}$$

that is,

$$L^A f \equiv \partial_{xx} f - \partial_x(p_1 f) + p_2 f.$$

LEMMA 1. *The two homogeneous solutions (ϕ^A, ϕ_2^A) of the adjoint operator L^A are*

$$\phi^A = -\frac{\phi_2}{W(\phi, \phi_2)}, \quad \phi_2^A = \frac{\phi}{W(\phi, \phi_2)}. \tag{50}$$

Proof. We first turn the second-order homogeneous equation of operator L into a system of first-order equations,

$$X' = \begin{bmatrix} 0 & 1 \\ -p_2 & -p_1 \end{bmatrix} X, \tag{51}$$

where the prime stands for derivative to x . The fundamental matrix solution to this system is

$$X = \begin{bmatrix} \phi & \phi_2 \\ \phi_x & \phi_{2x} \end{bmatrix}. \tag{52}$$

The adjoint system of Eq. (51) is

$$Y' = -\begin{bmatrix} 0 & -p_2 \\ 1 & -p_1 \end{bmatrix} Y. \tag{53}$$

Notice that if $Y = [y_1, y_2]^T$, where the superscript “ T ” represents vector or matrix transpose, then it is easy to verify that

$$L^A y_2 = 0, \tag{54}$$

that is, the second component of the vector solution Y is in the kernel of the adjoint operator L^A .

It is well known that the fundamental matrix solution to the adjoint vector system (53) is $(X^{-1})^T$. This can be proved by calculating $(XX^{-1})'$, where upon utilizing Eq. (51), a homogeneous differential equation for X^{-1} would be obtained. Taking the transpose of this equation would reveal that $(X^{-1})^T$ satisfies the adjoint equation (53). Notice that

$$(X^{-1})^T = \frac{1}{W(\phi, \phi_2)} \begin{bmatrix} \phi_{2x} & -\phi_x \\ -\phi_2 & \phi \end{bmatrix}. \tag{55}$$

Because the second-row functions in this matrix are in the kernel of the adjoint operator L^A , the functions ϕ^A and ϕ_2^A defined in Eq. (50) are then homogeneous solutions of the adjoint operator L^A . ■

In view of Lemma 1, if $g(x)$ is a localized function, then both adjoint homogeneous solutions ϕ^A, ϕ_2^A are unbounded, because the decay rates of ϕ and ϕ_2 at large $|x|$ are slower than that of the Wronskian $W(\phi, \phi_2)$.

The fact that L has only localized solutions and L^A has only unbounded solutions in their kernels makes the solvability condition for the first-order equation (40) novel, as we will delineate below.

4.3. Solvability conditions for certain potentials

In this subsection, we show how to impose the solvability condition on Eq. (40) under the following assumptions:

ASSUMPTION 1. *For the linear eigenmode (μ_0, ϕ) , $4\mu_0\phi^2 - \phi_x^2$ is strictly positive for all x .*

ASSUMPTION 2. *The function $g(x)$ decays exponentially at large x as*

$$g(x) \rightarrow b_{\pm}e^{-\beta|x|}, \quad x \rightarrow \pm\infty, \quad (56)$$

where b_{\pm} and β (> 0) are constants.

ASSUMPTION 3. *For these potentials, $\sqrt{4\mu_0} > \beta$.*

Assumption 1 assures that the coefficients in linear operators L and L^A are nonsingular. In addition, there will be no sign change on the x -interval in Eq. (28). This assumption will be made throughout the text.

Assumptions 2 and 3 are introduced to make our analysis more explicit. If the function $g(x)$ does not satisfy these assumptions, an alternative analysis will be outlined in the next subsection.

REMARK 1. *In some sense Assumptions 2 and 3 represent the most common case, because for practical purposes the exact decay rate at large x should have minimal effect on the dynamics of the system. Hence modifying the small tails of the potential to have a suitably exponentially decaying rate should not make a meaningful difference. However, we will still show in the next subsection that these perturbation calculations may be performed for general potentials, and then verify all results numerically.*

At large $|x|$, the asymptotics of the eigenfunction $\phi(x)$ can be readily seen from Eq. (32) as

$$\phi(x) \rightarrow \gamma_{\pm}e^{-\sqrt{4\mu_0}|x|}, \quad x \rightarrow \pm\infty, \quad (57)$$

where γ_{\pm} are constants. Then under Assumption 2, it is easy to see from Eq. (34) that the large- x asymptotics of $\sqrt{4\mu_0\phi^2 - \phi_x^2}$ is

$$\sqrt{4\mu_0\phi^2 - \phi_x^2} \rightarrow s_{\pm}e^{-(\beta+\sqrt{4\mu_0})|x|}, \quad x \rightarrow \pm\infty, \quad (58)$$

where

$$s_{\pm} = \frac{2b_{\pm}\gamma_{\pm}\sqrt{4\mu_0}}{\beta + \sqrt{4\mu_0}}.$$

Thus

$$p_1(x) \rightarrow \pm (\beta + \sqrt{4\mu_0}), \quad x \rightarrow \pm\infty, \quad (59)$$

$$p_2(x) \rightarrow \beta\sqrt{4\mu_0}, \quad x \rightarrow \pm\infty, \quad (60)$$

hence the asymptotics of operators L and L^A are

$$L \rightarrow \partial_{xx} \pm (\beta + \sqrt{4\mu_0})\partial_x + \beta\sqrt{4\mu_0}, \quad x \rightarrow \pm\infty \quad (61)$$

and

$$L^A \rightarrow \partial_{xx} \mp (\beta + \sqrt{4\mu_0})\partial_x + \beta\sqrt{4\mu_0}, \quad x \rightarrow \pm\infty. \quad (62)$$

From these asymptotics, it is seen more explicitly that all solutions of $Lf = 0$ are localized (with decaying tails as $e^{-\beta|x|}$, $e^{-\sqrt{4\mu_0}|x|}$, or their linear combinations), and all solutions of $L^A f = 0$ are unbounded (with growing tails as $e^{\beta|x|}$, $e^{\sqrt{4\mu_0}|x|}$, or their linear combinations).

Regarding the second homogeneous solution $\phi_2(x)$, in view of the asymptotics (57) of the first homogeneous solution $\phi(x)$, without loss of generality we can set the large negative- x asymptotics of $\phi_2(x)$ as

$$\phi_2(x) \rightarrow e^{\beta x}, \quad x \rightarrow -\infty. \quad (63)$$

Then its large positive- x asymptotics is

$$\phi_2(x) \rightarrow \kappa_1 e^{-\beta x} + \kappa_2 e^{-\sqrt{4\mu_0}x}, \quad x \rightarrow +\infty, \quad (64)$$

where κ_1, κ_2 are constants. Substituting these asymptotics into the Wronskian function and using the Wronskian formula (48), the value of κ_1 can be determined. However, this κ_1 value is not needed in our analysis.

From Lemma 1 and Eq. (47), we rewrite the adjoint homogeneous solutions ϕ^A and ϕ_2^A equivalently as

$$\phi^A = -\frac{\phi_2}{\sqrt{4\mu_0\phi^2 - \phi_x^2}}, \quad \phi_2^A = \frac{\phi}{\sqrt{4\mu_0\phi^2 - \phi_x^2}}. \quad (65)$$

Then using the asymptotics (57), (58), (63), and (64), we find that the large- x asymptotics of ϕ^A and ϕ_2^A are

$$\phi^A(x) \rightarrow \begin{cases} -d_- e^{-\sqrt{4\mu_0}x}, & x \rightarrow -\infty, \\ -d_+ \left(\kappa_1 e^{\sqrt{4\mu_0}x} + \kappa_2 e^{\beta x} \right), & x \rightarrow +\infty, \end{cases} \quad (66)$$

and

$$\phi_2^A(x) \rightarrow j_{\pm} e^{\beta|x|}, \quad x \rightarrow \pm\infty, \quad (67)$$

where

$$d_{\pm} = \frac{\beta + \sqrt{4\mu_0}}{2b_{\pm}\gamma_{\pm}\sqrt{4\mu_0}}, \quad j_{\pm} = d_{\pm}\gamma_{\pm}. \tag{68}$$

Notice that both adjoint solutions are unbounded and grow exponentially at large x .

The asymptotics of functions f_1 and f_2 in the first-order equation (40) can be similarly obtained as

$$f_1(x) \rightarrow q_{\pm}e^{-\sqrt{4\mu_0}|x|}, \quad x \rightarrow \pm\infty, \tag{69}$$

and

$$f_2(x) \rightarrow w_{\pm}e^{-2\sqrt{4\mu_0}|x|}, \quad x \rightarrow \pm\infty, \tag{70}$$

where

$$q_{\pm} = 2 \left(1 - \frac{\beta}{\sqrt{4\mu_0}} \right) \gamma_{\pm}, \quad w_{\pm} = \left(2 - \frac{\beta}{\sqrt{4\mu_0}} \right) \gamma_{\pm}^2. \tag{71}$$

Now we consider the solvability condition of the first-order equation (40). We see from Eqs. (31), (40), (43), and (70) that the large- x asymptotics of $R_1(x)$ must be

$$R_1(x) \rightarrow P_1^{\pm}(x)e^{-\sqrt{4\mu_0}|x|} + C_1^{\pm}e^{-2\sqrt{4\mu_0}|x|}, \quad x \rightarrow \pm\infty, \tag{72}$$

where $P_1^{\pm}(x)$ are certain linear functions of x (these linear functions come about when one expands the tail function $e^{-\sqrt{4\mu}|x|}$ of (31) into a perturbation series around $\mu = \mu_0$), and C_1^{\pm} are constants. Note that the tails $e^{-2\sqrt{4\mu_0}|x|}$ in the above equation are induced by the nonlinearity-related forcing term f_2 and are admissible. They do not contradict the leading-order $R(x)$ asymptotics (31) because they are of higher order. Enforcement of this tail behavior for $R_1(x)$ will yield the solvability condition, which determines the c_0 value.

We begin by taking the inner product of Eq. (40) with $\phi_2^A(x)$ to get

$$\langle \phi_2^A, LR_1 \rangle = \langle \phi_2^A, c_0(f_1 - c_0\sigma f_2) \rangle, \tag{73}$$

where the inner product is defined as

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} f^* g dx. \tag{74}$$

Performing integration by parts, the left side of this equation becomes

$$\begin{aligned} \langle \phi_2^A, LR_1 \rangle &= \langle L^A \phi_2^A, R_1 \rangle + (\phi_2^A R_{1x} - \phi_{2x}^A R_1 + p_1 \phi_2^A R_1) \Big|_{-\infty}^{+\infty} \\ &= (\phi_2^A R_{1x} - \phi_{2x}^A R_1 + p_1 \phi_2^A R_1) \Big|_{-\infty}^{+\infty}. \end{aligned} \tag{75}$$

In view of the asymptotics of ϕ_2^A , R_1 , and p_1 in Eqs. (59), (67), and (72), as well as Assumption 3, we see that the right side of the above equation is zero, hence we obtain a solvability condition from Eq. (73) as

$$\langle \phi_2^A, f_1 - c_0 \sigma f_2 \rangle = 0. \tag{76}$$

This solvability condition is the analog of Fredholm Alternatives condition, and it quickly yields the formula for c_0 as

$$c_0 = \frac{\langle \phi_2^A, f_1 \rangle}{\sigma \langle \phi_2^A, f_2 \rangle}. \tag{77}$$

Notice that f_1 and f_2 decay at large x as $e^{-\sqrt{4\mu_0}|x|}$ or faster (see Eqs. (69) and (70)), and $\phi_2^A(x)$ grows at large x as $e^{\beta|x|}$ (see Eq. (67)). Thus under Assumption 3, both integrals in the inner products of the above equation converge, and hence c_0 is well defined.

Equation (77) is a necessary condition for the existence of the first-order solution $R_1(x)$ with suitable asymptotics (72). Because c_0 must be positive, Eq. (77) then shows that, for the soliton bifurcation to occur to the right side of μ_0 (see Eq. (38)), the sign of nonlinearity σ must be chosen as the sign of the ratio $\langle \phi_2^A, f_1 \rangle / \langle \phi_2^A, f_2 \rangle$.

The above solvability condition (77) turns out to be also sufficient for the existence of solution $R_1(x)$ with suitable asymptotics (72). To show this, we notice that the general solution to the first-order equation (40) can be derived by variation of parameters as

$$R_1(x) = \phi(x) \int_0^x \phi^A(\xi) F(\xi) d\xi + \phi_2(x) \int_0^x \phi_2^A(\xi) F(\xi) d\xi + c_1 \phi(x) + c_2 \phi_2(x), \tag{78}$$

where c_1 and c_2 are real constants. Using the asymptotics detailed earlier in this section and under Assumption 3, we find that at large x , $\phi^A(x)F(x)$ approaches a constant, and $\phi_2^A(x)F(x)$ decays exponentially. Thus

$$\int_0^x \phi^A(\xi) F(\xi) d\xi \rightarrow \tilde{P}_1^\pm(x), \quad x \rightarrow \pm\infty, \tag{79}$$

where $\tilde{P}_1^\pm(x)$ are linear functions of x , and

$$\int_0^x \phi_2^A(\xi) F(\xi) d\xi \rightarrow \int_0^{\pm\infty} \phi_2^A(\xi) F(\xi) d\xi, \quad x \rightarrow \pm\infty. \tag{80}$$

In view of the large- x asymptotics of $\phi(x)$ and $\phi_2(x)$, for $R_1(x)$ in (78) to exhibit the suitable asymptotics (72), the necessary and sufficient conditions are

$$\int_0^{\pm\infty} \phi_2^A(\xi) F(\xi) d\xi + c_2 = 0, \tag{81}$$

which leads to the equation

$$\int_{-\infty}^{+\infty} \phi_2^A(x)F(x)dx = 0. \tag{82}$$

Substituting the expression (43) for $F(x)$ into this equation, we then obtain the c_0 formula (77). Hence this c_0 formula is a necessary and sufficient condition for the existence of solution $R_1(x)$ with suitable asymptotics (72).

In the R_1 formula (78), while c_0 is given by formula (77) and c_2 given by Eq. (81), c_1 is still a free parameter. This c_1 parameter will be fixed by requiring the second-order solution $R_2(x)$ to have suitable large- x asymptotics (similar to (72) but with linear functions $P_1^\pm(x)$ replaced by quadratic functions $P_2^\pm(x)$). This calculation of c_1 is in the same spirit of the c_0 calculation, thus details will not be pursued in this article.

4.4. Extension to general potentials

In the event that Assumptions 2 and 3 of the previous subsection do not hold, that is, the decay rate of the potential is not simply exponential, or the exponential decay rate is too fast, that is, $\beta > \sqrt{4\mu_0}$, then the simple c_0 formula (77) in the previous subsection will be invalid. For instance, when $\beta > \sqrt{4\mu_0}$, the integral in the numerator of (77) would be divergent in view of the asymptotics of its integrand. The quantity on the right side of Eq. (75) would not vanish either. Thus the solvability condition for these more general potentials needs a different treatment.

In our new treatment, we consider the $R_1(x)$ solution (78) and demand that its tail asymptotics match (72). Suppose the tail asymptotics of the second homogenous solution $\phi_2(x)$ has the form

$$\phi_2 \rightarrow \begin{cases} \tau^-(x), & x \rightarrow -\infty, \\ \kappa_2 e^{-\sqrt{4\mu_0}x} + \tau^+(x), & x \rightarrow +\infty, \end{cases} \tag{83}$$

where κ_2 is a certain real constant, and $\tau^\pm(x)$ are the other decaying tail functions. If $g(x)$ decays exponentially as $b_\pm e^{-\beta|x|}$ at large $|x|$, then it is seen from the Wronskian formula (48) that $\tau^\pm(x)$ also decays at the same exponential rate, which matches the asymptotics (63) and (64) in the previous subsection. If $g(x)$ decays as a Gaussian $b_\pm e^{-ax^2}$ at large $|x|$, where a is a positive constant, then from Eq. (48) and the asymptotics of $\phi(x)$ in Eq. (57), we can find through integration by parts that the leading-order asymptotics of $\tau^\pm(x)$ is

$$\tau^\pm(x) \rightarrow \alpha_\pm \frac{e^{-ax^2}}{x^2}, \quad x \rightarrow \pm\infty, \tag{84}$$

where α_{\pm} are certain constants. From these two examples, we see that the tails $\tau^{\pm}(x)$ decay at the same, or roughly the same, rates as the function $g(x)$.

We now substitute the F formula (43) into (78). Then, this R_1 solution can be rewritten as

$$R_1(x) = c_0 [R_{11}(x) - c_0\sigma R_{12}(x)] + c_1\phi(x) + c_2\phi_2(x), \tag{85}$$

where $R_{11}(x)$ and $R_{12}(x)$ are particular solutions of equations

$$LR_{11}(x) = f_1, \quad LR_{12}(x) = f_2. \tag{86}$$

For definiteness, we impose zero initial conditions on R_{11} and R_{12} at $x = 0$, that is,

$$R_{1k}(0) = R'_{1k}(0) = 0, \quad k = 1, 2. \tag{87}$$

Notice that both particular solutions R_{11} and R_{12} approach zero at large x , because the forcing terms f_1 and f_2 approach zero, and the homogeneous solutions are all localized.

The tails of the particular solutions R_{11} and R_{12} each have terms that decay exponentially and a term which decays like $\tau^{\pm}(x)$, due to the exponentially decaying forcing terms and exponential as well as $\tau^{\pm}(x)$ tails inside the homogeneous solutions. Specifically,

$$R_{11}(x) \rightarrow \begin{cases} P_{11}^-(x)e^{\sqrt{4\mu_0}x} + \chi_1^- \tau^-(x), & x \rightarrow -\infty, \\ P_{11}^+(x)e^{-\sqrt{4\mu_0}x} + \chi_1^+ \tau^+(x), & x \rightarrow +\infty, \end{cases} \tag{88a}$$

$$R_{12}(x) \rightarrow \begin{cases} P_{12}^- e^{\sqrt{4\mu_0}x} + C_1^- e^{2\sqrt{4\mu_0}x} + \chi_2^- \tau^-(x), & x \rightarrow -\infty, \\ P_{12}^+ e^{-\sqrt{4\mu_0}x} + C_1^+ e^{-2\sqrt{4\mu_0}x} + \chi_2^+ \tau^+(x), & x \rightarrow +\infty. \end{cases} \tag{88b}$$

Here P_{11}^{\pm} are linear functions and P_{12}^{\pm}, C_1^{\pm} are constants.

Substituting these asymptotics into the $R_1(x)$ formula (85) and comparing its tails with Eq. (72), we see that the coefficients on $\tau^{\pm}(x)$ must be zero as $x \rightarrow \pm\infty$, even if $\tau^{\pm}(x)$ decay more quickly than $e^{-\sqrt{4\mu_0}|x|}$ and $e^{-2\sqrt{4\mu_0}|x|}$, as is the case when $g(x)$ decays like a Gaussian, see Eq. (84). The reason for it is that the $R_1(x)$ term in the soliton solution can only contain tails $P_1^{\pm}(x)e^{-\sqrt{4\mu_0}|x|}$ and $C_1^{\pm}e^{-2\sqrt{4\mu_0}|x|}$, and nothing else (no matter how small they are). This leads to the following system of equations

$$c_0\chi_1^- - c_0^2\sigma\chi_2^- + c_2 = 0, \quad c_0\chi_1^+ - c_0^2\sigma\chi_2^+ + c_2 = 0. \tag{89}$$

From these, we obtain the necessary and sufficient solvability condition as

$$c_0 = \frac{\chi_1^+ - \chi_1^-}{\sigma(\chi_2^+ - \chi_2^-)}. \tag{90}$$

For potentials with exponential decay rates, the constants χ_1^\pm, χ_2^\pm can be found analytically with a bit of effort. In general, they may not be known analytically, but can be efficiently evaluated numerically, as examples in the next subsection will show.

In the above analysis, the c_0 coefficient was derived as a ratio (see Eqs. (77) and (90)), and we have tacitly assumed that the denominators in those ratios were nonzero. If those denominators are zero, then the solitons would be larger than $O(\epsilon)$, hence the perturbation expansion (37) of the solitons would need to be modified.

4.5. Numerical examples

Now, we numerically confirm the above analysis with two examples.

EXAMPLE 1. *For the first example, we choose the complex potential (5) with an uneven double-hump function*

$$g(x) = 0.8 [\operatorname{sech}(x + 2) + h \operatorname{sech}(x - 2)], \quad (91)$$

where h is a positive constant, and $\sigma = 1$ (focusing nonlinearity). Notice that this potential is exponentially decaying, satisfying our Assumption 2 with $\beta = 1$.

When $h = 1.2$, the function $g(x)$ and the corresponding complex potential $V(x)$ are displayed in Figs. 1(a) and (b), respectively. Notice that this potential is non- \mathcal{PT} -symmetric. Eigenvalues of the eigenmode problem (32) are shown in panel (c), where two real eigenvalues are found. The larger of these eigenvalues is $\mu_0 \approx 0.3708$, whose eigenfunction $\phi(x)$ is plotted in panel (d). For this eigenvalue, $\sqrt{4\mu_0} > 1$, thus Assumption 3 is met, and the analysis in Section 4.3 applies.

From this linear eigenmode, we have verified numerically that a continuous family of solitons bifurcates out. The power curve of this soliton family is shown in panel (e). Here the power is defined as $P = \int_{-\infty}^{\infty} |\psi|^2 dx$. The analytical prediction for the power slope $P'(\mu_0)$ at the bifurcation point can be obtained from Eqs. (37)–(39) as

$$P'_{\text{anal}}(\mu_0) = c_0 \int_{-\infty}^{\infty} \phi dx, \quad (92)$$

where c_0 is given by formula (77). For $h = 1.2$, this analytical power slope is found to be approximately 5.8961. The line with this power slope is plotted as dashed red line in panel (e), and good agreement with the numerical power slope can be seen. In panel (f), the amplitude profile $R = |\psi|^2$ of the soliton at the marked point of the power curve (with $\mu = 0.6$) is displayed.

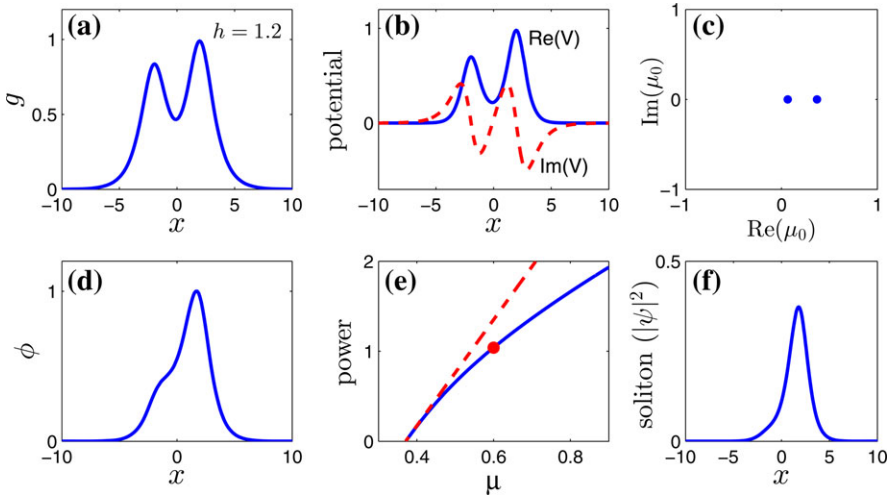


Figure 1. (a) The example function $g(x)$ in Eq. (91) with $h = 1.2$; (b) complex potential $V(x)$ for the $g(x)$ function in (a); (c) eigenvalues μ_0 of the eigenmode problem (32); (d) eigenfunction ϕ of the largest eigenvalue in (c); (e) power curve of solitons bifurcating from the linear mode in (d) under focusing nonlinearity (solid blue: numerical values; dashed red: the line with the theoretical slope value (92)); and (f) soliton ($|\psi|^2$) at the marked point of the power curve.

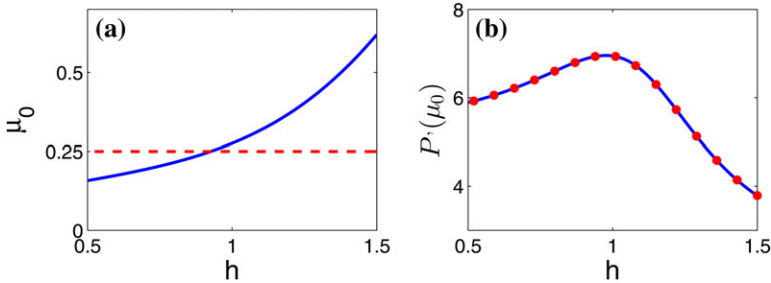


Figure 2. (a) Linear eigenvalue μ_0 in Eq. (32) for the $g(x)$ function (91) at various h values; the horizontal dashed line is at $\mu_0 = 1/4$; (b) comparison on the slope of the power curve at the bifurcation point μ_0 between numerical values (solid blue) and analytical predictions (red dots).

As parameter h in the $g(x)$ function (91) varies, the discrete eigenvalue μ_0 will change (see Fig. 2(a)). When h drops below 0.926, μ_0 will fall under 0.25, entering the $\sqrt{4\mu_0} < \beta$ regime (where Assumption 3 does not hold). To test our theory for both $\sqrt{4\mu_0} > \beta$ and $\sqrt{4\mu_0} < \beta$ cases, we have plotted in Fig. 2(b) the theoretical predictions for the power slope $P'_{\text{anal}}(\mu_0)$ in Eq. (92) for $0.5 \leq h \leq 1.5$, which encompasses both cases. The reader is reminded that the c_0 formula is given by Eq. (77) when

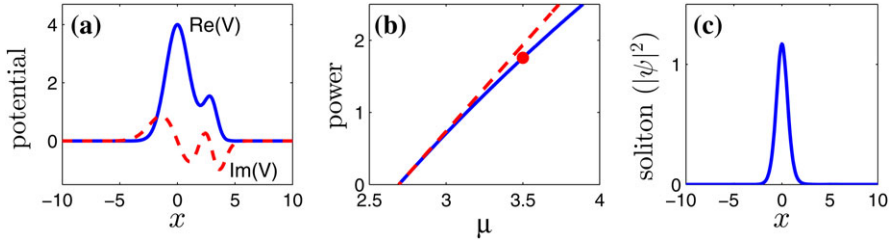


Figure 3. (a) Complex potential $V(x)$ for the $g(x)$ function in Example 2; (b) power curve of solitons under focusing nonlinearity (solid blue: numerical values; dashed red: the line with the theoretical slope (92)); and (c) soliton $(|\psi|^2)$ at the marked point of the power curve.

$\sqrt{4\mu_0} > \beta$ and by Eq. (90) when $\sqrt{4\mu_0} < \beta$. In the same fig., numerically obtained power slopes $P'(\mu_0)$ for each h value are shown as well. It is seen that numerical and analytical slope values exactly match each other, confirming the accuracy of our theoretical analysis in Sections 4.3 and 4.4.

EXAMPLE 2. As for the second example, we consider the potential (5) with

$$g(x) = 2e^{-x^2/4} + e^{-(x-3)^2}. \quad (93)$$

The resulting potential $V(x)$ is displayed in Fig. 3(a). The tails of this potential decay like a Gaussian, which is faster than exponential. Thus the results in Section 4.4 apply. In this case, we first select the $\phi_2(x)$ function by requiring that for $x \rightarrow -\infty$ the function decay as in Eq. (84) and take this tail to be $\tau^-(x)$. Now for $x \rightarrow +\infty$, the dominant decay of this tail is exponential, that is, the tail term $\tau^+(x)$ decays faster than $e^{-\sqrt{4\mu_0}x}$ in Eq. (83), thus one must first find the coefficient κ_2 of the exponential tail from large- x values of $\phi_2(x)$. Then subtracting away this exponential tail from $\phi_2(x)$, the remaining tail is $\tau^+(x)$. To obtain χ_1^\pm and χ_2^\pm in Eq. (88), we first compute $R_{11}(x)$ and $R_{12}(x)$ from the inhomogeneous equation (40), with F replaced by f_1 and f_2 , under the initial conditions (87). This is done by integrating the inhomogeneous equation from $x = 0$ out to $x = \pm\infty$. By subtracting their (slower decaying) exponential tails and comparing the remaining tails with $\tau^\pm(x)$ in ϕ_2 , χ_1^\pm and χ_2^\pm can then be ascertained. From these numbers, the c_0 value is calculated from formula (90).

Now, we compare these analytical predictions against numerical results. Solving the eigenvalue problem (32), we find three discrete real eigenvalues, the largest being $\mu_0 \approx 2.6923$. From this eigenmode, we have confirmed that a soliton family indeed bifurcates out. If the nonlinearity is focusing ($\sigma = 1$), the power curve of this soliton family is plotted in Fig. 3(b), and

the profile of the soliton at the marked point of the power curve (with $\mu = 3.5$) is illustrated in panel (c). On the power curve, the line with analytically predicted power slope at the bifurcation point from Eqs. (90) and (92) is also plotted. It is seen that this analytical power slope matches the numerical one very well.

5. Extension to More General Nonlinearities

In this section, we show that the results in the previous sections can be readily extended to a wider class of nonlinearities

$$i\Psi_t + \Psi_{xx} + V(x)\Psi + G(|\Psi|^2)\Psi = 0, \quad (94)$$

where $G(\cdot)$ is an arbitrary real function, and $V(x)$ is a complex potential. Solitons (3) in this equation satisfy the stationary equation

$$\psi_{xx} - \mu\psi + V(x)\psi + G(|\psi|^2)\psi = 0. \quad (95)$$

Just as in the case of cubic nonlinearity, in the absence of the potential [$V(x) = 0$], this soliton equation admits two constants of motion. Assuming that the constant of motion in the presence of the complex potential $V(x)$ is a continuous deformation of those without the potential, we can show by the same technique employed in Section 3 that the only complex potentials which admit a constant of motion are those in the special form of (5), and the corresponding constant of motion is

$$J = r_x^2 - \mu r^2 + H(r^2) + r^2(\theta + g)^2, \quad (96)$$

where $H(z) = \int G(z)dz$, and $dJ/dx = 0$.

We can also show that for these general nonlinearities, with potentials of the form (5), continuous families of solitons still bifurcate out from linear discrete eigenmodes. Without loss of generality, we require $H(0) = 0$. Then for solitons, $J = 0$. Using this relation, the equation for the complex soliton $\psi(x)$ is reduced to the following second-order equation for the real amplitude variable $R(x) = |\psi(x)|^2$:

$$R_{xx} - 4\mu R + 2R G(R) + 2H(R) \pm 2g\sqrt{4\mu R^2 - 4R H(R) - R_x^2} = 0. \quad (97)$$

This equation is the analog of Eq. (28) for the cubic NLS equation (1). Repeating similar analysis as in the earlier text, these soliton bifurcations can be explicitly calculated.

To illustrate these analytical results for general nonlinearities, we consider the following example with a saturable nonlinearity.

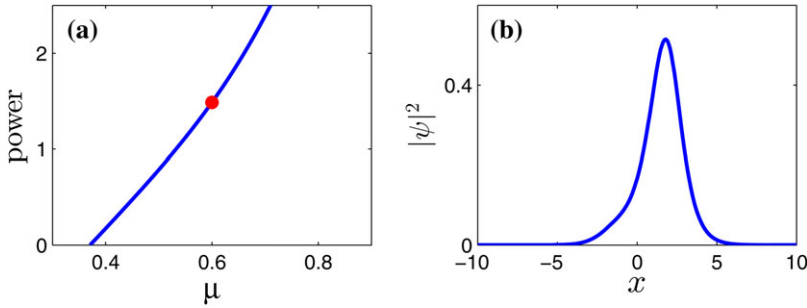


Figure 4. (a) Power curve of solitons bifurcating from a linear mode in Example 3; and (b) soliton ($|\psi|^2$) at the marked point of the power curve.

EXAMPLE 3. Consider the NLS equation (94) with a saturable nonlinearity and complex potential,

$$i\Psi_t + \Psi_{xx} + V(x)\Psi + \frac{|\Psi|^2}{1 + |\Psi|^2}\Psi = 0, \quad (98)$$

where the potential $V(x)$ is of the special form (5) with $g(x)$ chosen the same as in Example 1 (i.e., $g(x)$ is given by Eq. (91)), with h fixed as $h = 1.2$. Solitons in this equation are sought of the form (3), where $\psi(x)$ solves

$$\psi_{xx} - \mu\psi + V(x)\psi + \frac{|\psi|^2}{1 + |\psi|^2}\psi = 0. \quad (99)$$

Because the potential here is the same as that in Example 1, discrete eigenvalues in the linear equation (32) remain the same as those shown in Fig. 1(c), with the larger one being $\mu_0 \approx 0.3708$. From this eigenmode, we have confirmed that a continuous family of solitons bifurcates out, whose power curve is displayed in Fig. 4(a). At the marked point of the power curve, the corresponding soliton is plotted in Fig. 4(b). This example verifies that the bifurcation of soliton families in complex potentials (5) occurs for a wider class of nonlinearities (94).

6. Summary and Discussion

In this paper, we have analyzed soliton families in NLS-type equations with non- \mathcal{PT} -symmetric complex potentials. Under a weak assumption, we have shown that stationary forms of these equations admit a constant of motion if and only if the complex potential is of the special form (5). Using this constant of motion, we reduced the second-order complex soliton equation to a new second-order real equation for the amplitude of the soliton.

From this new soliton equation, we showed, by perturbation methods, that continuous families of solitons bifurcate out from linear eigenmodes for this special form of complex potentials. These results hold not only for the cubic nonlinearity, but also for all nonlinearities of the form $G(|\Psi|^2)\Psi$, where $G(\cdot)$ is an arbitrary real function. While it has been known that \mathcal{PT} -symmetric dissipative systems share some important properties with conservative systems, the results in this paper reveal that certain types of non- \mathcal{PT} -symmetric dissipative systems can also share such properties of conservative systems (such as the existence of soliton families).

Our results also shed light on a more general question: what non- \mathcal{PT} -symmetric complex potentials in the NLS-type equations (1) and (94) admit continuous families of solitons? In the absence of \mathcal{PT} symmetry, the existence of a constant of motion in the stationary soliton equation is critical for the existence of soliton families. We have shown that such a constant of motion exists only for special potentials of the form (5), assuming this constant of motion is a continuous deformation of that from the potential-free equation. Because this assumption is reasonable, we conjecture that *the only non- \mathcal{PT} -symmetric complex potentials which admit soliton families are those of the special form (5)*.

It should be pointed out that the question of solitons in non- \mathcal{PT} -symmetric potentials is closely related to the question of non- \mathcal{PT} -symmetric solitons in \mathcal{PT} -symmetric potentials. Indeed, for \mathcal{PT} -symmetric potentials of the same special form (5), where $g(x)$ is taken to be even, it has been shown numerically that symmetry breaking of solitons can occur [30]. As a consequence, continuous families of non- \mathcal{PT} -symmetric solitons exist in a \mathcal{PT} -symmetric potential. This symmetry breaking is surprising because it is forbidden in generic \mathcal{PT} -symmetric potentials [31, 32]. Analytical understanding of this symmetry breaking is still an open question; however, based on the analysis in this paper, it is hopeful that this symmetry breaking can now be analytically studied. However, this lies outside the scope of the present article.

In the end, we mention that bifurcation of soliton families from linear modes occurs in special forms of two-dimensional non- \mathcal{PT} -symmetric complex potentials as well [33]. Analytical understanding of such bifurcations in two spatial dimensions is a more challenging question, which merits further investigation.

Acknowledgments

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