

# Eigenfunctions of Linearized Integrable Equations Expanded Around an Arbitrary Solution

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Eigenfunctions of linearized integrable equations expanded around an arbitrary solution are obtained for the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy and the Korteweg–de Vries (KdV) hierarchy. It is shown that the linearization operators and the integrodifferential operator that generates the hierarchy are commutable. Consequently, eigenfunctions of the linearization operators are precisely squared eigenfunctions of the associated eigenvalue problem. Similar results are obtained for the adjoint linearization operators as well. These results make a simple connection between the direct soliton/multisoliton perturbation theory and the inverse-scattering based perturbation theory for these hierarchy equations.

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## 1. Introduction

Integrable equations are nonlinear evolution equations that can be solved exactly by the inverse scattering method. Over the past few decades, it has been discovered that many physically important equations such as the Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS), and sine-Gordon equations are integrable (see [1] and the references therein). Linearization of an integrable equation around its solution arises in many important applications, most notably in a direct soliton/multisoliton perturbation theory.

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In such situations, eigenfunctions of linearization operators and their completeness are fundamental questions. For linearization around single-soliton solutions, these complete eigenfunctions have been obtained for a large class of integrable equations, such as the KdV hierarchy, NLS hierarchy, modified-KdV hierarchy, sine-Gordon, and Benjamin–Ono equations [2–9]. It has been found that these eigenfunctions are related to squared eigenfunctions of the associated eigenvalue problem (except for the Benjamin–Ono equation). However, for linearization around a general solution such as a multisoliton solution, complete eigenfunctions are known for much less integrable equations [2, 8]. However, some general ideas have been proposed to determine these eigenfunctions. One idea by Keener and McLaughlin [2] is that eigenfunctions of a linearization operator expanded around an arbitrary solution are the variations of the solution with respect to each parameter in the scattering data. Another idea by Herman [6] is to utilize the Lax pair of the integrable equation and find special combinations of squared eigenfunctions of the associated eigenvalue problem, so that these combinations satisfy the linearized equation of the evolution equation. However, in both approaches, each equation has to be treated separately. In addition, for each equation, much work is needed to find eigenfunctions of the linearization operator, or relate them to squared eigenfunctions of the associated eigenvalue problem. The idea by Yang [9], however, is free of these problems. This idea is to show that linearization operators of a hierarchy and the integrodifferential operator that generates the hierarchy are commutable; thus, they share the same set of eigenfunctions. Furthermore, these eigenfunctions are simply squared eigenfunctions of the associated eigenvalue problem. Compared to the other two approaches, this method explicitly gives the eigenfunctions of linearization operators in the simplest way. It treats an entire hierarchy all at once as well. In [9], this idea was applied only to linearizations of the KdV, NLS and modified-KdV hierarchies around single-soliton solutions. In that special case, Yang's results went beyond commutability of the operators. He also showed that linearization operators of the hierarchy equations could be factored into the integrodifferential operator that generates the hierarchy and the linearization operator of the lowest-order equation in the hierarchy. Commutability of linearization operators and the integrodifferential operator is a simple consequence of this factorization representation for the linearization operators.

In this article, we extend the results of Yang [9] to the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy and KdV hierarchy linearized around an *arbitrary* solution. The AKNS hierarchy is the family of integrable equations associated with the Zakharov–Shabat eigenvalue problem [10, 11]. In this general case, we can still show that linearization operators are commutable with the integrodifferential operator that generates the hierarchy (the factorization result of linearization operators for single-soliton solutions no longer hold).

This commutability allows us to establish that complete eigenfunctions of linearization operators in the AKNS or KdV hierarchy are simply squared eigenfunctions of the Zakharov–Shabat or Schrödinger operator. Similar results can be obtained for adjoint linearization operators as well. In a direct soliton/multisoliton perturbation theory, these squared eigenfunctions will then serve as the expansion basis for perturbation solutions. Interestingly, these same squared eigenfunctions were also, used to expand perturbation solutions in the inverse-scattering-based perturbation theory [12–14]. Thus, our results in this article indicate that, at a deeper level, the direct soliton/multisoliton perturbation theory and the inverse-scattering-based perturbation theory are actually equivalent.

## 2. Eigenfunctions of linearization operators for the AKNS hierarchy

The AKNS hierarchy associated with the Zakharov–Shabat eigenvalue problem is [11]:

$$\begin{bmatrix} r_t \\ -q_t \end{bmatrix} + i\omega(2L_z^+) \begin{bmatrix} r \\ q \end{bmatrix} = 0, \quad (1)$$

where the integrodifferential operator  $L_z^+$  is

$$L_z^+ = \frac{1}{2i} \begin{bmatrix} \frac{\partial}{\partial x} - 2r \int_{-\infty}^x dyq & 2r \int_{-\infty}^x dyr \\ -2q \int_{-\infty}^x dyq & -\frac{\partial}{\partial x} + 2q \int_{-\infty}^x dyr \end{bmatrix}, \quad (2)$$

and  $\omega(k)$  is the dispersion relation of the linear equation in the  $r$ -component. In this section, we require that  $\omega(k)$  is an entire function. When  $\omega(k) = k^2$  and  $q = -r^*$ , Equation (1) reduces to the NLS equation; when  $\omega(k) = k^3$  and  $q = -r$ , the modified-KdV equation results. The adjoint operator of  $L_z^+$  is

$$L_z = \frac{1}{2i} \begin{bmatrix} -\frac{\partial}{\partial x} - 2q \int_x^\infty dyr & -2q \int_x^\infty dyq \\ 2r \int_x^\infty dyr & \frac{\partial}{\partial x} + 2r \int_x^\infty dyq \end{bmatrix}. \quad (3)$$

Now suppose  $[r_0(x, t), q_0(x, t)]^T$  is an arbitrary solution of the evolution Equation (1). To avoid dealing with divergent integrals in the following analysis, we require that this solution vanish as  $|x|$  goes to infinity. But generalization to nonvanishing solutions is also possible by appropriately defining divergent integrals, as we did in [9]. Next we linearize Equation (1) around this arbitrary solution. For this purpose, we write

$$r = r_0(x, t) + \tilde{r}(x, t), \quad q = q_0(x, t) - \tilde{q}(x, t), \quad (4)$$

where  $\tilde{r}, \tilde{q} \ll 1$ . Note that we deliberately introduced opposite signs in front of  $r$  and  $q$ 's perturbations. This is important for obtaining the commutability

relations that we present later in this section. When Equation (4) is substituted into Equation (1), linearization of Equation (1) is:

$$L \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix} = 0, \tag{5}$$

where  $L$  is the linearization operator. We denote the adjoint operator of  $L$  as  $L^A$ . We also denote  $L_0^+$  and  $L_0$  as the integrodifferential operators  $L_z^+$  and  $L_z$  with  $r$  and  $q$  replaced by the solutions  $r_0(x,t)$  and  $q_0(x,t)$ . The primary objective of this section is to show that operators  $L$  and  $L_0^+$  commute, and  $L^A$  and  $L_0$  commute; i.e.,

$$LL_0^+ = L_0^+L, \tag{6}$$

and

$$L^AL_0 = L_0L^A. \tag{7}$$

We prove relation (6) first. Relation (7) then follows naturally.

Without loss of generality, we assume that the dispersion relation  $\omega(k)$  is a power function,  $\omega(k) = k^n$ , where  $n$  is a nonnegative integer. The reason is that any entire function of  $\omega(k)$  can be expanded into a power series. Linearization of operator  $L_z^+$  around the solution  $(r_0, q_0)$  is

$$L_z^+ = L_0^+ + \frac{1}{2i} \mathcal{F} \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix} + O(\tilde{r}^2, \tilde{r}\tilde{q}, \tilde{q}^2), \tag{8}$$

where the operator  $\mathcal{F}$  is defined as

$$\mathcal{F} \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix} = \begin{pmatrix} -2\tilde{r} \int_{-\infty}^x dyq_0 + 2r_0 \int_{-\infty}^x dy\tilde{q} & 2\tilde{r} \int_{-\infty}^x dyr_0 + 2r_0 \int_{-\infty}^x dy\tilde{r} \\ 2\tilde{q} \int_{-\infty}^x dyq_0 + 2q_0 \int_{-\infty}^x dy\tilde{q} & -2\tilde{q} \int_{-\infty}^x dyr_0 + 2q_0 \int_{-\infty}^x dy\tilde{r} \end{pmatrix}. \tag{9}$$

Then, for power functions of  $\omega(k)$ , the linearization operator  $L$  is simply:

$$L \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix}_t + i(2L_0^+)^n \begin{pmatrix} \tilde{r} \\ -\tilde{q} \end{pmatrix} + \sum_{k=1}^n (2L_0^+)^{k-1} \mathcal{F} \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix} (2L_0^+)^{n-k} \begin{pmatrix} r_0 \\ q_0 \end{pmatrix}. \tag{10}$$

Denoting

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = -i(2L_0^+)^n \begin{pmatrix} r_0 \\ q_0 \end{pmatrix}, \tag{11}$$

then the evolution of  $(r_0, q_0)$  becomes

$$\begin{pmatrix} r_0 \\ -q_0 \end{pmatrix}_t = \begin{pmatrix} P_n \\ Q_n \end{pmatrix}, \tag{12}$$

and the functions  $(P_n, Q_n)$  satisfy the recursion relation

$$\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} = 2L_0^+ \begin{pmatrix} P_n \\ Q_n \end{pmatrix}. \tag{13}$$

To establish the commutability of operators  $L$  and  $L_0^+$ , we examine the function

$$\mathcal{H}_n \equiv 2i(L_0^+L - LL_0^+) \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix}. \tag{14}$$

Simple calculations show that  $\mathcal{H}_n$  has the following expression:

$$\begin{aligned} \mathcal{H}_n = & \begin{pmatrix} 2P_n \int_{-\infty}^x (q_0\tilde{r} - r_0\tilde{q})dy - 2r_0 \int_{-\infty}^x (Q_n\tilde{r} + P_n\tilde{q})dy \\ -2Q_n \int_{-\infty}^x (q_0\tilde{r} - r_0\tilde{q})dy - 2q_0 \int_{-\infty}^x (Q_n\tilde{r} + P_n\tilde{q})dy \end{pmatrix} \\ & - (2L_0^+)^{n+1} \begin{pmatrix} \tilde{r} \\ -\tilde{q} \end{pmatrix} + (2L_0^+)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (2L_0^+) \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix} \\ & + \sum_{k=1}^n (2L_0^+)^{k-1} \left\{ 2iL_0^+ \mathcal{F} \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix} - \mathcal{F} \left[ 2iL_0^+ \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix} \right] \right\} (2L_0^+)^{n-k} \begin{pmatrix} r_0 \\ q_0 \end{pmatrix}. \end{aligned} \tag{15}$$

Here, we have replaced the time derivatives  $r_{0t}$  and  $-q_{0t}$  in  $\mathcal{H}_n$  by  $P_n$  and  $Q_n$  in view of Equation (12). It is important to realize that the above  $\mathcal{H}_n$  expression (15) is now purely algebraic and is independent of the evolution Equation (12). We use algebraic manipulations and the induction method to prove that  $\mathcal{H}_n$  is zero for all  $n \geq 0$ .

When  $n=0$  or  $1$ , we can verify directly that  $\mathcal{H}_n$  is, indeed, zero. Now, we assume that  $\mathcal{H}_n=0$  for some  $n \geq 0$ . Then, we try to show that  $\mathcal{H}_{n+1}=0$ . For this purpose, we calculate the quantity  $\mathcal{H}_{n+1} - 2L_0^+ \mathcal{H}_n$ . It turns out that most of the summation terms in  $\mathcal{H}_{n+1}$  and  $2L_0^+ \mathcal{H}_n$  cancel each other out. The terms remaining are

$$\begin{aligned} \mathcal{H}_{n+1} - 2L_0^+ \mathcal{H}_n = & \begin{pmatrix} 2P_{n+1} \int_{-\infty}^x (q_0\tilde{r} - r_0\tilde{q})dy - 2r_0 \int_{-\infty}^x (Q_{n+1}\tilde{r} + P_{n+1}\tilde{q})dy \\ -2Q_{n+1} \int_{-\infty}^x (q_0\tilde{r} - r_0\tilde{q})dy - 2q_0 \int_{-\infty}^x (Q_{n+1}\tilde{r} + P_{n+1}\tilde{q})dy \end{pmatrix} \\ & - 2L_0^+ \begin{pmatrix} 2P_n \int_{-\infty}^x (q_0\tilde{r} - r_0\tilde{q})dy - 2r_0 \int_{-\infty}^x (Q_n\tilde{r} + P_n\tilde{q})dy \\ -2Q_n \int_{-\infty}^x (q_0\tilde{r} - r_0\tilde{q})dy - 2q_0 \int_{-\infty}^x (Q_n\tilde{r} + P_n\tilde{q})dy \end{pmatrix} \\ & - \left\{ 2L_0^+ \mathcal{F} \begin{bmatrix} \tilde{r} \\ \tilde{q} \end{bmatrix} + i\mathcal{F} \left[ 2iL_0^+ \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix} \right] \right\} \begin{pmatrix} P_n \\ Q_n \end{pmatrix}. \end{aligned} \tag{16}$$

When the recursion relation (13) for  $P_{n+1}$  and  $Q_{n+1}$  is substituted into the above expression, algebraic simplifications immediately reveal that

$$\mathcal{H}_{n+1} - 2L_0^+ \mathcal{H}_n = 0. \tag{17}$$

Because  $\mathcal{H}_n$  is zero by assumption, it then follows that  $\mathcal{H}_{n+1} = 0$ . Thus,  $\mathcal{H}_n = 0$  for all  $n \geq 0$ , which means that  $L$  and  $L_0^+$  are commutable. For a general entire

function of the dispersion relation  $\omega(k)$ , this result still holds, as an entire function can be expanded into a power series.

The proof for the commutability of  $L^A$  and  $L_0$  is trivial once the commutability of  $L$  and  $L_0^+$  has been established. The adjoint operator of  $LL_0^+$  is  $L_0L^A$ , and the adjoint of  $L_0^+L$  is  $L^AL_0$ . Because  $LL_0^+ = L_0^+L$ , their adjoints are certainly the same; i.e.,  $L_0L^A = L^AL_0$ . Thus,  $L^A$  and  $L_0$  are also commutable.

An important consequence of the commutability relations (6) and (7) is that  $L$  ( $L^A$ ) and  $L_0^+$  ( $L_0$ ) share the same set of eigenfunctions. To see how this comes about, let us assume that  $\Psi(x, t, \zeta)$  is a continuous eigenfunction of  $L_0^+$  with real eigenvalue  $\zeta$ , i.e.,

$$L_0^+\Psi = \zeta\Psi. \quad (18)$$

Under the condition that  $[r_0(x, t), q_0(x, t)]$  vanishes as  $|x|$  goes to infinity, we can impose the boundary condition for  $\Psi$  as

$$\Psi(x, t, \zeta) \rightarrow \begin{pmatrix} 0 \\ -e^{-2i\zeta x} \end{pmatrix}, \quad x \rightarrow -\infty. \quad (19)$$

Because  $L$  and  $L_0^+$  are commutable, we have

$$L_0^+L\Psi = \zeta L\Psi. \quad (20)$$

Thus,  $L\Psi$  is also an eigenfunction of  $L_0^+$  with eigenvalue  $\zeta$ . As  $x$  goes to infinity, the linearization operator  $L$  becomes

$$L \rightarrow \begin{bmatrix} \partial_t + i\omega(-i\partial_x) & 0 \\ 0 & \partial_t + i\omega(-i\partial_x) \end{bmatrix}, \quad |x| \rightarrow \infty. \quad (21)$$

Consequently, the boundary condition for  $L\Psi$  can be obtained from Equations (19) and (21) as

$$L\Psi(x, t, \zeta) \rightarrow -i\omega(2\zeta) \begin{pmatrix} 0 \\ -e^{-2i\zeta x} \end{pmatrix}, \quad x \rightarrow -\infty, \quad (22)$$

which is proportional to the boundary condition (19) of eigenfunction  $\Psi$ . Then, it becomes clear that  $L\Psi$  and  $\Psi$  are the same eigenfunction of operator  $L_0^+$  with eigenvalue  $\zeta$  (i.e., they are linearly dependent). In view of their boundary conditions, we see that

$$L\Psi = -i\omega(2\zeta)\Psi, \quad (23)$$

i.e.,  $\Psi(x, t, \zeta)$  is also a continuous eigenfunction of operator  $L$  with eigenvalue  $-i\omega(2\zeta)$ .

For the same real eigenvalue  $\zeta$ ,  $L_0^+$  has another linearly independent eigenfunction  $\bar{\Psi}$  with boundary condition

$$\bar{\Psi}(x, t, \zeta) \rightarrow \begin{pmatrix} e^{2i\zeta x} \\ 0 \end{pmatrix}, \quad x \rightarrow -\infty. \quad (24)$$

Similar analysis shows that  $\bar{\Psi}$  is also a continuous eigenfunction of  $L$ , but with eigenvalue  $i\omega(2\zeta)$ , i.e.,

$$L\bar{\Psi} = i\omega(2\zeta)\bar{\Psi}. \quad (25)$$

For the discrete eigenfunctions and generalized eigenfunctions of  $L_0^+$ , same analysis indicates that they are also discrete eigenfunctions and generalized eigenfunctions of  $L$ . Thus  $L_0^+$  and  $L$ , indeed, share the same set of eigenfunctions. Naturally, the same statement applies to  $L_0$  and  $L_A$  as well.

What exactly are the sets of eigenfunctions for  $L$  and  $L^A$ ? Are these sets complete? In view of our results above, we only need to find the answers for operators  $L_0^+$  and  $L_0$ . The eigenfunctions for  $L_0^+$  and  $L_0$  and their closure have been known for over 20 years from the celebrated work by Ablowitz *et al.* [11] and by Kaup [15]. The results can be summarized as follows.

Consider the Zakharov–Shabat eigenvalue problem with potential  $[q_0(x, t), r_0(x, t)]$ :

$$v_{1x} + i\zeta v_1 = q_0(x, t)v_2, \quad (26)$$

$$v_{2x} - i\zeta v_2 = r_0(x, t)v_1, \quad (27)$$

and define Jost functions for real  $\zeta$  as

$$\psi(x, t, \zeta) = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\zeta x}, \quad x \rightarrow \infty, \quad (28)$$

$$\bar{\psi}(x, t, \zeta) = \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\zeta x}, \quad x \rightarrow \infty, \quad (29)$$

$$\phi(x, t, \zeta) = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\zeta x}, \quad x \rightarrow -\infty, \quad (30)$$

$$\bar{\phi}(x, t, \zeta) = \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -0 \end{bmatrix} e^{i\zeta x}, \quad x \rightarrow -\infty. \quad (31)$$

The right and left solutions are related by

$$\phi(x, t, \zeta) = a(t, \zeta)\bar{\psi}(x, t, \zeta) + b(t, \zeta)\psi(x, t, \zeta), \quad (32)$$

$$\bar{\phi}(x, t, \zeta) = -\bar{a}(t, \zeta)\psi(x, t, \zeta) + \bar{b}(t, \zeta)\bar{\psi}(x, t, \zeta), \quad (33)$$

where

$$\bar{a}(t, \zeta)a(t, \zeta) + \bar{b}(t, \zeta)b(t, \zeta) = 1 \quad (34)$$

from Wronskian relations. With Equation (34), the inverse of Equations (32) and (33) is

$$\psi = -a\bar{\phi} + \bar{b}\phi, \quad (35)$$

$$\bar{\psi} = \bar{a}\phi + b\bar{\phi}, \quad (36)$$

where we have suppressed the dependent variables  $x$ ,  $t$ , and  $\zeta$ . In addition to the continuous spectrum ( $\zeta$  real), Equations (26) and (27) may also possess discrete eigenvalues (bound states) in the upper and the lower half  $\zeta$ -plane. In the upper half plane, these occur whenever  $a(t, \zeta) = 0$ , and we designate them by  $\zeta_k$ ,  $k = 1, 2, \dots, n$ , where  $n$  is the total number of bound states in the upper half  $\zeta$ -plane. At  $\zeta = \zeta_k$ ,  $\phi$  and  $\psi$  become linearly dependent and

$$\phi(x, t, \zeta_k) = b(t, \zeta_k)\psi(x, t, \zeta_k), \quad k = 1, 2, \dots, n. \quad (37)$$

In the lower half  $\zeta$ -plane, bound states correspond to zeros of  $\bar{a}(t, \zeta)$ , which we designate by  $\bar{\zeta}_k$ ,  $k = 1, 2, \dots, n$ . At  $\zeta = \bar{\zeta}_k$ ,

$$\bar{\phi}(x, t, \bar{\zeta}_k) = \bar{b}(t, \bar{\zeta}_k)\bar{\psi}(x, t, \bar{\zeta}_k), \quad k = 1, 2, \dots, n. \quad (38)$$

It is important to note that when  $[q_0(x, t), r_0(x, t)]$  is a solution of the AKNS hierarchy (1), the discrete eigenvalues  $\zeta_k$  and  $\bar{\zeta}_k$  are independent of time  $t$ .

With the above notations, the eigenfunctions and generalized eigenfunctions of operators  $L_0^+$  and  $L_0$  are simply squared eigenstates of the Zakharov–Shabat system (26) and (27) [11]. Specifically, the set of eigenfunctions and generalized eigenfunctions for  $L_0^+$  is

$$\left\{ \left[ \begin{array}{c} \phi_2^2 \\ -\phi_1^2 \end{array} \right]_{\zeta}, \left[ \begin{array}{c} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{array} \right]_{\zeta}, \zeta \text{ real}; \left[ \begin{array}{c} \phi_2^2 \\ -\phi_1^2 \end{array} \right]_{\zeta_k}, \frac{\partial}{\partial \zeta} \left[ \begin{array}{c} \phi_2^2 \\ -\phi_1^2 \end{array} \right]_{\zeta_k}, 1 \leq k \leq n; \left[ \begin{array}{c} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{array} \right]_{\bar{\zeta}_k}, \frac{\partial}{\partial \zeta} \left[ \begin{array}{c} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{array} \right]_{\bar{\zeta}_k}, 1 \leq k \leq n \right\}, \quad (39)$$

and the set of such eigenfunctions for  $L_0$  is

$$\left\{ \left[ \begin{array}{c} \psi_1^2 \\ \psi_2^2 \end{array} \right]_{\zeta}, \left[ \begin{array}{c} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{array} \right]_{\zeta}, \zeta \text{ real}; \left[ \begin{array}{c} \psi_1^2 \\ \psi_2^2 \end{array} \right]_{\zeta_k}, \frac{\partial}{\partial \zeta} \left[ \begin{array}{c} \psi_1^2 \\ \psi_2^2 \end{array} \right]_{\zeta_k}, 1 \leq k \leq n; \left[ \begin{array}{c} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{array} \right]_{\bar{\zeta}_k}, \frac{\partial}{\partial \zeta} \left[ \begin{array}{c} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{array} \right]_{\bar{\zeta}_k}, 1 \leq k \leq n \right\}. \quad (40)$$

It has been shown by Kaup [15] that each of these two sets is complete. The orthogonality and inner products of functions in these sets have also been obtained there. In view of these facts, we then conclude that the sets (39) and (40) are also the complete sets of eigenfunctions and generalized eigenfunctions for linearization operators  $L$  and  $L^A$ , respectively. What about the corresponding eigenvalues? The eigenvalues are actually quite easy to obtain from the asymptotic behaviors of these eigenfunctions. For operator  $L$ , the results are:

$$L \left[ \begin{array}{c} \phi_2^2 \\ -\phi_1^2 \end{array} \right]_{\zeta} = -i\omega(2\zeta) \left[ \begin{array}{c} \phi_2^2 \\ -\phi_1^2 \end{array} \right]_{\zeta}, \quad \zeta \text{ real}; \quad (41)$$



$$L \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\zeta} = i\omega(2\zeta) \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\zeta}, \quad \zeta \text{ real}; \quad (42)$$

$$L \begin{bmatrix} \phi_2^2 \\ -\phi_1^2 \end{bmatrix}_{\zeta_k} = -i\omega(2\zeta_k) \begin{bmatrix} \phi_2^2 \\ -\phi_1^2 \end{bmatrix}_{\zeta_k}, \quad 1 \leq k \leq n; \quad (43)$$

$$L \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\bar{\zeta}_k} = i\omega(2\bar{\zeta}_k) \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\bar{\zeta}_k}, \quad 1 \leq k \leq \bar{n}; \quad (44)$$

$$L \frac{\partial}{\partial \zeta} \begin{bmatrix} \phi_2^2 \\ -\phi_1^2 \end{bmatrix}_{\zeta_k} = -i\omega(2\zeta_k) \frac{\partial}{\partial \zeta} \begin{bmatrix} \phi_2^2 \\ -\phi_1^2 \end{bmatrix}_{\zeta_k} - 2i\omega'(2\zeta_k) \begin{bmatrix} \phi_2^2 \\ -\phi_1^2 \end{bmatrix}_{\zeta_k}, \quad 1 \leq k \leq n; \quad (45)$$

and

$$L \frac{\partial}{\partial \zeta} \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\bar{\zeta}_k} = i\omega(2\bar{\zeta}_k) \frac{\partial}{\partial \zeta} \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\bar{\zeta}_k} + 2i\omega'(2\bar{\zeta}_k) \begin{bmatrix} \bar{\phi}_2^2 \\ -\bar{\phi}_1^2 \end{bmatrix}_{\bar{\zeta}_k}, \quad 1 \leq k \leq \bar{n}. \quad (46)$$

The results for  $L^A$  are:

$$L^A \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta} = -i\omega(2\zeta) \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta}, \quad \zeta \text{ real}; \quad (47)$$

$$L^A \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\zeta} = i\omega(2\zeta) \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\zeta}, \quad \zeta \text{ real}; \quad (48)$$

$$L^A \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta_k} = -i\omega(2\zeta_k) \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta_k}, \quad 1 \leq k \leq n; \quad (49)$$

$$L^A \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\bar{\zeta}_k} = i\omega(2\bar{\zeta}_k) \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\bar{\zeta}_k}, \quad 1 \leq k \leq \bar{n}; \quad (50)$$

$$L^A \frac{\partial}{\partial \zeta} \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta_k} = -i\omega(2\zeta_k) \frac{\partial}{\partial \zeta} \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta_k} - 2i\omega'(2\zeta_k) \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix}_{\zeta_k}, \quad 1 \leq k \leq n; \quad (51)$$

and

$$L^A \frac{\partial}{\partial \zeta} \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\bar{\zeta}_k} = i\omega(2\bar{\zeta}_k) \frac{\partial}{\partial \zeta} \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\bar{\zeta}_k} + 2i\omega'(2\bar{\zeta}_k) \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix}_{\bar{\zeta}_k}, \quad 1 \leq k \leq \bar{n}. \quad (52)$$

Last, we note that in the development of a direct soliton/multisoliton perturbation theory, it is often convenient to use the derivatives of soliton/multisoliton solutions with respect to soliton parameters as discrete

eigenfunctions and generalized eigenfunctions of the linearization operator [2, 5, 9]. These derivative states span the same space as the discrete eigenfunctions in the set (39) do. Thus, use of either discrete set is sufficient.

### 3. Eigenfunctions of linearization operators for the KdV hierarchy

For the KdV hierarchy, similar results hold. However, analysis is simpler, because we have only a scalar equation to consider. This hierarchy can be written as in [11]:

$$q_t + C(4L_s^+)q_x = 0, \quad (53)$$

where  $q(x, t)$  is a real function,  $C(k^2)$  is the phase velocity of the linear equation, and the integrodifferential operator  $L_s^+$  is:

$$L_s^+ = -\frac{1}{4} \frac{\partial^2}{\partial x^2} - q + \frac{1}{2} q_x \int_x^\infty dy. \quad (54)$$

Here, the subscript  $s$  in  $L_s^+$  refers to Schrödinger, because the associated eigenvalue problem for the KdV hierarchy (53) is the Schrödinger equation [11, 16]. In this section, we require the phase velocity function  $C(z)$  to be entire. The adjoint operator of  $L_s^+$  is:

$$L_s = -\frac{1}{4} \frac{\partial^2}{\partial x^2} - q + \frac{1}{2} \int_{-\infty}^x dy q_y. \quad (55)$$

Suppose  $q_0(x, t)$  is an arbitrary solution of the evolution Equation (53). To linearize Equation (53) around this solution, we write

$$q = q_0(x, t) + \tilde{q}(x, t), \quad (56)$$

where  $\tilde{q} \ll 1$ . When Equation (56) is substituted into the evolution Equation (53) and higher-order terms in  $\tilde{q}$  neglected, the linearized equation is

$$L_{\text{kh}} \tilde{q} = 0, \quad (57)$$

where  $L_{\text{kh}}$  is the linearization operator. Here, the subscript kh is the abbreviation of the KdV hierarchy. The adjoint operator of  $L_{\text{kh}}$  is denoted as  $L_{\text{kh}}^A$ . We also denote  $L_{s0}^+$  and  $L_{s0}$  as the operators  $L_s^+$  and  $L_s$  with  $q(x, t)$  replaced by the solution  $q_0(x, t)$ . The objective of this section is to show that  $L_{\text{kh}}$  and  $L_{s0}^+$  are commutable, and  $L_{\text{kh}}$  and  $L_{s0}$  are commutable; i.e.,

$$L_{\text{kh}} L_{s0}^+ = L_{s0}^+ L_{\text{kh}}, \quad (58)$$

and

$$L_{\text{kh}}^A L_{s0} = L_{s0} L_{\text{kh}}^A. \quad (59)$$

These results are analogous to Equations (6) and (7) for the AKNS hierarchy.

Without loss of generality, we will only prove relations (58) and (59) for power functions of the phase velocity function,  $C(z) = z^n$ , where  $n$  is a non-negative integer. For this power function, it is easy to check that the linearization operator  $L_{\text{kh}}$  is:

$$L_{\text{kh}}\tilde{q} = \tilde{q}_t + (4L_{s_0}^+)^n \tilde{q}_x + \sum_{k=1}^n (4L_{s_0}^+)^{k-1} \mathcal{F}_{\text{kh}}[\tilde{q}] (4L_{s_0}^+)^{n-k} q_{0x}, \quad (60)$$

where

$$\mathcal{F}_{\text{kh}}[\tilde{q}] = -4\tilde{q} + 2\tilde{q}_x \int_x^\infty dy, \quad (61)$$

and  $q_0(x, t)$  is a solution of the evolution equation (53). Denoting

$$W_n = -(4L_{s_0}^+)^n q_{0x}, \quad (62)$$

then  $q_{0t}$  is simply

$$q_{0t} = W_n, \quad (63)$$

where functions  $W_n$  satisfy the recursion relation

$$W_{n+1} = 4L_{s_0}^+ W_n. \quad (64)$$

To show that  $L_{\text{kh}}$  and  $L_{s_0}^+$  are commutable, we calculate the quantity

$$\mathcal{L}_n \equiv -4(L_{s_0}^+ L_{\text{kh}} - L_{\text{kh}} L_{s_0}^+) \tilde{q}, \quad (65)$$

which has the expression

$$\begin{aligned} \mathcal{L}_n = & 2W_{nx} \int_x^\infty \tilde{q} dy - 4W_n \tilde{q} - (4L_{s_0}^+)^n \left[ 4L_{s_0}^+ \tilde{q}_x - 4(L_{s_0}^+ \tilde{q})_x \right] \\ & - \sum_{k=1}^n (4L_{s_0}^+)^{k-1} \{ 4L_{s_0}^+ \mathcal{F}_{\text{kh}}[\tilde{q}] - \mathcal{F}_{\text{kh}}[4L_{s_0}^+ \tilde{q}] \} (4L_{s_0}^+)^{n-k} q_{0x}. \end{aligned} \quad (66)$$

Now we show that  $\mathcal{L}_n = 0$  for all  $n \geq 0$  by the induction method.

When  $n = 0$  or  $1$ , trivial calculations show that  $\mathcal{L}_n$  is, indeed, zero. Now assume that  $\mathcal{L}_n = 0$  for some  $n \geq 0$ . Note that

$$\begin{aligned} \mathcal{L}_{n+1} - 4L_{s_0}^+ \mathcal{L}_n = & 2W_{n+1,x} \int_x^\infty \tilde{q} dy - 4W_{n+1} \tilde{q} - 4L_{s_0}^+ \left[ 2W_{nx} \int_x^\infty \tilde{q} dy - 4W_n \tilde{q} \right] \\ & + \{ 4L_{s_0}^+ \mathcal{F}_{\text{kh}}[\tilde{q}] - \mathcal{F}_{\text{kh}}[4L_{s_0}^+ \tilde{q}] \} W_n. \end{aligned} \quad (67)$$

Substituting the recursion relation (64) for  $W_{n+1}$  into the above Equation (67) and carrying out some algebraic simplifications including integration by parts, we find that

$$\mathcal{L}_{n+1} - 4L_{s_0}^+ \mathcal{L}_n = 0. \quad (68)$$

Because  $\mathcal{L}_n = 0$  by assumption, we see that  $\mathcal{L}_{n+1} = 0$ . This induction procedure proves that  $\mathcal{L}_n = 0$  for all  $n \geq 0$ . Thus,  $L_{\text{kh}}$  and  $L_{s_0}^+$  are commutable

for any power function of the phase velocity  $C(z)$ . The commutability for general entire functions of  $C(z)$  follows from the fact that an entire function can be expanded into a power series.

Now  $L_{kh}$  and  $L_{s0}^+$  are commutable. Taking the adjoint of the commutability relation (57), we find that  $L_{kh}^A$  and  $L_{s0}$  are also commutable.

Commutability of  $L_{kh}$  ( $L_{kh}^A$ ) and  $L_{s0}^+$  ( $L_{s0}$ ) implies that these operators share the same set of eigenfunctions and generalized eigenfunctions. The eigenfunctions of  $L_{s0}^+$  and  $L_{s0}$  are well known [11, 14]. They are simply squared eigenfunctions of the Schrödinger operator with potential  $q_0(x, t)$ . Specifically, consider the Schrödinger equation

$$v_{xx} + [\zeta^2 + q_0(x, t)]v = 0. \tag{69}$$

Using conventional notation, we define the eigenstates  $\psi(x, t, \zeta)$  and  $\phi(x, t, \zeta)$  of Equation (69) as

$$\psi(x, t, \zeta) \rightarrow \begin{cases} e^{i\zeta x}, & x \rightarrow \infty; \\ a(t, \zeta)e^{i\zeta x} - b(t, -\zeta)e^{-i\zeta x}, & x \rightarrow -\infty; \end{cases} \tag{70}$$

and

$$\phi(x, t, \zeta) \rightarrow \begin{cases} e^{-i\zeta x}, & x \rightarrow -\infty; \\ a(t, \zeta)e^{-i\zeta x} + b(t, \zeta)e^{i\zeta x}, & x \rightarrow \infty. \end{cases} \tag{71}$$

In addition to the above continuous spectrum (real  $\zeta$ ), Equation (69) may also possess discrete eigenvalues in the upper half  $\zeta$ -plane (on the imaginary axis for real potential  $q_0$ ) where  $a(t, \zeta_k) = 0, k = 1, 2, \dots, n$ . Note that if  $q_0(x, t)$  is a solution of the KdV hierarchy (53), then these discrete eigenvalues  $\zeta_k$  are independent of time  $t$  [11, 16]. With the above notations, the set of eigenfunctions and generalized eigenfunctions for the operator  $L_{s0}^+$  is

$$\left\{ \frac{\partial \psi^2}{\partial x} \Big|_{\zeta}, \zeta \text{ real}; \frac{\partial \psi^2}{\partial x} \Big|_{\zeta_k}, \frac{\partial^2 \psi^2}{\partial x \partial \zeta} \Big|_{\zeta_k}, 1 \leq k \leq N \right\}, \tag{72}$$

and the set of such eigenfunctions for  $L_{s0}$  is

$$\left\{ \phi^2 \Big|_{\zeta}, \zeta \text{ real}; \phi^2 \Big|_{\zeta_k}, \frac{\partial \phi^2}{\partial \zeta} \Big|_{\zeta_k}, 1 \leq k \leq N \right\}. \tag{73}$$

Commutability of  $L_{s0}^+$  ( $L_{s0}$ ) and  $L_{kh}$  ( $L_{kh}^A$ ) shows that the sets (72) and (73) are also eigenfunctions and generalized eigenfunctions for the linearization operators  $L_{kh}$  and  $L_{kh}^A$ , respectively. In addition, we can readily show that the eigenvalue relations are

$$L_{kh} \frac{\partial \psi^2}{\partial x} \Big|_{\zeta} = 2i\zeta C(4\zeta^2) \frac{\partial \psi^2}{\partial x} \Big|_{\zeta}, \quad \zeta \text{ real}; \tag{74}$$

$$L_{\text{kh}} \frac{\partial \psi^2}{\partial x} \Big|_{\zeta_k} = 2i\zeta_k C(4\zeta_k^2) \frac{\partial \psi^2}{\partial x} \Big|_{\zeta_k}, \quad 1 \leq k \leq n; \quad (75)$$

$$L_{\text{kh}} \frac{\partial^2 \psi^2}{\partial x \partial \zeta} \Big|_{\zeta_k} = 2i\zeta_k C(4\zeta_k^2) \frac{\partial^2 \psi^2}{\partial x \partial \zeta} \Big|_{\zeta_k} + [2iC(4\zeta_k^2) + 16i\zeta_k^2 C'(4\zeta_k^2)] \\ \times \frac{\partial \psi^2}{\partial x} \Big|_{\zeta_k}, \quad 1 \leq k \leq n; \quad (76)$$

and

$$L_{\text{kh}}^A \phi^2 \Big|_{\zeta} = 2i\zeta C(4\zeta^2) \phi^2 \Big|_{\zeta}, \quad \zeta \text{ real}; \quad (77)$$

$$L_{\text{kh}}^A \phi^2 \Big|_{\zeta_k} = 2i\zeta_k C(4\zeta_k^2) \phi^2 \Big|_{\zeta_k}, \quad 1 \leq k \leq n. \quad (78)$$

$$L_{\text{kh}}^A \frac{\partial \phi^2}{\partial \zeta} \Big|_{\zeta_k} = 2i\zeta_k C(4\zeta_k^2) \frac{\partial \phi^2}{\partial \zeta} \Big|_{\zeta_k} \\ + [2iC(4\zeta_k^2) + 16i\zeta_k^2 C'(4\zeta_k^2)] \phi^2 \Big|_{\zeta_k}, \quad 1 \leq k \leq n. \quad (79)$$

The completeness of the two sets (72) and (73) and their inner products have been derived in [14, 17]. Thus, these sets can be used to expand the perturbation solutions in a direct soliton/multisoliton perturbation theory [6, 7, 9].

#### 4. Concluding remarks

In this article, we have studied the linearization operators of the AKNS hierarchy and KdV hierarchy equations expanded around an arbitrary solution. We have found that these linearization operators and the integrodifferential operator that generates the hierarchy are commutable. This commutability relation immediately reveals that linearization operators and the integrodifferential operator share the same set of eigenfunctions, and these eigenfunctions are simply squared eigenfunctions of the Zakharov–Shabat or Schrödinger equations. Compared with the other methods for determining eigenfunctions of the linearization operators [2, 6], our method is the simplest. Our method also gives the eigenfunctions for the entire AKNS and KdV hierarchies, not one equation at a time. In addition, our result makes a clear connection between the direct soliton/multisoliton perturbation theory and the inverse-scattering-based perturbation theory, as perturbation solutions in both theories are expanded onto the same complete set of squared eigenfunctions. With the eigenfunctions of linearization operators now available, we can proceed to develop a direct soliton/multisoliton perturbation theory for the

AKNS and KdV hierarchies, which should reproduce the results of [13, 14] obtained by the inverse-scattering-based perturbation method. This problem falls outside the scope of the present article. Another interesting question is whether the idea of this paper can be extended to derive eigenfunctions of linearization operators for integrable equations that do not fall into the AKNS and KdV hierarchies. Examples of such equations include the Benjamin–Ono equation, the derivative NLS equation, the massive Thirring equation, the Kadomtsev–Petviashvili (K–P) equation, etc. This equation will be left for future studies.

We learned recently about the work [18, 19] where a symmetry approach was developed for scalar integrable equations. For the KdV hierarchy, the commutability between the linearization operator and the recursion operator has been established there. Our commutability result for vector integrable equations such as AKNS hierarchy is new.

### Acknowledgments

The author thanks Professor D. J. Kaup for very helpful discussions on this subject. This work was supported in part by the Air Force Office of Scientific Research under contract number USAF F49620-99-1-0174 and by National Science Foundation under Grant DMS-9971712.

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(Received August 1, 2000)