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# STABILITY AND EVOLUTION OF SOLITARY WAVES IN PERTURBED GENERALIZED NONLINEAR SCHRÖDINGER EQUATIONS*

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**Abstract.** In this paper, we study the stability and evolution of solitary waves in perturbed generalized nonlinear Schrödinger (NLS) equations. Our method is based on the completeness of the bounded eigenstates of the associated linear operator in $L_2$ space and a standard multiple-scale perturbation technique. Unlike the adiabatic perturbation method, our method details all instability mechanisms caused by perturbations of such equations and explicitly specifies when such instabilities will occur. In particular, our method uncovers the instability caused by bifurcation of nonzero discrete eigenvalues of the linearization operator. As an example, we consider the perturbed cubic-quintic NLS equation in detail and determine the stability regions of its solitary waves. In the instability region, we also specify where the solitary waves decay, collapse, develop moving fronts, or evolve into a stable spatially localized and temporally periodic state. The generalization of this method to other perturbed nonlinear wave systems is also discussed.

**Key words.** perturbed generalized NLS equations, solitary waves, stability

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## 1. Introduction.

In recent years, the perturbed generalized nonlinear Schrödinger (NLS) equation has attracted a great deal of attention. This equation is of the form

$$iA_t + A_{xx} + f(|A|^2)A = \epsilon p(A, A^*), \tag{1.1}$$

where $f$ is a real-valued algebraic function, $p$ is a spatial differential operator, and $\epsilon$ ($> 0$) is a small parameter. It has been shown to govern the evolution of a wave packet in a weakly nonlinear and dispersive medium and has thus arisen in diverse fields such as water waves, plasma, and nonlinear optics [1, 2, 3, 4, 5]. In particular, this equation is now widely accepted in optics field as a good model for optical pulse propagation in nonlinear fibers (see [6, 7] and the references therein). The rapid advances in optical-soliton based fast-rate telecommunication systems in recent years has stimulated intensive research on it. Another application of (1.1) is in pattern formation, where it has been used to model some nonequilibrium pattern forming systems (see [8] and the references therein).

The unperturbed form of this equation ($\epsilon = 0$) supports solitary waves of the form

$$A = e^{iVx/2+i(\omega-V^2/4)t-i\rho_0} a_0(x - Vt - x_0), \tag{1.2}$$

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where $a_0(\theta)$ is a real-valued function and satisfies the equation

$$a_{0\theta\theta} - \omega a_0 + f(a_0^2)a_0 = 0, \tag{1.3a}$$

$$a_0 \longrightarrow 0, \qquad |\theta| \to \infty, \tag{1.3b}$$

and $V, \omega(> 0), x_0$ and $\rho_0$ are arbitrary real constants. Note that the existence of these solitary waves is the basis for telecommunication systems using optical solitons as information bits. When perturbations are present, one very important concern is whether or not these solitary waves will persist. This question has been studied extensively in the literature. The linear stability of the solitary waves in the unperturbed equation (1.1) has been investigated in [9], where a criteria for instability was given. When $f(x) = x$, (1.1) is the perturbed NLS equation, whose solitary waves were examined in numerous articles such as [7, 10, 11, 12, 13, 14, 29, 30, 31], among others. The dynamical behavior of the solitary waves in this equation is now well understood. But for more general forms of $f$, the results are few and far from complete. This case has been investigated in [8, 15, 16, 17] using various methods. In [8], the adiabatic perturbation technique was employed (see also [12]). These authors assumed a quasi-stationary form for the solitary wave, determined the slow evolution of the parameters of this wave, and then discovered certain types of instability from those evolution equations. But as they pointed out, the stability they established refers only to the particular class of perturbations compatible with the quasi-stationary solution assumption. It was recognized that there could be other instability mechanisms which could not be revealed by this adiabatic method (this is indeed the case). In [15], the Evans function approach was used. These authors studied the spectrum structure of the cubic-quintic NLS equation. They found that a discrete eigenvalue bifurcates out of the continuous spectrum when a small quintic term is present. They also showed that these bifurcated discrete modes (internal modes) cause soliton oscillations and radiation-induced damping (see also [18]). The numerical approach to this problem was taken in [16, 17]. In this work, the authors investigated the stability of analytic solitary waves of the cubic-quintic complex Ginzburg–Landau equation and found that they are generally unstable, except in a few special cases. The instability was caused by the existence of growing disturbances, whose largest growth rates were numerically estimated. The authors also obtained parameter regions in which stable solitary waves exist for various choices of parameter values. In their work, $\epsilon$ was not small, but since the work was numerical, it was not clear just exactly what was the source of their instability.

In this paper, we develop a new analytical method for studying the stability and evolution of the solitary waves in (1.1). This method is complete in that it can reveal all possible instability mechanisms. It is based on knowing the closure of the bounded eigenstates of the associated linear operator in $L_2$ space, combined with a standard multiple-scale perturbation method. $L_2$ is the space of all the square-integrable functions. In essence, this method is similar to the one developed in [11] (see also [13, 14]) for solitons in perturbed integrable equations. But here the new feature is that, since the unperturbed equation (1.1) is nonintegrable in general, the completeness of the bounded eigenstates (or equivalently, the Green's function) of the associated linear operator has to be established anew. We use a direct scattering technique, analogous to that in [19, 20], to accomplish this task. In the process, the structure of the spectra of this linear operator will also be obtained and detailed.

Using this new method, we can detail all possible instabilities of the solitary waves in (1.1) and give a complete account of the stability and evolution of any solitary wave of this equation. In particular, we find an instability which has never been detailed before. This instability arises due to a nonzero, discrete eigenvalue of the associated linear operator bifurcating. We would like to point out here that, in principle, this method can be applied to any perturbed nonlinear wave equation for uncovering all the instabilities of its permanent waves. We will come back to this point in section 4.

After the general procedure of this method is introduced, we will apply it to the perturbed cubic-quintic NLS equation and carry out the analysis in detail. Assuming that the perturbation contains only terms of the Ginzburg–Landau type (as in [8, 16, 17]), we will show that the perturbed cubic-quintic nonlinear Schrödinger equation allows at most two solitary waves, of which at most one is stable. We also find that the solitary waves of the model (1.1) have three instability mechanisms which are related to perturbations of, respectively, the zero, nonzero (discrete), and continuous eigenvalues of the associated linear operator in the unperturbed equation (1.1). Of these instabilities, the instability related to perturbations of the nonzero discrete eigenvalues has never before been studied. Its capture requires expansion of the perturbation series of the solution out to second order, $\epsilon^2$. With this information, we derive the necessary and sufficient conditions for these solitary waves to be stable. For two typical sets of parameters, we further specify the regions of parameter space, inside of which this equation does have stable solitary waves. In addition, we also identify the various parameter regions, wherein an unstable solitary wave would exhibit any of a variety of instability scenarios. These scenarios include a simple decay into radiation, a collapse of the main pulse, a breaking apart into two separating fronts, and even the evolution into a stable, spatially localized, and temporally periodic state. The last evolution scenario will be shown to be induced by an oscillating instability, which is itself caused by the bifurcation of any nonzero discrete eigenvalues. Finally, we point out how this method can be generalized so that one could study the stability of permanent waves in other nonlinear wave systems.

**2. The procedure.** In this section, we detail the procedure for studying the stability and evolution of the solitary waves (1.2) in (1.1). For simplicity, we consider the case where the perturbation term, $p$, is of the form

$$p(A, A^*) = \sum_{k=0}^{n} p_k(|A|^2)\frac{\partial^k A}{\partial x^k}, \tag{2.1}$$

where $p_k$ $(k = 1, \dots, n)$ are complex functions. This will exclude parametrically forced perturbations (see [21]). But even in such cases, the analysis given here can be readily modified. Anticipating the slow evolution of the free parameters of the solitary wave when a perturbation is present, we write the solution of this equation in the form

$$A = e^{iV\theta/2+i\rho}a(\theta, t, T_1, T_2, \dots, ; \epsilon), \tag{2.2}$$

where

$$\theta = x - \int_0^t V dt - \theta_0, \qquad \rho = \int_0^t \left(\omega + \frac{1}{4}V^2\right) dt - \rho_0, \tag{2.3}$$

and $\omega$, $V$, $\theta_0$, and $\rho_0$ are all functions of slow time $T_1 = \epsilon t$, $T_2 = \epsilon^2 t, \dots$. When (2.2) is substituted into (1.1), the equation for $a$ is found to be

$$ia_t - \omega a + a_{\theta\theta} + f(|a|^2)a = \epsilon F - \epsilon\left\{ia_{T_1} - i\theta_{0T_1}a_\theta + \left(\frac{1}{2}V\theta_{0T_1} - \frac{1}{2}V_{T_1}\theta + \rho_{0T_1}\right)a\right\}$$
$$- \epsilon^2\left\{ia_{T_2} - i\theta_{0T_2}a_\theta + \left(\frac{1}{2}V\theta_{0T_2} - \frac{1}{2}V_{T_2}\theta + \rho_{0T_2}\right)a\right\} + O(\epsilon^3), \tag{2.4}$$

where

$$F = p(A, A^*)e^{-iV\theta/2 - i\rho}. \tag{2.5}$$

To solve this equation, we expand $a$ into a perturbation series

$$a = a_0(\theta) + \epsilon a_1 + \epsilon^2 a_2 + \cdots, \tag{2.6}$$

and take $a_0$ to satisfy (1.3). Thus, the zeroth order of (2.4) is now trivially satisfied. At order $\epsilon$, $a_1$ is governed by the linear equation

$$ia_{1t} - \omega a_1 + a_{1\theta\theta} + r(\theta)a_1 + q(\theta)a_1^* = w_1, \tag{2.7a}$$

$$a_1|_{t=0} = 0, \tag{2.7b}$$

where

$$r = f(a_0^2) + a_0^2 f'(a_0^2), \qquad q = a_0^2 f'(a_0^2), \tag{2.8}$$

$$w_1 = F_0 - ia_{0T_1} + i\theta_{0T_1}a_{0\theta} - \left(\frac{1}{2}V\theta_{0T_1} - \frac{1}{2}V_{T_1}\theta + \rho_{0T_1}\right)a_0, \tag{2.9}$$

$$F_0 = p(A_0, A_0^*)e^{-iV\theta/2 - i\rho}, \tag{2.10}$$

and $A_0 = e^{iV\theta/2 + i\rho}a_0$. Note that $F_0$ appears to have a fast time $t$ dependence, but it actually does not, due to the form of $p$ in (2.1). This fact will be used in later analysis. Denoting $U_1 = (a_1, a_1^*)^T$, (2.7a) can be rewritten as

$$(i\partial_t + L)U_1 = (w_1, -w_1^*)^T, \tag{2.11a}$$

$$U_1|_{t=0} = 0, \tag{2.11b}$$

where the linear operator $L$ is

$$L = \sigma_3 \begin{pmatrix} \partial_{\theta\theta} - \omega + r & q \\ q & \partial_{\theta\theta} - \omega + r \end{pmatrix}, \tag{2.12}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.13}$$

are Pauli spin matrices.

The linear operator $L$ is the key to solving this problem. If we could expand all quantities in the eigenstates of this operator, then we can expand and solve (2.4) to all orders. Thus, first we need the eigenstates, $\psi(x,\lambda)$, where $\lambda$ is the eigenvalue, and the eigenvalue spectra of the operator $L$, where

$$L\psi = \lambda\psi. \tag{2.14}$$

We will now discuss the main features of this eigenvalue problem. Due to the form of $L$, if $\lambda$ is an eigenvalue and $\psi$ the corresponding eigenfunction, then it follows that $-\lambda$, $\lambda^*$, and $-\lambda^*$ are also eigenvalues, with corresponding eigenfunctions as $\sigma_1\psi$, $\psi^*$, and $\sigma_1\psi^*$. In the appendix, we detail the structure of the spectra of the operator $L$, using the direct scattering technique. We show that the discrete eigenvalues of $L$ are zeros of the analytical function $\triangle_2$ defined there. Now, there are four free parameters in the unperturbed solitary wave (1.2). Perturbations of, or shifts in, these four free parameters correspond to the four degrees of freedom represented by the degenerate $\lambda = 0$ eigenvalue. Thus $\lambda = 0$ is at least a fourfold eigenvalue of $L$. Two of these degenerate eigenvalues have two discrete eigenfunctions

$$\psi_{01} = a_{0\theta}(1,1)^T, \qquad \psi_{02} = a_0(1,-1)^T, \tag{2.15}$$

which exactly satisfy the eigenvalue equations

$$L\psi_{01} = 0, \qquad L\psi_{02} = 0. \tag{2.16}$$

The other two eigenvalues correspond to two generalized eigenfunctions, called "derivative states" in [13, 14] and elsewhere, and are given by

$$\phi_{01} = \frac{1}{2}\theta a_0(1,-1)^T, \qquad \phi_{02} = a_{0\omega}(1,1)^T. \tag{2.17}$$

These states are not true eigenfunctions, but they are necessary for closure [22]. They satisfy the modified eigenvalue equations

$$L\phi_{01} = \psi_{01}, \qquad L\phi_{02} = \psi_{02}. \tag{2.18}$$

The total number of the discrete eigenvalues of $L$ (with the multiplicity of all degenerate eigenvalues included) is given by the "winding number" of $\triangle_2$, as its argument moves along the path, $P$, described in the appendix and shown in Figure A.1. The continuous eigenvalues of $L$ are found along the two half-lines $\{\lambda : \lambda > \omega\}$ and $\{\lambda : \lambda < -\omega\}$. For each value of $\lambda$ in those intervals, there are two continuous eigenstates, one symmetric in $\theta$ (denoted as $\psi_s(\theta,\lambda)$) and the other antisymmetric in $\theta$ (denoted as $\psi_a(\theta,\lambda)$). In the appendix, we also argue that the union of the discrete regular eigenstates, the discrete generalized eigenstates, and the continuous eigenstates, form a complete set in the space of $L_2$. Therefore, (2.11) can be solved by expanding the solution, $U_1$, and the inhomogeneous term, in this complete set of functions.

For simplicity, in the rest of the paper we assume that zero is a fourfold discrete eigenvalue of $L$. In addition, we assume that $L$ has only two other simple, discrete, nonzero eigenvalues, denoted as $\lambda_d$ and $-\lambda_d$, with the corresponding eigenfunctions denoted by $\psi_d$ and $\psi_{-d} = \sigma_1\psi_d$. This is the case for the perturbed cubic-quintic NLS equation, to be discussed in more detail later in this paper. If $L$ has more than the above eigenvalues, or if the nonzero discrete eigenvalues are not simple, then the following analysis can be easily and appropriately modified.

Under the above assumptions, we then can expand $(w_1, -w_1^*)^T$ and $U_1$ in this closed set. Using arbitrary coefficients, we can take

$$(2.19)\qquad \begin{aligned}(w_1, -w_1^*)^T &= c_{01}\psi_{01}(\theta) + c_{02}\psi_{02}(\theta) + d_{01}\phi_{01}(\theta) + d_{02}\phi_{02}(\theta)\\ &\quad + c_d\psi_d(\theta) + c_{-d}\psi_{-d}(\theta) + \textstyle\int_I \{c_a(\lambda)\psi_a(\theta,\lambda) + c_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda,\end{aligned}$$

$$(2.20)\qquad \begin{aligned}U_1 &= h_{01}\psi_{01}(\theta) + h_{02}\psi_{02}(\theta) + g_{01}\phi_{01}(\theta) + g_{02}\phi_{02}(\theta)\\ &\quad + h_d\psi_d(\theta) + h_{-d}\psi_{-d}(\theta) + \textstyle\int_I \{h_a(\lambda)\psi_a(\theta,\lambda) + h_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda,\end{aligned}$$

where the interval $I = (-\infty, -\omega] \cup [\omega, \infty)$. We will define an inner product by

$$(2.21)\qquad \langle f, g\rangle = \int_{-\infty}^{\infty} f^T(\theta)\sigma_3 g(\theta)d\theta.$$

Then it is easy to show that the only nonzero inner products of these bounded eigenstates are

$$\langle\psi_{01},\phi_{01}\rangle,\ \langle\psi_{02},\phi_{02}\rangle,\ \langle\psi_d,\psi_d\rangle,\ \langle\psi_{-d},\psi_{-d}\rangle, \langle\psi_a,\psi_a\rangle,\ \text{and}\ \langle\psi_s,\psi_s\rangle.$$

In particular,

$$(2.22a)\qquad \langle\psi_a(\cdot,\lambda), \psi_a(\cdot,\lambda')\rangle = k_a(\lambda)\delta(\lambda-\lambda'),$$

$$(2.22b)\qquad \langle\psi_s(\cdot,\lambda), \psi_s(\cdot,\lambda')\rangle = k_s(\lambda)\delta(\lambda-\lambda'),$$

where $k_a(\lambda)$ and $k_s(\lambda)$ can be related to the scattering data of (2.14). However, their exact form will not be required here. When the expansions (2.19) and (2.20) are substituted into (2.11), and the above inner products are used, then we obtain the following equations for the coefficients in $U_1$:

$$(2.23a)\qquad i\frac{\partial h_{01}}{\partial t} + g_{01} = c_{01}, \qquad i\frac{\partial h_{02}}{\partial t} + g_{02} = c_{02},$$

$$(2.23b)\qquad i\frac{\partial g_{01}}{\partial t} = d_{01}, \qquad i\frac{\partial g_{02}}{\partial t} = d_{02},$$

$$(2.23c)\qquad i\frac{\partial h_d}{\partial t} + \lambda_d h_d = c_d, \qquad i\frac{\partial h_{-d}}{\partial t} - \lambda_d h_{-d} = c_{-d},$$

$$(2.23d)\qquad i\frac{\partial h_a}{\partial t} + \lambda h_a = c_a, \qquad i\frac{\partial h_s}{\partial t} + \lambda h_s = c_s,$$

$$(2.23e)\qquad h_{01} = h_{02} = g_{01} = g_{02} = h_d = h_{-d} = h_a = h_s = 0 \quad \text{at } t = 0.$$

Here

$$(2.24a)\qquad c_{01} = \frac{\langle(w_1, -w_1^*)^T, \phi_{01}\rangle}{\langle\psi_{01},\phi_{01}\rangle}, \qquad c_{02} = \frac{\langle(w_1, -w_1^*)^T, \phi_{02}\rangle}{\langle\psi_{02},\phi_{02}\rangle},$$

$$d_{01} = \frac{\langle (w_1, -w_1^*)^T, \psi_{01} \rangle}{\langle \psi_{01}, \phi_{01} \rangle}, \qquad d_{02} = \frac{\langle (w_1, -w_1^*)^T, \psi_{02} \rangle}{\langle \psi_{02}, \phi_{02} \rangle}, \tag{2.24b}$$

$$c_d = \frac{\langle (w_1, -w_1^*)^T, \psi_d \rangle}{\langle \psi_d, \psi_d \rangle}, \qquad c_{-d} = \frac{\langle (w_1, -w_1^*)^T, \psi_{-d} \rangle}{\langle \psi_{-d}, \psi_{-d} \rangle}, \tag{2.24c}$$

$$c_a = \frac{\langle (w_1, -w_1^*)^T, \psi_a \rangle}{k_a}, \qquad c_s = \frac{\langle (w_1, -w_1^*)^T, \psi_s \rangle}{k_s}. \tag{2.24d}$$

Note that $c_{-d} = -c_d^*$, since $\psi_{-d} = \sigma_1 \psi_d$ and is real. For the same reason, $h_{-d} = h_d^*$. Since $w_1$ does not depend on the fast time $t$, neither do the quantities in (2.24a). To suppress the secular terms in $h_{01}$, $h_{02}$, $g_{01}$, and $g_{02}$, we need to require that

$$c_{01} = c_{02} = d_{01} = d_{02} = 0. \tag{2.25}$$

In view of (2.24a) and (2.9), these four conditions will produce the following slow evolution equations for $V$, $\omega$, $\theta_0$, and $\rho_0$ on the $T_1$ time scale:

$$\frac{dV}{dT_1} = \frac{4 \int_{-\infty}^{\infty} a_{0\theta} \ \mathrm{Re}(F_0) d\theta}{\int_{-\infty}^{\infty} a_0^2 \ d\theta}, \tag{2.26a}$$

$$\frac{d\omega}{dT_1} = \frac{2 \int_{-\infty}^{\infty} a_0 \ \mathrm{Im}(F_0) d\theta}{\int_{-\infty}^{\infty} (a_0^2)_\omega d\theta}, \tag{2.26b}$$

$$\frac{d\theta_0}{dT_1} \int_{-\infty}^{\infty} a_0^2 \ d\theta + \frac{d\omega}{dT_1} \int_{-\infty}^{\infty} \theta (a_0^2)_\omega d\theta = 2 \int_{-\infty}^{\infty} \theta a_0 \ \mathrm{Im}(F_0) d\theta, \tag{2.26c}$$

$$\left( V \frac{d\theta_0}{dT_1} + 2 \frac{d\rho_0}{dT_1} \right) \int_{-\infty}^{\infty} (a_0^2)_\omega d\theta - \frac{dV}{dT_1} \int_{-\infty}^{\infty} \theta (a_0^2)_\omega d\theta = 4 \int_{-\infty}^{\infty} a_{0\omega} \mathrm{Re}(F_0) d\theta. \tag{2.26d}$$

Here "Re" and "Im" represent the real and imaginary parts of a complex number. It is noted that (2.26a,b) have been previously obtained by the adiabatic perturbation method in [8]. Similar equations were also derived for solitons in perturbed nonlinear Schrödinger equations (see [7, 10, 14]). In order for the solitary waves of the model (1.1) to be stable, these equations must have stable fixed points. Otherwise, an instability will arise. Such an instability would be due to the zero eigenvalue of the linear operator $L$ bifurcating and moving into the unstable region because of the perturbations. This has been discussed in [8].

When conditions (2.26a) are satisfied, solving (2.23a), we get

$$h_{01} = h_{02} = g_{01} = g_{02} = 0, \tag{2.27a}$$

$$h_d = c_d \{ 1 - \alpha_d(T_1) e^{i\lambda_d t} \} / \lambda_d, \qquad h_{-d} = h_d^*, \tag{2.27b}$$

$$h_a = c_a \{ 1 - \alpha_a(T_1) e^{i\lambda t} \} / \lambda, \tag{2.27c}$$

$$h_s = c_s\{1 - \alpha_s(T_1)e^{i\lambda t}\}/\lambda, \tag{2.27d}$$

and

$$\alpha_d(0) = \alpha_a(0) = \alpha_s(0) = 1. \tag{2.28}$$

Then the solution $U_1$ is

$$U_1 = h_d\psi_d(\theta) + h_{-d}\psi_{-d}(\theta) + \int_I \{h_a(\lambda)\psi_a(\theta,\lambda) + h_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda. \tag{2.29}$$

Here the $\alpha$'s are constants of the integration, and possibly functions of $T_1$, as indicated. The $c$'s are slowly varying with $T_1$ when $V$ and $\omega$ are also.

It is important to realize here that, in order for the solitary wave (1.2) to be stable, in addition to the conditions (2.25), we also need to require that the coefficients $h_d$, $h_{-d}$, $h_a$, and $h_s$ in $U_1$ do not grow unbounded on either the $t$ or the $T_1$ scales. On the $t$ scale, $h_a$ and $h_s$ are already bounded since the continuous eigenvalues of the operator $L$ are always real (see (2.27c,d)). But in order for $h_d$ and $h_{-d}$ to remain bounded on this scale, it is necessary for $\lambda_d$ to be real. If this is not so, then the solitary wave is unstable. Consequently, we shall now assume that $\lambda_d$ is real.

On the $T_1$ scale, we need to ensure that $\alpha_d$, $\alpha_a$, and $\alpha_s$ in (2.27a) remain bounded. To obtain the evolution equations for these coefficients, we need to expand (2.4) out to second order, $\epsilon^2$. When (2.6) is substituted into (2.4) and terms of order $\epsilon^2$ collected, an equation for $a_2$ will be obtained. Denoting $U_2 = (a_2, a_2^*)^T$, the equation for $U_2$ is

$$(i\partial_t + L)U_2 = (w_2, -w_2^*)^T, \tag{2.30a}$$

$$U_2|_{t=0} = 0, \tag{2.30b}$$

where

$$\begin{aligned} w_2 = {} & F_1 - ia_{1T_1} + i\theta_{0T_1}a_{1\theta} - (\tfrac{1}{2}V\theta_{0T_1} - \tfrac{1}{2}V_{T_1}\theta + \rho_{0T_1})a_1 \\ & - ia_{0T_2} + i\theta_{0T_2}a_{0\theta} - (\tfrac{1}{2}V\theta_{0T_2} - \tfrac{1}{2}V_{T_2}\theta + \rho_{0T_2})a_0 \\ & - a_0 f'(a_0^2)a_1(a_1 + 2a_1^*) - a_0^3 f''(a_0^2)(a_1 + a_1^*)^2/2, \end{aligned} \tag{2.31}$$

$$F_1 = \{p_A(A_0, A_0^*)A_1 + p_{A^*}(A_0, A_0^*)A_1^*\}e^{-iV\theta/2 - i\rho}, \tag{2.32}$$

and $A_1 = e^{iV\theta/2 + i\rho}a_1$.

This equation can be solved analogously to (2.11). We expand $(w_2, -w_2^*)^T$ and $U_2$ as

$$\begin{aligned} (w_2, -w_2^*)^T = {} & \hat{c}_{01}\psi_{01}(\theta) + \hat{c}_{02}\psi_{02}(\theta) + \hat{d}_{01}\phi_{01}(\theta) + \hat{d}_{02}\phi_{02}(\theta) \\ & + \hat{c}_d\psi_d(\theta) + \hat{c}_{-d}\psi_{-d}(\theta) + \textstyle\int_I \{\hat{c}_a(\lambda)\psi_a(\theta,\lambda) + \hat{c}_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} U_2 = {} & \hat{h}_{01}\psi_{01}(\theta) + \hat{h}_{02}\psi_{02}(\theta) + \hat{g}_{01}\phi_{01}(\theta) + \hat{g}_{02}\phi_{02}(\theta) \\ & + \hat{h}_d\psi_d(\theta) + \hat{h}_{-d}\psi_{-d}(\theta) + \textstyle\int_I \{\hat{h}_a(\lambda)\psi_a(\theta,\lambda) + \hat{h}_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda. \end{aligned} \tag{2.34}$$

The coefficients in $U_2$ are governed by equations similar to (2.23a) with only a hat added to each quantity. To suppress the secular terms in $\hat{h}_{01}$, $\hat{h}_{02}$, $\hat{g}_{01}$, and $\hat{g}_{02}$, we will obtain the evolution equations for the parameters $V$, $\omega$, $\theta_0$, and $\rho_0$ on the $T_2$ time scale.

The coefficient $\hat{h}_d$ in $U_2$ is governed by the equation

$$i\frac{\partial \hat{h}_d}{\partial t} + \lambda_d \hat{h}_d = \hat{c}_d, \tag{2.35}$$

where

$$\hat{c}_d = \frac{\langle (w_2, -w_2^*)^T, \psi_d\rangle}{\langle \psi_d, \psi_d\rangle}. \tag{2.36}$$

Now $\hat{c}_d$ has resonant terms which are proportional to $e^{i\lambda_d t}$. To see this, we put (2.1) into (2.32) and get

$$F_1 = \sum_{k=0}^{n} \left\{ p_k'(a_0^2)\frac{\partial^k A_0}{\partial \theta^k}(A_0^* A_1 + A_0 A_1^*) + p_k(a_0^2)\frac{\partial^k A_1}{\partial \theta^k} \right\} e^{-iV\theta/2 - i\rho}. \tag{2.37}$$

When (2.27a) and (2.29) are substituted into (2.37), we find that the $e^{i\lambda_d t}$ and $e^{-i\lambda_d t}$ coefficients in $F_1$ are proportional to $c_d\alpha_d/\lambda_d$ and $(c_d\alpha_d)^*/\lambda_d$, respectively. Suppose such terms in $F_1$ are

$$\frac{c_d\alpha_d}{\lambda_d} e_1(\theta) e^{i\lambda_d t} + \left[\frac{c_d\alpha_d}{\lambda_d} e_2(\theta)\right]^* e^{-i\lambda_d t}; \tag{2.38}$$

then the coefficient of the $e^{i\lambda_d t}$ term in $\hat{c}_d$ is found, from (2.36), to be

$$K = \{ i(c_d\alpha_d)_{T_1} + (k_1 + k_2 + k_3) c_d\alpha_d \}/\lambda_d, \tag{2.39}$$

where

$$k_1 = (V\theta_{0T_1}/2 - V_{T_1}\theta/2 + \rho_{0T_1}) \frac{\int_{-\infty}^{\infty} (\psi_{d1}^2 + \psi_{d2}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{2.40a}$$

$$k_2 = \frac{(c_d + c_d^*)}{\lambda_d} \frac{\int_{-\infty}^{\infty} (\psi_{d1} + \psi_{d2}) \{2a_0 f'(a_0^2)(\psi_{d1}^2 + \psi_{d1}\psi_{d2} + \psi_{d2}^2) + a_0^3 f''(a_0^2)(\psi_{d1} + \psi_{d2})^2\} d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{2.40b}$$

$$k_3 = \frac{\int_{-\infty}^{\infty} (e_1\psi_{d1} + e_2\psi_{d2}) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{2.40c}$$

and $e_1$ and $e_2$ were introduced in (2.38). Since the $Ke^{i\lambda_d t}$ term in $\hat{c}_d$ is a homogeneous solution of (2.35), then in order to suppress the secular growth in $\hat{h}_d$, we must have $K = 0$. This gives us a slow evolution equation for $c_d\alpha_d$, which is

$$\frac{d(c_d\alpha_d)}{dT_1} = i(k_1 + k_2 + k_3) c_d\alpha_d. \tag{2.41}$$

Note that both $k_1$ and $k_2$ are real quantities. Thus if $\mathrm{Im}(k_3)$ is negative, $c_d\alpha_d$ will exponentially grow and the solitary wave (1.2) will be unstable.

There are two ways to look at this instability. First, one could consider that the perturbation has coupled the solitary wave to the discrete eigenmodes $\psi_{\pm d}$ so that energy is pumped from the solitary wave into the eigenmode. It could also be interpreted as the initially real discrete eigenvalues, $\pm\lambda_d$, becoming unstable by moving off into an unstable region, in the presence of perturbations. In fact, in view of (2.27a,b), one can consider that $\lambda_d$ has been shifted to $\lambda_d + \epsilon(k_1 + k_2 + k_3)$. This instability is new and has yet not been analyzed in the literature.

A similar argument applies to the coefficients $\hat{h}_a$ and $\hat{h}_s$ of the continuous eigenstates in $U_2$. Suppression of the secular terms in those coefficients will produce evolution equations for $\alpha_a(T_1, \lambda)$ and $\alpha_s(T_1, \lambda)$ on the $T_1$ scale. If $\alpha_a$ or $\alpha_s$ grows unbounded, instability will also arise. There are also two ways to view this instability. First, it is clear that if this mode goes unstable, then energy will be injected into the continuous eigenmodes of $L$. This can also be viewed as the continuous eigenvalues of $L$ moving off into the unstable region, due to the perturbations. However, the evolution equations for $\alpha_a$ and $\alpha_s$ in $U_1$ are very difficult to analyze, since they are integrodifferential equations. As one can see, these equations involve convolutions which couple together $\alpha_a$ and $\alpha_s$ over all continuous eigenvalues. Nevertheless, let us take the second viewpoint and consider that this type of instability is simply the continuous eigenvalues of $L$ moving into the unstable region.

The reader is reminded that $L$'s continuous spectrum is the union of the two separated intervals $(-\infty, -\omega)$ and $(\omega, \infty)$. There are two ways in which this instability can happen. One is that $L$'s continuous spectrum simply distorts over into an unstable region. The other way is that new discrete eigenvalues could bifurcate out of the continuous spectrum and cross over into an unstable region. The first case is easy to handle. Recall that the continuous eigenvalues of the operator linearized around a solitary wave, even for the perturbed equation (1.1), as well as for the unperturbed version, can easily be specified (see [23]). Thus, this type of instability can be determined by simply looking at the continuous spectra and without the necessity of deriving and examining the evolution equations for $\alpha_a$ and $\alpha_s$.

Now for the second case, where new discrete eigenvalues bifurcate out of $L$'s continuous spectrum. Inside each of the two continuum intervals $(-\infty, -\omega)$ and $(\omega, \infty)$, the wavenumbers of the continuous eigenfunctions at infinity are nonzero. Thus no new discrete eigenvalues can bifurcate from inside either of the continuum intervals. It is only possible for them to bifurcate from the edge points of the continuous spectrum, at $\lambda = \pm\omega$, and then, only if the edge points do actually lie in the continuous spectrum. The condition for the edge points to lie inside the continuous spectrum, for generalized NLS equations, has been detailed in [15], where it was also shown that this condition is not satisfied for the cubic-quintic NLS equation (provided that the coefficient of the quintic term is nonzero). Basically, the reason that the edge points are not in the continuous spectrum is that, at either edge point, the eigenfunctions grow linearly in $x$ as $|x|$ goes to infinity. This happens when the scattering coefficients have a simple pole in the wavenumber. Other generalized NLS equations will need to be checked individually for this condition. If it is not satisfied, then the edge points will not be a part of the continuous spectrum, and then it follows that bifurcation of new discrete eigenvalues from the continuous spectrum will not take place under perturbations. If the condition is satisfied, such as in the (integrable) NLS equation [14], such bifurcation becomes possible [15]. In this case, further analysis would be needed to determine instability from this eigenvalue bifurcation.

It can be seen that the above procedure allows one to determine the stability in

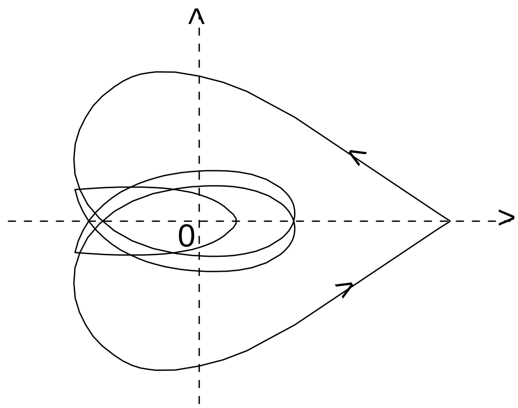


FIG. 3.1. *The trajectory of* $\triangle_2(\zeta)$ *as* $\zeta$ $P$ *for* $c_3 = 1$ *and* $c_5 < 0$. *The path* $P$ *is specified in and shown in Figure* A.1.

all possibilities. With these possibilities detailed and resolved as described above, our procedure can then be fully carried out, and the stability regions of these solitary waves, under perturbations, can be specified.

**3. The perturbed cubic-quintic NLS equation.** In this section, we use the perturbed cubic-quintic NLS equation of Ginzburg–Landau type as an example and carry out the detailed analysis. This equation is of the form

$$iA_t + A_{xx} + c_3|A|^2A + c_5|A|^4A = \epsilon i(b_1A_{xx} + \gamma A - b_3|A|^2A - b_5|A|^4A), \tag{3.1}$$

where all the coefficients are real-valued, $c_3 = \pm 1$ by scaling, and $0 < \epsilon \ll 1$. When $\epsilon = 0$, (3.1) supports solitary waves of the form (1.2), where

$$a_0(\theta) = \left[\frac{4\omega}{c_3 + \sqrt{1 + 16c_5\omega/3}\,\cosh(2\sqrt{\omega}\,\theta)}\right]^{1/2}, \tag{3.2}$$

and $\omega > 0$. If $c_3 = 1$, this wave exists when $c_5\omega > -3/16$; if $c_3 = -1$, it exists when $c_5 > 0$. The linear operator $L$ is given by (2.8) and (2.12), where $f(x) = c_3x + c_5x^2$.

We first establish the spectrum structure of $L$. We have shown that its continuous eigenvalues are the intervals $I = (-\infty, -\omega] \cup [\omega, \infty)$. To determine the total number of its discrete eigenvalues, we numerically calculated $\triangle_2$ along the path $P$ shown in the appendix for $c_3 = \pm 1$ and $c_5, \omega$ being allowed various values. For each of the following three cases: (1) $c_3 = 1$, $c_5 > 0$, (2) $c_3 = 1$, $c_5 < 0$, and (3) $c_3 = -1$, $c_5 > 0$, the results are always qualitatively the same. In case (2), the orbit of $\triangle_2$, as $\zeta$ moves along $P$, is sketched in Figure 3.1. We see that in this case, the winding number of $\triangle_2$ is four, thus $L$ has four discrete eigenvalues (multiplicity of eigenvalues included). In the other two cases, we find that the winding number of $\triangle_2$ is six. Recall that $\lambda = 0$ is always a discrete eigenvalue of $L$. To determine its multiplicity, we chose a small closed path around $\lambda = 0$ (i.e., $\zeta = e^{i\pi/4}$) and find that the winding number of $\triangle_2$ is always four for all three cases. This means that $\lambda = 0$ is always a fourfold discrete eigenvalue of $L$. We then conclude that in case (2), $\lambda = 0$ is the only discrete eigenvalue of $L$, while in the other two cases, $L$ has two additional nonzero, discrete eigenvalues. Due to the symmetry of the eigenvalues, these two nonzero eigenvalues have to be either real or purely imaginary. In addition, one is always the negative of the other. We will denote them as $\lambda_d$ and $-\lambda_d$, as before. Closer examination reveals that in case (1) $\lambda_d$ is real and in case (3) it is purely imaginary. This is consistent with the results in [9]. It indicates that in case (1) the solitary wave (1.2) is linearly neutrally stable in the unperturbed equation (3.1). But in case (3), it is linearly unstable, and thus also unstable under weak perturbations. In the rest of the paper, we assume $c_3 = 1$.

Next we determine the slow evolution equations for the solitary wave parameters $V$, $\omega$, $\theta_0$, and $\rho_0$. In view of the perturbation term in (3.1), and after some algebra, we find that (2.26a) become

$$\frac{dV}{dT_1} = -\frac{4b_1 \int_{-\infty}^{\infty} a_{0\theta}^2 d\theta}{\int_{-\infty}^{\infty} a_0^2 d\theta} V, \tag{3.3}$$

$$\frac{d\omega}{dT_1} = \frac{S(\omega) - b_1 V^2/2}{\frac{d}{d\omega} \ln \int_{-\infty}^{\infty} a_0^2 d\theta}, \tag{3.4}$$

$$\frac{d\theta_0}{dT_1} = \frac{d\rho_0}{dT_1} = 0, \tag{3.5}$$

where

$$S(\omega) = \left(2\gamma - b_1\omega - \frac{3b_5\omega}{c_5}\right) + \left(\frac{b_1}{4} - 2b_3 + \frac{9b_5}{4c_5}\right) \frac{\int_{-\infty}^{\infty} a_0^4 d\theta}{\int_{-\infty}^{\infty} a_0^2 d\theta}. \tag{3.6}$$

One should observe that $S$ depends on the parameters of the perturbation as well as on the parameters of the solitary wave. The fixed points of (3.3) and (3.4) are $V = 0$ and

$$S(\omega) = 0. \tag{3.7}$$

In order for these fixed points to be stable, we need to require that

$$b_1 > 0, \qquad \frac{S'(\omega)}{\frac{d}{d\omega} \ln \int_{-\infty}^{\infty} a_0^2 d\theta} < 0. \tag{3.8}$$

The first condition ensures that the velocity, $V$, will not grow away from zero, and the second condition ensures that the frequency, $\omega$, also does not move away from its initial value. It will be shown below that $\frac{d}{d\omega} \ln \int_{-\infty}^{\infty} a_0^2 d\theta > 0$ (see (3.10) and (3.14)). Thus the second stability condition is simply

$$S'(\omega) < 0. \tag{3.9}$$

If $b_1 < 0$, we see from (3.3) that, as $t \to \infty$, $|V| \to \infty$. Then from (3.4), we see that $\omega \to \infty$. This means the solitary wave gets steeper and steeper, and more narrow (see (3.2)). We call this case "wave collapse." If $b_1 > 0$, as $t \to \infty$, $V$ approaches its fixed point $V = 0$. This implies that the solitary wave slows down and stops moving.

In the following, we derive explicit expressions for $S(\omega)$ and $\int_{-\infty}^{\infty} a_0^2 d\theta$. These formulas are dependent on the sign of $c_5$.

(1) $c_5 > 0$: In this case,

$$\int_{-\infty}^{\infty} a_0^2 d\theta = \sqrt{3/c_5}\ (\pi/2 - \arctan u^{-1}), \tag{3.10}$$

$$S(\omega) = -\frac{3}{16c_5^2}\left(s_1 + s_2\omega + \frac{s_3 u}{\pi/2 - \arctan u^{-1}}\right), \tag{3.11}$$

where

$$s_1 = b_1c_5 - 8b_3c_5 + 9b_5 - 32\gamma c_5^2/3, \tag{3.12a}$$

$$s_2 = 16c_5(b_5 + b_1c_5/3), \tag{3.12b}$$

$$s_3 = -b_1c_5 + 8b_3c_5 - 9b_5, \tag{3.12c}$$

and

$$u = \sqrt{16c_5\omega/3}. \tag{3.13}$$

Taking the frequency, $\omega$, to be always positive, it is easy to verify that $S''(\omega)$ never changes sign, and that the sign is in fact the same as $s_3$, i.e., when $s_3 > 0$, $S''(\omega) > 0$, and when $s_3 < 0$, $S''(\omega) < 0$. Hence, when the system parameters in (3.1) are fixed, the concavity of $S(\omega)$ is also fixed. As a result, (3.4) has at most two fixed points, and (3.1) therefore has at most two solitary waves.

Now, let us say that there are two different and distinct fixed points. Then it follows that $S'(\omega)$ will have opposite signs at these points. From (3.10), we see that the sign of $\frac{d}{d\omega}\ln\int_{-\infty}^{\infty} a_0^2 d\theta$ is always positive. Thus if there are two fixed points, then one of these two fixed points must be stable and the other unstable.

Note the following. As $\omega \to 0$, $S(\omega) \to 2\gamma$, and as $\omega \to +\infty$, $S(\omega) \to -sgn(s_2)\infty$. Now, we must have $\gamma < 0$, or else the background would not be stable. Thus if $s_2 < 0$, (3.4) has only a single unstable fixed point, $\omega_u$. In this case, the unstable evolution would be as follows. If the initial solitary wave frequency $\omega_0$ was less than $\omega_u$, then the evolution would be $\omega \to 0$. In this case, the solitary wave decays. If $\omega_0 > \omega_u$, then we have $\omega \to +\infty$, i.e., the wave collapses. Thus if $s_2 < 0$, we have no stable solitary wave.

Conversely, if $s_2 > 0$, (3.4) has either zero or two fixed points. When it has no fixed points, $S(\omega) < 0$ and $\omega \to 0$, i.e., the solitary wave decays. When it has two fixed points, the larger one of them is stable, and the smaller one unstable. This fact is independent of the sign of $s_3$. In this case, $\omega$ approaches either zero or the stable fixed point, depending on the initial value.

(2) $c_5 < 0$: In this case, similar results can be obtained. Here

$$\int_{-\infty}^{\infty} a_0^2 d\theta = \sqrt{-3/c_5}\ \tanh^{-1} v, \tag{3.14}$$

$$S(\omega) = -\frac{3}{16c_5^2}\left(s_1 + s_2\omega + \frac{s_3 v}{\tanh^{-1} v}\right), \tag{3.15}$$

where $s_i$ $(i = 1, 2, 3)$ are given in (3.12a), and $v = \sqrt{-16c_5\omega/3}$. Notice that (3.14) is an increasing function. Thus $\frac{d}{d\omega} \ln \int_{-\infty}^{\infty} a_0^2 d\theta > 0$. As in the previous case, we can also show here that $S''(\omega)$ and $s_3$ always have the same sign. Thus (3.1) has at most two solitary waves, with at most one stable. As $\omega \to 0$, $S(\omega) \to 2\gamma$, and as $\omega \to -3/16c_5$, $S(\omega) \to -\frac{3}{16c_5^2}(s_1 - \frac{3}{16c_5}s_2)$.

Now we have two cases. If $s_1 - \frac{3}{16c_5}s_2 < 0$, and taking $\gamma < 0$, (3.4) has a single unstable fixed point $\omega_u$. Thus when $\omega_0 < \omega_u$, $\omega \to 0$, and the wave decays. But when $\omega_0 > \omega_u$, $\omega \to -3/16c_5$. In this case, we have a very different evolution where the wave splits into two symmetric fronts, each propagating in opposite directions (see (3.2)). The height of these fronts is $\sqrt{-3/4c_5}$ (to leading order of the perturbation expansion).

Now consider the other case where $s_1 - \frac{3}{16c_5}s_2 > 0$. Then (3.4) has zero or two fixed points. When it has none, the solitary wave decays. When it has two, the larger one is stable, and the smaller one unstable. The wave frequency $\omega$ then approaches either zero or the stable fixed point.

When (3.3) and (3.4) allow stable fixed points, the corresponding solitary wave may still be unstable due to the nonzero discrete eigenvalues moving into the unstable region under perturbations. We have shown that when $c_5 < 0$, nonzero discrete eigenvalues do not exist, but when $c_5 > 0$, two such eigenvalues of opposite sign exist and are real. Suppose $\lambda_d$ and $-\lambda_d$ are these two nonzero eigenvalues, and $\psi_d = (\psi_{d1}, \psi_{d2})^T$ and $\psi_{-d} = \sigma_1\psi_d$ the corresponding eigenfunctions. Upon inserting the perturbation terms of (3.1) into (2.40c) and after some simplifications, we find that

$$\text{Im}(k_3) = b_1(m_1 + V^2/4) - \gamma + 2m_3b_3 + 3m_5b_5, \tag{3.16}$$

where

$$m_1 = \frac{\int_{-\infty}^{\infty} (\psi_{d1\theta}^2 - \psi_{d2\theta}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{3.17a}$$

$$m_3 = \frac{\int_{-\infty}^{\infty} a_0^2(\psi_{d1}^2 - \psi_{d2}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{3.17b}$$

and

$$m_5 = \frac{\int_{-\infty}^{\infty} a_0^4(\psi_{d1}^2 - \psi_{d2}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}. \tag{3.17c}$$

Note that these $m_k$ values are functions of $c_3$, $c_5$, and $\omega$, because the solitary wave $a_0$ and the discrete eigenmode $\psi_d$ are also.

Now, the only case of interest is when we do have a stable fixed point. Let $V(= 0)$ and $\omega$ be the stable fixed points of their evolution (3.3) and (3.4). Then we simply evaluate the discrete eigenfunction $\psi_d$ (from (2.14)), the integrals in (3.17), and thereby $\text{Im}(k_3)$ in (3.16). If it is negative, then according to (2.41), the solitary wave in (3.1) will be unstable to this mechanism.

Last, we consider the instability of the solitary waves (1.2) in (3.1) caused by the shift of the continuous eigenvalues of $L$ under perturbations. As discussed in the end of section 2, in this case, no new discrete eigenvalues can bifurcate out of the continuous spectrum. Thus, there is no instability due to this mechanism. Conversely, the results in [23] indicate that when $b_1 < 0$ or $\gamma > 0$, the continuous eigenvalues will move into an unstable region due to perturbations, and under those conditions, we do have an instability. Otherwise, this type of instability is absent.

Now we summarize the above results on the stability of the solitary waves (1.2) in the perturbed cubic-quintic NLS equation (3.1). When $c_3 = -1$, all the solitary waves are unstable. When $c_3 = 1$ and $c_5 < 0$, the solitary wave is stable if and only if $b_1 > 0$, $\gamma < 0$, $V = 0$, and $\omega$ is the stable fixed point of (3.4). When $c_3 = 1$ and $c_5 > 0$, it is stable if and only if $b_1 > 0$, $\gamma < 0$, $V = 0$, $\omega$ is the stable fixed point of (3.4), and $\mathrm{Im}(k_3)$ given by (3.16) is positive. The conditions $b_1 > 0$, $\gamma < 0$, and $V = 0$ also follow directly from the physics of the problem: $b_1 > 0$ is required to stabilize the soliton in the frequency domain, $\gamma$ must be negative to stabilize the zero background, and $V = 0$ is required for any symmetric solution [16, 17]. Comparison of these results with those by the adiabatic perturbation method [8] shows that, when $c_3 = 1$ and $c_5 < 0$, the adiabatic method yields the correct stability conditions; but when $c_3 = 1$ and $c_5 > 0$, it does not. This is because it misses the instability caused by the additional mode with the nonzero discrete eigenvalue, $\lambda_d$, into which the solitary wave could emit energy, as manifested by (2.41).

As two examples, we choose $c_3 = 1$, $c_5 = 1$, and $-1$, $\gamma = -0.1$, $b_3 = -1$ and determine the regions in the $(b_1, b_5)$ plane, for the existence of stable solitary waves, by using the above results. For the given $b_1$ and $b_5$ values, we first plot the graph of $S(\omega)$. This curve will readily show if a fixed point $\omega_c$ of (3.4) exists. Notice that $\omega_c$ is now a function of $b_1$ and $b_5$. If it exists, we then determine it numerically by applying Newton's method to (3.7). Its stability can be visually determined by checking the slope of the graph $S(\omega)$ at $\omega_c$. For positive slopes, $\omega_c$ is unstable. For negative ones, it is stable. When $c_5 = -1$, each stable fixed point corresponds to a stable solitary wave. The $(b_1, b_5)$ region where a stable fixed point $\omega_s$ exists constitutes the parameter region where a stable solitary wave (3.2) exists. This region is shown in Figure 3.2(II) as the shaded area. Its lower boundary is $b_5 = 4b_3c_5 + 16\gamma c_5^2/9 = 1.1556$. Its upper boundary is determined numerically. Note that in this region of parameter space, an unstable fixed point $\omega_u$ also exists, and $\omega_u < \omega_s$. If the initial wave frequency $\omega_0$ is less than the unstable fixed point, the wave would decay ($\omega \to 0$). Otherwise the wave approaches the stable solitary wave ($\omega \to \omega_s$). Above this region, no fixed points $\omega$ exist in (3.4). Any solitary wave (3.2) will decay to zero. Below this region, a single unstable fixed point $\omega_u$ exists. Any solitary wave (3.2) either decays ($\omega \to 0$) or forms two separating fronts ($\omega \to -3/16c_5 = 3/16$), depending on whether the initial frequency $\omega_0$ is less than or greater than $\omega_u$.

When $c_5 = 1$, the $(b_1, b_5)$ region where a stable fixed point $\omega_s$ of (3.4) exists is shown in Figure 3.2(I) between two solid lines. This is the region captured by the adiabatic perturbation method [8]. Its lower boundary is given by equation $b_5 = -c_5b_1/3 = -b_1/3$. Its upper boundary is determined numerically. Above the upper solid curve, no fixed point exists in (3.4). Any solitary wave (3.2) decays to zero. Below the lower solid curve, a single unstable fixed point $\omega_u$ exists. Here, any solitary wave either decays to zero ($\omega \to 0$) or collapses ($\omega \to \infty$), depending on whether the initial frequency $\omega_0$ is less than or greater than $\omega_u$. As we previously discussed, for $c_5 > 0$, a stable fixed point $\omega_s$ in (3.4) may not correspond to a stable solitary wave due to a possible unstable bifurcation of nonzero discrete eigenvalue $\lambda_d$. To exclude this instability, for every point between the lower and upper solid lines we take the stable fixed point $\omega_s$ of (3.4) and numerically determine the discrete eigenmode $\lambda_d$ and

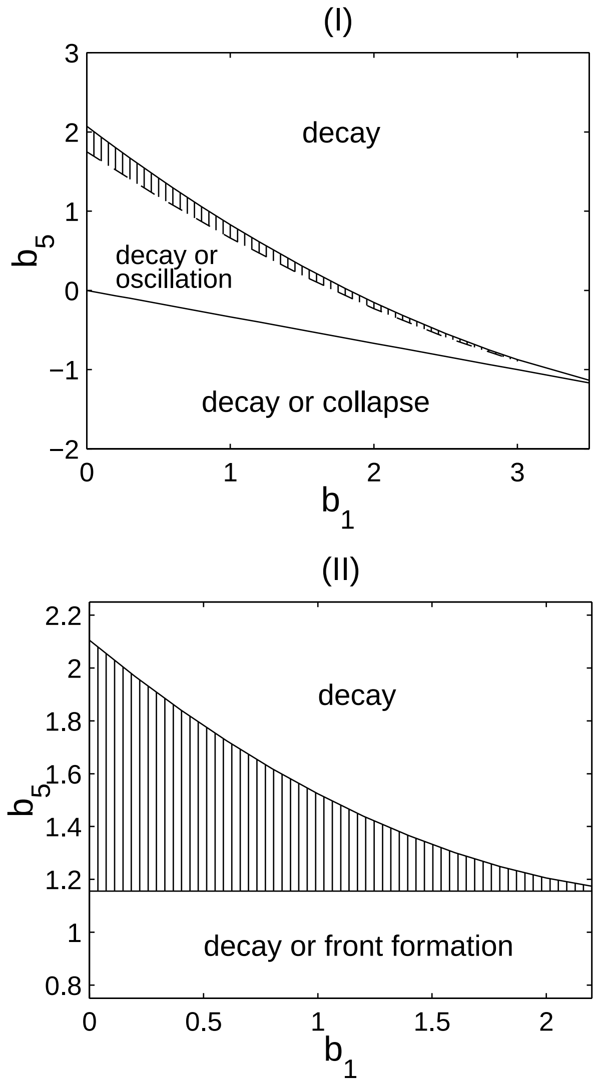


FIG. 3.2. *Regions (shaded) where a stable solitary wave* (3.2) *exists in* (3.1) *for* $c_3 = 1$, $\gamma = -0.1$, *and* $b_3 = -1$. (I) $c_5 = 1$; (II) $c_5 = -1$. *Note that in this region, a solitary wave* (3.2) *either approaches this stable wave or decays to zero, depending on the initial frequency* $\omega_0$. *Regions where a solitary wave decays, collapses, forms moving fronts, or evolves into a localized periodic state are also labeled.*

$\psi_d$ from (2.14), using the shooting method. We then evaluate $m_k$ $(k = 1, 2, 3)$ from (3.17) and $\text{Im}(k_3)$ from (3.16), with $V$ being taken as zero. The stable region is the set of $(b_1, b_5)$ points where the stable fixed point, $\omega_s$, exists and $\text{Im}(k_3)$ is positive. This region is shown in Figure 3.2(I) as the shaded area. Inside this region, a solitary wave (3.2) may approach this (unique) stable solitary wave or decay to zero, depending on its initial frequency $\omega_0$. In the region below the shaded area and above the lower solid line, $\text{Im}(k_3)$ is negative. In this region, the frequency $\omega$ of the solitary wave (3.2) approaches either zero or $\omega_s$. In the former case, the solitary wave decays. In the latter case, it still suffers the oscillating instability due to bifurcation of eigenvalue

$\lambda_d$. This instability region is missed by the adiabatic perturbation method [8]. Its capture is one of the major new results in this paper. An interesting fact is that, in the $c_5 = 1$ case, stable solitary waves exist even when $b_5 < 0$ (see Figure 3.2(I)). In this case, both nonlinear effects in the perturbations are amplifying, but they are offset by strong diffusion. Thus a stable pulse is still possible. However, such stable regions are very small.

As we have mentioned above, in the region bounded by the dashed and lower solid curves in Figure 3.2(I), the solitary wave (3.2) suffers an oscillating instability caused by $\lambda_d$ bifurcation. This bifurcation is analogous to Hopf bifurcation in dynamical systems. One important question is: What is the final state of this instability? We answer this question numerically here. For this purpose, we fix $c_3 = c_5 = 1$, $\gamma = -0.1$, $b_3 = -1$, $\epsilon = 0.1$, and select values $b_1$ and $b_5$ under the dashed line and above the lower solid line in Figure 3.2(I). We also select $\omega$ as the unique stable fixed point of (3.4) and $V = 0$. This selection is consistent with our objective. We then integrate (3.1) numerically with the initial condition as the unperturbed solitary wave (3.2). For $b_1 = 1$ and $b_3 = 0.5$, the stable fixed point $\omega_s = 0.6848$. The evolution of the solitary wave (3.2) is shown in Figure 3.3. We observe that the solitary wave undergoes an oscillating instability, as predicted. After transient evolution, the solution eventually approaches a stable spatially localized and temporally periodic state. This phenomenon is analogous to supercritical Hopf bifurcation in dynamical systems. We also tested a few other $(b_1, b_5)$ values in the same region, and the results are similar. It is relevant to remark here that similar stable oscillating states have been reported before in Ginzburg–Landau equations which cannot be considered as perturbed cubic-quintic NLS equations [24, 25]. Thus, these localized oscillating states are coherent structures in a large parameter domain of the Ginzburg–Landau equation.

During the review of this paper, one anonymous referee brought to our attention a recent independent work [26], which studies the same (3.1), but with the assumption that $c_5 \ll 1$. Basically, their equation is just the perturbed NLS equation. In that work, the authors studied the instability caused by bifurcation of discrete eigenvalues from the edge points of the continuous spectrum, using the Evan's function approach. This is somewhat similar to the instability due to bifurcation of the discrete eigenvalue $\lambda_d$, which we identified in this work. However, our results are more general, since here $c_5$ is arbitrary. We also want to emphasize that, due to the arbitrariness of $c_5$, our unperturbed equation is nonintegrable. To our knowledge, the procedure presented in this paper is the first procedure which identifies all instability mechanisms of solitary waves in perturbed *nonintegrable* systems.

**4. Discussion.** In this paper, we studied the stability and evolution of the solitary waves in perturbed generalized NLS equation (1.1), and the perturbed cubic-quintic NLS equation of Ginzburg–Landau type (3.1) in particular. We found that the solitary waves in (1.1) are subject to three types of instability which are associated with the bifurcations of the zero, nonzero (discrete), and continuous eigenvalues of the linear operator $L$ in the presence of perturbations. When specializing to the perturbed cubic-quintic NLS equation of Ginzburg–Landau type, we proved that for any set of parameters, (3.1) has at most one stable solitary wave. For two particular examples, we also specified the parameter regions of stable solitary waves, and regions where the solitary wave decays, collapses, forms moving fronts, or evolves into a stable localized oscillating state. When compared to the adiabatic results in [8], we have shown that the adiabatic method misses the instability caused by bifurcation of the nonzero discrete eigenvalues of the operator $L$.

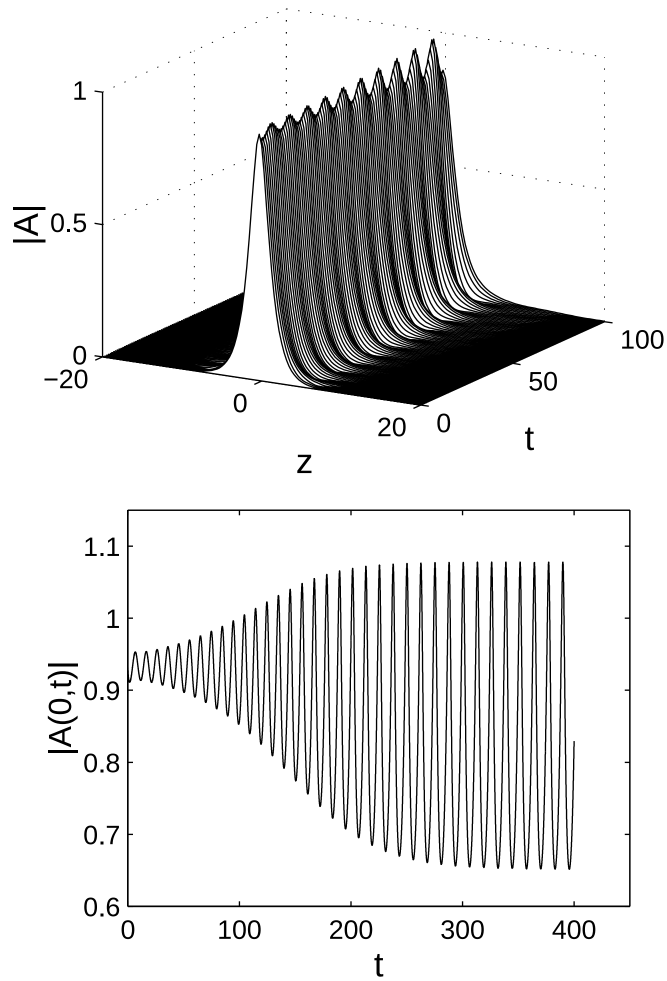


FIG. 3.3. *Evolution of a solitary wave* (3.2) *with* $c_3 = c_5 = 1$, $\gamma = -0.1$, $b_1 = 1$, $b_3 = -1$, $b_5 = 0.5$, *and* $\epsilon = 0.1$. *This point lies in the region marked as "decay or oscillation" in Figure* 3.2(I). *The initial frequency* $\omega_0$ *in* (3.2) *is the chosen stable fixed point of* (3.4).

The method we employed in this work is based on the completeness of the bounded eigenstates of the operator $L$ and a standard multiple-scale perturbation technique. The key in this analysis is the completeness of $L$'s bounded eigenstates in $L_2$ space. It allowed us to solve the relevant linearized equations at various orders and detect secularities in the linear solutions, which then set the stage for the multiple-scale perturbation method to come into play. For general perturbed nonlinear wave systems, if this completeness of the bounded eigenstates of the associated linear operator can be established, then the analysis in this paper can be adapted to those systems, and a full account of the stability and evolution of permanent waves in the presence of perturbations can be provided as well. The completeness of the bounded eigenstates of a linear operator has been studied extensively in the literature (see [19, 27, 28], for example). It has been well established for self-adjoint operators. For generic nonself-adjoint operators, as discussed in the appendix, one can still show the completeness using the direct scattering technique similar to that in [19]. For nongeneric operators, corresponding to discrete eigenvalues, generalized eigenstates as well as the regular eigenstates may exist. In this case, both the regular and generalized eigenstates of the operator will be needed for closure. In the appendix, such a closure relation is heuristically established for the (nongeneric) linear operator (2.12) of the generalized NLS equation. We believe that such a relation holds for a much larger class of linear operators. In this light, our recipe for the study of stability and evolution of permanent waves in perturbed nonlinear systems can be widely applied.

**Appendix.** In this appendix, we study the spectrum structure of the operator $L$ given by (2.12) and establish the completeness of its bounded eigenstates in $L_2$ space. For the exactly integrable NLS equation, the eigenstates of $L$ are related to the squared Zakharov–Shabat eigenstates. Thus, the completeness of $L$'s bounded eigenfunctions can be established by the inverse scattering technique [14, 22]. However, for the generalized nonlinear Schrödinger equation that connection breaks down. In this case, we will use the direct scattering method as developed in [19, 20] to accomplish this task. For convenience, we will replace $\theta$ by $x$. We first consider the general potentials $q(x)$ and $r(x)$ which vanish at infinity, then specialize to the present case where $q$ and $r$ are given by (2.8).

The eigenvalue problem

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \tag{A.1}$$

can be written as

$$u_{xx} - (\omega+\lambda)u = -r(x)u - q(x)v, \tag{A.2a}$$

$$v_{xx} - (\omega-\lambda)v = -r(x)v - q(x)u. \tag{A.2b}$$

To avoid dealing with the branch cuts at $\lambda = \pm\omega$, we make the following parameter transform:

$$\lambda = \omega(\zeta^2 + \zeta^{-2})/2. \tag{A.3}$$

Then (A.2) becomes

$$Y_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \delta^2 - r & 0 & -q & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & \eta^2 - r & 0 \end{pmatrix} Y, \tag{A.4}$$

where $Y = (u, u_x, v, v_x)^T$, and

$$\delta = \sqrt{\omega/2}\,(\zeta + \zeta^{-1}), \qquad \eta = i\sqrt{\omega/2}\,(\zeta - \zeta^{-1}). \tag{A.5}$$

The rest of the analysis is analogous to that for a $n$th order scalar scattering problem considered in [19]. Thus, the results will be only sketched here with the proofs omitted.

We define the singular set $\Sigma$ as

$$\Sigma = \{\zeta : \text{the real parts of any two numbers of } \delta,\ -\delta,\ \eta, \text{ and } -\eta \text{ are equal}\}. \tag{A.6}$$

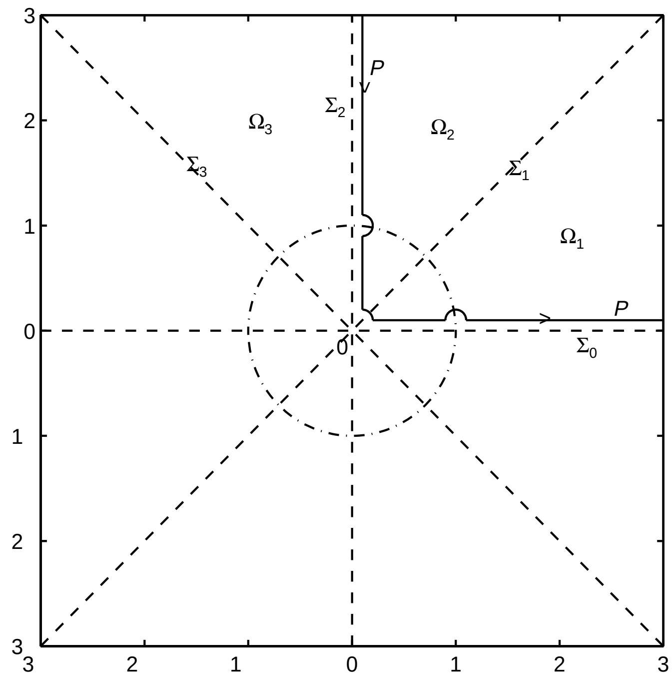


Fig. A.1. *The complex* $\zeta$*-plane.*

It is easy to see that $\Sigma$ is the set of all the rays originating from $\zeta = 0$ with angles being the multiples of $\pi/4$. We number these rays cyclically as $\Sigma_0, \Sigma_1, \ldots$ and the sectors $C \setminus \Sigma$ as $\Omega_1, \Omega_2, \ldots$ (shown in Figure A.1). On $\Sigma_0$, $\mathrm{Re}(\eta) = 0$; on $\Sigma_1$, $\mathrm{Re}(\delta) = \mathrm{Re}(-\eta)(\neq 0)$; on $\Sigma_2$, $\mathrm{Re}(\delta) = 0$; in sector $\Omega_1$, $\mathrm{Re}(\delta) > \mathrm{Re}(-\eta) > \mathrm{Re}(\eta) > \mathrm{Re}(-\delta)$; in $\Omega_2$, $\mathrm{Re}(-\eta) > \mathrm{Re}(\delta) > \mathrm{Re}(-\delta) > \mathrm{Re}(\eta)$; etc. In each sector, we define two fundamental matrices $\Phi^+$ and $\Phi^-$ of (A.4) according to the ordering of $\delta$, $-\delta$, $\eta$, and $-\eta$ in that sector. For instance, in $\Omega_1$, we define

$$\Phi^{\pm}(x,\zeta) = m^{\pm}(x,\zeta)e^{xJ}, \tag{A.7}$$

where $J = \mathrm{diag}(\delta, -\eta, \eta, -\delta)$,

$$m^{\pm}(x,\zeta) \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ \delta & 0 & 0 & -\delta \\ 0 & 1 & 1 & 0 \\ 0 & -\eta & \eta & 0 \end{pmatrix}, \qquad x \to \pm\infty, \tag{A.8}$$

and $m^{\pm}$ are bounded as $|x| \to \infty$. In other sectors, $\Phi^{\pm}$ can be similarly defined. It is easy to see that $\Phi^{\pm}$ so defined are unique. They exist for all $\zeta \in C \setminus \Sigma$, apart from a discrete set $Z$ which is all the zeros of $\triangle_k$ $(k = 1, 2, 3)$ to be defined below. At the points $\zeta_k \in Z$, $\Phi^{\pm}$ have pole singularity.

Next we define the functions

$$\triangle_1 = m_1^- \wedge m_2^+ \wedge m_3^+ \wedge m_4^+/(-4\delta\eta), \tag{A.9a}$$

$$\triangle_2 = m_1^- \wedge m_2^- \wedge m_3^+ \wedge m_4^+/(-4\delta\eta), \tag{A.9b}$$

(A.9c) $$\triangle_3 = m_1^- \wedge m_2^- \wedge m_3^- \wedge m_4^+ /(-4\delta\eta),$$

where "$\wedge$" represents the wedge products of vectors. It can be shown that $\triangle_k$ ($k = 1, 2, 3$) are independent of $x$, analytic in each sector $\Omega_i$, and $\triangle_k \to 1$ as $|\zeta| \to \infty$. Furthermore, $\triangle_2$ is analytic across the boundary $\Sigma_1$. The discrete eigenvalues $\lambda$ of the operator $L$ correspond to the zeros of $\triangle_2$ through relation (A.3). The continuous eigenvalues of $L$ correspond to the two rays $\Sigma_0$ and $\Sigma_2$. We choose a path $P$ in the $\zeta$ plane as shown in Figure A.1, with its direction counterclockwise. This path starts at $\zeta = 0^+ + \infty i$, moves down vertically to $\zeta = i$, half-circles around it, and moves downward again until it reaches $\zeta = 0$. Then it quarter-circles around $\zeta = 0$ and moves horizontally along the upper side of the positive $\mathrm{Re}(\zeta)$ axis, until it arrives at $\zeta = 1$. Then it half-circles around $\zeta = 1$, keeps on moving horizontally, and eventually ends at $\zeta = \infty + 0^+ i$. In the $\lambda$ plane, this path corresponds to one which encloses the entire $\lambda$ plane except for the continuous spectrum $\{\lambda : \lambda > \omega \text{ or } \lambda < -\omega\}$. Thus the winding number of $\triangle_2$ along $P$,

(A.10) $$N = \frac{1}{2\pi i}\int_P \frac{\triangle_2'(\zeta)}{\triangle_2(\zeta)} d\zeta = \frac{1}{2\pi}\{\arg\{\triangle_2(\infty)\} - \arg\{\triangle_2(\infty + i)\}\},$$

gives the total number of the discrete eigenvalues of $L$ (with multiplicity of nonsimple eigenvalues counted).

The completeness of the bounded eigenstates of $L$ in $L_2$ space can be established by constructing the Green's function to the equation

(A.11) $$(L - \lambda)G(x, y, \zeta) = \delta_y(x)\mathrm{diag}(1, -1)$$

and proving that $G$ can be expressed as a linear combination of $L$'s bounded eigenstates. Following [19], we call the operator $L$ generic if (1) $\triangle_k$ ($k = 1, 2, 3$) have no common zeros and no multiple zeros; (2) they have no zeros on $\Sigma$; and (3) the set of their zeros is finite. For a generic self-adjoint operator, the completeness of its bounded eigenstates in $L_2$ space was proved in [19] using this approach. If the operator is generic, but not self-adjoint, slight modification to the analysis in [19] can be made to establish the completeness relation as well. Thus non-self-adjointness would pose no difficulty. If the operator $L$ is nongeneric, the problem can be more difficult.

We briefly discuss this problem here, and to avoid treating the most general operator $L$, we will focus on $L$ where $q(x)$ and $r(x)$ are given by (2.8). This is the operator relevant in our stability analysis. It is easy to see that this operator is not self-adjoint. As we said, this causes little difficulty. More serious is the fact that $L$ is not generic. The reason is doublefold. First, $\lambda = 0$ is a discrete eigenvalue of $L$. Thus $\triangle_2$ is zero at $\zeta = e^{\pi i/4}$ (see (A.3)), which lies on $\Sigma$. Second, $\lambda = 0$ has algebraic multiplicity 4 and geometric multiplicity 2. The 4 algebraic multiplicity is due to the four free parameters in the solitary wave (1.2): spatial position ($x_0$), phase ($\rho_0$), velocity ($V$), and amplitude ($\omega$). Thus $\triangle_2$ has multiple zeros at $\lambda = 0$. It is well known that, for a general operator, if a discrete eigenvalue is multifold, and its algebraic multiplicity is larger than its geometric multiplicity, then the generalized eigenstates, as well as the regular eigenstates, of this eigenvalue are also needed for closure (see [14]). In fact, if a discrete eigenvalue is a $k$th fold root of $\triangle_2$, then correspondingly $k$ regular or generalized eigenstates should be included. Thus in the present case, what we need to establish is that the bounded eigenstates of $L$, including the generalized discrete eigenstates, form a complete set. Because $L$ is nongeneric, obviously, a more powerful proof must be devised and used to demonstrate closure. However, we present a heuristic argument below.

The main difficulty in the completeness proof is due to $L$ being nongeneric, which in turn is caused mainly by the discrete eigenvalue $\lambda = 0$. However, we recall that this is also exactly the case in the linearization operator $L_0$ of the integrable NLS equation. Note that $L_0$ is a special case of the present operator $L$. In the NLS case, the closure of bounded eigenstates of $L_0$, including generalized eigenstates of $\lambda = 0$, has been successfully established in [14] by using explicitly analytic eigensolutions and generalized (derivative) states. Unfortunately, for the generalized NLS equation, explicit analytic eigenstates are not available. Nevertheless, it is still reasonable to expect that, for the present operator $L$, being nongeneric due to $\lambda = 0$ does not break down the closure relation either. One major difference between the present $L$ and $L_0$ of the NLS equation is that $L$ may have additional nonzero, real-valued, discrete eigenvalues $\lambda_d$, where $|\lambda_d| < \omega$, while $L_0$ does not. However, these nonzero eigenvalues do not lie on $\Sigma$ (see (A.3)). Furthermore, they are usually single zeros of $\triangle_2$, as in the case of the cubic-quintic NLS equation (see section 3). Thus those additional eigenvalues do not make $L$ nongeneric and therefore do not break down the closure relation at all.

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