The Stability and Nonlinear Evolution of Edge Waves

By Jianke Yang

The stability of steady-state standing edge waves to large-scale disturbances is studied. Both stable and unstable edge-wave modes are found. These modes strongly affect the dynamics of edge-wave evolution.

1. Introduction

The classical linear water-wave problem on a wedge-shaped beach can be formulated in terms of a velocity potential ϕ, which satisfies the Laplace equation, a free surface boundary condition, and a seabed condition, namely,

$$\Delta\phi = 0, \tag{1.1}$$

$$g\phi_y + \phi_{tt} = 0, \qquad y = 0, \tag{1.2}$$

$$\phi_y = -\phi_x \tan\alpha, \qquad y = -x\tan\alpha, \tag{1.3}$$

where x is out to sea, y is vertical, z is along the beach, g is the gravitational acceleration, and α is the angle of the beach. Stokes [1] first noted a solution of these equations that represents edge waves, which are propagating along the beach with their crests perpendicular to the shoreline and have an amplitude that

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decays exponentially off the coast. This solution for the velocity potential ϕ is

$$\phi = \frac{ga}{\omega} e^{-kx \cos\alpha + ky \sin\alpha} \sin(kz - \omega t), \tag{1.4}$$

where a is the amplitude of the edge wave, and k and ω are the wavenumber and frequency. The dispersion relation between ω and k is

$$\omega^2 = gk \sin\alpha. \tag{1.5}$$

Ursell [2] further discovered that the Stokes solution is only one of many possible edge-wave modes and that successively more possible modes arise as α decreases. A second mode is possible for $\alpha < \pi/6$, a third for $\alpha < \pi/10$, and so on. The nth mode appears as α drops below $\pi/2(2n + 1)$, and its dispersion relation is

$$\omega^2 = gk \sin(2n + 1)\alpha. \tag{1.6}$$

There is also a continuous spectrum of solutions with $\omega^2 > gk$ to complete the representation of general disturbances.

Edge waves are very distinctive on a beach because their maximum amplitude is on the shoreline. It is now believed that they are responsible for the formation of beach cusps [3] and the generation of rip currents and periodic circulation cells in the nearshore region [4].

The generation of edge waves has been intensively studied both experimentally and theoretically over the last forty years. Greenspan [5] first demonstrated that large-scale edge waves can be excited by atmospheric forcing due to storms moving along the coastline. For smaller scale edge waves, in an attempt to explain the experimental observations of Galvin [6] and Bowen and Inman [4], Guza and Davis [7] proposed the nonlinear interaction mechanism of edge waves with incoming wavetrains. Using the shallow-water approximation, they showed that a monochromatic harmonic wavetrain of frequency ω, normally incident and strongly reflected on a beach, is unstable to subharmonic standing edge-wave perturbations of frequency $\frac{1}{2}\omega$. Guza and Inman's [3] experiments on a bounded beach indicated that this subharmonic resonance was the strongest, and a subharnomic standing edge-wave was preferentially excited. A synchronous edge wave (same period as the incident wave) was sometimes also excited, but the generation was a higher order, weaker resonance and was evident only when the subharmonic resonance was excluded by the beach geometry. If the edge-wave coastline antinode number was low, edge waves would reach a steady state. More interesting was the fact that if the wavenumber was high, the number of edge-wave antinodes sometimes alternated between adjacent integers. In a further development of the theory, Whitham [8] calculated the leading-order nonlinear corrections to the linear dispersion relation of traveling Stokes edge waves and thus deduced that propagating finite-amplitude edge waves are always unstable to large-scale modulations. Later, Minzoni and Whitham [9] studied the excitation of standing subharmonic edge waves by a normally incident, strongly reflected wavetrain.

They formulated the problem in the full water-wave theory, without making the shallow-water approximation, and solved it for beach angles $\alpha = \pi/2N$, where N is an integer. Their work confirms the results from the shallow-water theory in the small-beach-angle limit. Akylas [10] studied the large-scale temporal and spatial modulations of subharmonic edge waves excited by resonant interactions with normally incident, strongly reflected wavetrains and derived equations governing the modulations of edge-wave envelopes. He then reexamined the modulational stability of a propagating edge-wave train and confirmed that the instability, predicted by Whitham [8], indeed leads to a series of envelope solitons. He also found that the steady state standing subharmonic edge wave with the wavenumber at exact subharmonic resonance is unstable to large-scale modulations.

Although much work has been done on edge waves, some important questions remain open. First, the effect of the beach geometry on edge waves has not been studied analytically. All the previous analytical work was done on an open beach. But if the beach is bounded by two sidewalls, which is always the case in experiments, this beach geometry affects the edge-wave dynamics, sometimes even excluding the excitation of edge waves. This effect shows clearly in Guza and Inman's [3] experiments. Second, the nonlinear evolution of subharmonic edge waves on a wide beach is still not clear. Since in this situation the spatial large-scale modulation arises as well as the temporal one, the evolution equations of these modulations have been derived by Akylas [10]. But what these equations imply about the edge-wave evolution is not known. One steady state standing subharmonic edge wave was found by Akylas [10] to be unstable to large-scale modulations. The significance of this finding is not clear. The relevant experiments conducted by Guza and Inman [3] show that the number of edge-wave antinodes sometimes alternated between adjacent integers. This phenomenon has yet to be explained.

In this paper, the two problems of beach geometry effect and nonlinear evolution are studied. As to the first, beach geometry is found to introduce an additional detuning term to the governing equations which affects the edge-wave dynamics. As to the second, the stability of all possible steady state standing edge-wave modes to large-scale disturbances is examined, and regions of stable and unstable modes are analytically specified. The unstable mode found by Akylas [10] is shown to fall in the unstable-mode region. The significance of these stable and unstable modes is also discussed. Finally, numerical calculations of the equations governing the large-scale edge-wave modulations are carried out. The antinode-number alternation phenomenon is found in the numerical results. The nonlinear evolution of edge waves on a wide beach is commented on at the end of this paper.

2. Formulation

Equations governing the edge-wave amplitudes are a little different on a wide bounded beach and on an open beach. We treat them separately.

2.1. Edge waves on an open beach

Consider a normally incident and strongly reflected wavetrain of frequency ω interacting with two Stokes edge-wave packets of frequency $\frac{1}{2}\omega$ propagating in opposite directions along an open beach of angle α. The undisturbed incident wave and its reflection are described by a potential

$$\phi_{inc} = \frac{ga}{\omega} S_1(x, y)e^{-i\omega t} + c.c., \tag{2.1}$$

where g is the gravitational acceleration, a is the amplitude scale of the incident wave, and $S_1(x, y)$ is a real-valued function that satisfies the Laplace equation and corresponding boundary conditions (see Minzoni and Whitham [9] for details). This incident-wave field is modified when edge waves are excited.

Following Akylas [10], a suitable expansion of the velocity potential for the two Stokes edge-wave packets and the incident wavetrain is of the form (dimensions have been restored except as noted)

$$\begin{aligned}
\phi &= \frac{ga}{\omega}\{\epsilon^{-\frac{1}{2}}e^{-kx\,\cos\alpha + ky\,\sin\alpha}[A(X, Y, Z, T)e^{i(kz-\frac{1}{2}\omega t)} \\
&\quad + B(X, Y, Z, T)e^{i(-kz-\frac{1}{2}\omega t)} + c.c.] \\
&\quad + [S(x, y; X, Y, Z, T)e^{-i\omega t} + c.c.] + \text{higher order terms}\},
\end{aligned} \tag{2.2}$$

where

$$k = \frac{\omega^2}{4g\,\sin\alpha}, \qquad \epsilon = \frac{4ka}{\sin\alpha} = \frac{a\omega^2}{g\,\sin^2\alpha}, \tag{2.3}$$

which is assumed small,

$$X = \mu 4kx, \qquad Y = \mu 4ky, \qquad Z = \mu 4kz, \qquad T = \mu\omega t, \tag{2.4}$$

and $\mu^{-1} \gg 1$ is the dimensionless modulation scale.

A multiple-scale perturbation method is used to determine the evolution of the edge-wave amplitudes A and B. It is found that A, B satisfy the following equations on the shoreline ($X = 0, Y = 0$):

$$A_T + A_Z = -i\mu A_{ZZ} + \frac{\epsilon}{\mu}\left\{\frac{1}{2}\cos\alpha\chi(\alpha)S_0 B^* - \frac{i}{128}A^2 A^* + \delta BB^* A\right\}, \tag{2.5}$$

$$B_T - B_Z = -i\mu B_{ZZ} + \frac{\epsilon}{\mu}\left\{\frac{1}{2}\cos\alpha\chi(\alpha)S_0 A^* - \frac{i}{128}B^2 B^* + \delta AA^* B\right\}, \tag{2.6}$$

where

$$S_0 = S_1(0, 0), \tag{2.7}$$

$$\chi(\alpha)S_0 = k\int_0^\infty S_1(x, 0)e^{-2kx\,\cos\alpha}dx, \tag{2.8}$$

and when $\alpha = 2\pi/N$,

$$\delta = \frac{1}{64}\left\{-32N\chi^2\sin 2\alpha + i\left(3 + \frac{32N\sin 2\alpha}{\pi}\int_0^\infty \frac{C_l^2}{l-\omega^2}dl\right)\right\}, \quad (2.9)$$

where the integral in (2.9) is interpreted as a principle value (see Minzoni and Whitham [9]).

If the beach angle α is small, χ and δ can be evaluated asymptotically (see [9, 10]):

$$\chi(\alpha) \sim \frac{1}{2e^2}, \qquad \delta(\alpha) \sim \frac{1}{256}(-1.8413 + 0.4942i)$$

$$= -0.72 \times 10^{-2} + 0.19 \times 10^{-2}i \ (\alpha \to 0). \quad (2.10)$$

When the balance $\epsilon = \mu$ is chosen, the dispersive terms in equations (2.5) and (2.6) are relatively small and can be neglected. Further, if we simply denote $\frac{1}{2}\cos\alpha\chi(\alpha)S_0$ as S_0, equations for A and B on the shoreline ($X = 0, Y = 0$) reduce to

$$A_T + CgA_Z = S_0B^* + i\gamma A^2A^* + \delta BB^*A, \quad (2.11)$$

$$B_T - CgB_Z = S_0A^* + i\gamma B^2B^* + \delta AA^*B, \quad (2.12)$$

where

$$Cg = 1, \qquad \gamma = -\frac{1}{128}, \quad (2.13)$$

and δ is as given by (2.9), or (2.10) if the beach angle α is small.

2.2. Edge waves on a wide bounded beach

When the beach is wide but bounded by two sidewalls normal to the shoreline, the forced edge-wave wavelength and the free edge-wave wavelength mismatch will also affect the dynamics of edge waves.

Consider a beach of angle α, which is bounded by two sidewalls at $z = 0$ and $z = b$. A normally incident wave of frequency ω comes to the shore and is strongly reflected. The generated edge wave has the primary wavenumber $k_0 = N\pi/b$, where N is the number of the coastline edge-wave antinodes. We assume that b and N are large, but $k_0 = N\pi/b$ remains of order $\omega^2/4g\sin\alpha$. This edge wave is both temporally and spatially modulated.

Suppose the undisturbed incident wave field is described by a potential

$$\phi_{inc} = \frac{ga}{\omega}S_1(x, y)e^{-i\omega t} + c.c., \quad (2.14)$$

where $S_1(x, y)$ is a real-valued function, and a is the incident-wave amplitude scale, which is assumed small. Introduce the small perturbation parameter

$$\epsilon = \frac{a\omega^2}{g\,\sin^2\alpha}, \tag{2.15}$$

and assume that

$$\epsilon N = h \sim O(1), \tag{2.16}$$

where h is a dimensionless measure of the beach width. The suitable expansion of the velocity potential for the edge-wave packets and the incident wave is of the following dimensional form:

$$\phi = \frac{ga}{\omega}\{\epsilon^{-\frac{1}{2}}e^{-kx\,\cos\alpha+ky\,\sin\alpha}[A(X, Y, Z, T)e^{i(k_0z-\frac{1}{2}\omega t)}$$

$$+ B(X, Y, Z, T)e^{i(-k_0z-\frac{1}{2}\omega t)} + c.c.]$$

$$+ [S(x, y; X, Y, Z, T)e^{-i\omega t} + c.c.] + \text{higher order terms}\}, \tag{2.17}$$

where

$$k = \frac{\omega^2}{4g\,\sin\alpha}, \qquad k_0 = \frac{N\pi}{b}, \tag{2.18}$$

$$X = \epsilon\,4kx, \qquad Y = \epsilon\,4ky, \qquad Z = \epsilon\,4k_0z, \qquad T = \epsilon\,\omega t. \tag{2.19}$$

A perturbation analysis results in the following equations for A and B on the shoreline ($X = 0$, $Y = 0$):

$$A_T + A_Z = iJA + S_0B^* + i\gamma A^2A^* + \delta BB^*A, \tag{2.20}$$

$$B_T - B_Z = iJB + S_0A^* + i\gamma B^2B^* + \delta AA^*B, \tag{2.21}$$

$$A = B, \qquad Z = 0, 4\pi h, \tag{2.22}$$

where

$$J = \frac{1}{4\epsilon}\left(1 - \frac{k_0}{k}\right) \text{ is the detuning parameter}, \tag{2.23}$$

$$\gamma = -\frac{1}{128},$$

$$\delta = \frac{1}{64}\left\{-32N\chi^2\,\sin 2\alpha + i\left(3 + \frac{32N\,\sin 2\alpha}{\pi}\int_0^\infty \frac{C_l^2}{l-\omega^2}dl\right)\right\} \tag{2.24}$$

With a stretching of the Z coordinate, the above equations can be rewritten as

$$A_T + CgA_Z = iJA + S_0 B^* + i\gamma A^2 A^* + \delta BB^* A, \qquad (2.25)$$

$$B_T - CgB_Z = iJB + S_0 A^* + i\gamma B^2 B^* + \delta AA^* B, \qquad (2.26)$$

$$A = B, \qquad Z = 0, \pi, \qquad (2.27)$$

with

$$C_g = \frac{1}{4h}. \qquad (2.28)$$

The value of C_g depends on the actual width of the beach. If the beach is very wide, C_g is quite small.

When the beach angle α is small,

$$\gamma = -\frac{1}{128}, \qquad \delta = -0.72 \times 10^{-2} + 0.19 \times 10^{-2}i. \qquad (2.29)$$

Dissipation may be introduced by adding a linear damping term to the equations (2.25) and (2.26).

3. Steady state standing edge waves and their stability

On a bounded beach, since the possible free edge-wave frequencies are far apart, an incident wave is only able to excite a single subharmonic standing edge wave, if any. When it does, this edge wave reaches a steady final state (see [3, 9]). But if the beach is open or bounded but wide, since the possible resonant frequencies are so close together, an incident wave usually excites several adjacent standing edge-wave modes, so that large-scale amplitude modulations arise. If this is the case, there are serious and important questions as to how edge waves evolve and what their final states may be.

A first step toward answering these questions is to study the stability of steady state standing edge-wave modes to large-scale disturbances. The equations governing these large-scale disturbances on the shoreline are as derived earlier. If dissipation is also included, they take the form

$$A_T + CgA_Z = i(J + iL)A + S_0 B^* + i\gamma A^2 A^* + \delta BB^* A, \qquad (3.1)$$

$$B_T - CgB_Z = i(J + iL)B + S_0 A^* + i\gamma B^2 B^* + \delta AA^* B \qquad (3.2)$$

$$(A = B, \qquad Z = 0, \pi \text{ on a wide bounded beach}),$$

where J is the detuning parameter (on an open beach, $J = 0$), and $L > 0$ is the linear damping coefficient. Physical arguments show that $\text{Re}(\delta) < 0$ [11].

3.1. Steady state standing edge waves

The steady state standing edge waves are described by

$$A = a_K e^{-iKZ}, \qquad B = a_K e^{iKZ}, \qquad (3.3)$$

where K is any integer if the beach is wide but bounded, any real number if it is unbounded (or open). Each K represents one steady state standing edge-wave mode. a_K is a constant and is determined by the equation

$$i(J + KC_g + iL)a_K + S_0 a_K^* + (\delta + i\gamma)a_K^2 a_K^* = 0. \qquad (3.4)$$

If $a_K = re^{i\theta}$, it is a simple matter to show that r is given by

$$r^2 = \frac{L\delta_r - (J + KC_g)(\delta_i + \gamma) + \sqrt{S_0^2|\delta + i\gamma|^2 - [L(\delta_i + \gamma) + (J + KC_g)\delta_r]^2}}{|\delta + i\gamma|^2}, \qquad (3.5)$$

and θ is given by the equation

$$-i(J + KC_g) + L - (\delta + i\gamma)r^2 = S_0 e^{-2i\theta}, \qquad (3.6)$$

where δ_r and δ_i are the real and imaginary parts of δ.

Clearly such steady edge waves exist only for a limited range of the parameter combination $J + KC_g$. If $\delta_i + \gamma < 0$, which is the case for small-angle beaches, such steady states exist only when

$$-\sqrt{S_0^2 - L^2} \equiv \bar{J}_{\min} < J + KC_g < \bar{J}_{\max} \equiv \frac{|S_0(\delta + i\gamma)| - |L(\delta_i + \gamma)|}{|\delta_r|}. \qquad (3.7)$$

If $\delta_i + \gamma > 0$, such steady states exist when

$$-\frac{|S_0(\delta + i\gamma)| - |L(\delta_i + \gamma)|}{|\delta_r|} \equiv \bar{J}_{\min} < J + KC_g < \bar{J}_{\max} \equiv \sqrt{S_0^2 - L^2}. \qquad (3.8)$$

3.2. Linear stability analysis

The stability of the steady mode (3.3) with $K = 0$ on an open beach was studied by Akylas [10] and was found unstable to large-scale disturbances. The present study examines the stability of all possible steady modes of the form (3.3) on both wide bounded and open beaches.

Assume that A, B as given by (3.3) are slightly disturbed and are written as

$$A = (a_K + \tilde{a}(Z, T))e^{-iKZ},$$
$$B = (a_K + \tilde{b}(Z, T))e^{iKZ}, \qquad (3.9)$$

where $\tilde{a}$, $\tilde{b}$ are infinitesimal disturbances. After (3.9) is substituted into the equations (3.1) and (3.2), and higher order terms are neglected, $\tilde{a}, \tilde{b}$ are found to satisfy the following linear equations:

$$\tilde{a}_T + Cg\tilde{a}_Z = [i(J + KCg + iL) + (\delta + 2i\gamma)a_K a_K^*]\tilde{a}$$
$$+ i\gamma a_K^2 \tilde{a}^* + \delta a_K a_K^* \tilde{b} + (S_0 + \delta a_K^2)\tilde{b}^*, \qquad (3.10)$$

$$\tilde{b}_T - Cg\tilde{b}_Z = [i(J + KCg + iL) + (\delta + 2i\gamma)a_K a_K^*]\tilde{b}$$
$$+ i\gamma a_K^2 \tilde{b}^* + \delta a_K a_K^* \tilde{a} + (S_0 + \delta a_K^2)\tilde{a}^*, \qquad (3.11)$$

$$(\tilde{a} = \tilde{b}, \qquad Z = 0, \pi \text{ on a wide bounded beach}).$$

With the notation $\tilde{c} \equiv \tilde{a}^*, \tilde{d} \equiv \tilde{b}^*$, equations for $\tilde{a}, \tilde{b}, \tilde{c}$ and $\tilde{d}$ are easily obtained from (3.10) and (3.11). When the following change of variables

$$W_1 = \tilde{a} + \tilde{b}, \qquad W_2 = \tilde{a} - \tilde{b}, \qquad W_3 = \tilde{c} + \tilde{d}, \qquad W_4 = \tilde{c} - \tilde{d} \quad (3.12)$$

is made, W_1, W_2, W_3, W_4 are found to satisfy the equations

$$W_{1T} + C_g W_{2Z} = [i(J + KC_g + iL) + (2\delta + 2i\gamma)a_K a_K^*]W_1$$
$$+ (S_0 + (\delta + i\gamma)a_K^2)W_3,$$

$$W_{2T} + C_g W_{1Z} = [i(J + KC_g + iL) + 2i\gamma a_K a_K^*]W_2$$
$$+ ((i\gamma - \delta)a_K^2 - S_0)W_4,$$

$$W_{3T} + C_g W_{4Z} = (S_0 + (\delta + i\gamma)a_K^2)^* W_1$$
$$+ [i(J + KC_g + iL) + (2\delta + 2i\gamma)a_K a_K^*]^* W_3,$$

$$W_{4T} + C_g W_{3Z} = ((i\gamma - \delta)a_K^2 - S_0)^* W_2$$
$$+ [i(J + KC_g + iL) + 2i\gamma a_K a_K^*]^* W_4.$$

$$(W_2 = 0, \qquad Z = 0, \pi \text{ on a wide bounded beach}).$$

$$(W_4 = 0, \qquad Z = 0, \pi \text{ on a wide bounded beach}). \qquad (3.13)$$

On an open beach, the conventional normal mode analysis assumes that

$$(W_1, W_2, W_3, W_4) = (\overline{W}_1, \overline{W}_2, \overline{W}_3, \overline{W}_4)e^{imZ + \sigma T}, \qquad (3.14)$$

where the disturbance wavenumber m takes on any real value. On a wide bounded beach, due to the sidewall boundary conditions, the normal mode analysis assumes that

$$(W_1, W_2, W_3, W_4) = (\overline{W}_1 \cos mZ, \ \overline{W}_2 \sin mZ,$$
$$\overline{W}_3 \cos mZ, \ \overline{W}_4 \sin mZ)e^{\sigma T}, \qquad (3.15)$$

where the disturbance wavenumber m takes on any integer value. In both cases, the eigenvalue σ is related to the wavenumber m by the same quartic equation

$$
\begin{aligned}
\sigma^4 &- [P_1 + P_1^* - 2L]\sigma^3 + [P_1 P_1^* - P_2 P_2^* - 2L(P_1 + P_1^*) + 2(mCg)^2]\sigma^2 \\
&- [(mCg)^2(P_1 + P_1^* - 2L) - 2L(P_1 P_1^* - P_2 P_2^*)]\sigma \\
&+ (mCg)^2[P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4 + (mCg)^2] = 0,
\end{aligned} \qquad (3.16)
$$

where

$$
\begin{aligned}
P_1 &= P_1(K) = i(J + KCg) - L + 2(\delta + i\gamma)a_K a_K^*, \\
P_2 &= P_2(K) = S_0 + (\delta + i\gamma)a_K^2, \\
P_3 &= P_3(K) = i(J + KCg) - L + 2i\gamma a_K a_K^*, \\
P_4 &= P_4(K) = (i\gamma - \delta)a_K^2 - S_0,
\end{aligned} \qquad (3.17)
$$

and a_K is as given by the equation (3.4). Notice that due to (3.4),

$$
P_2 = -\frac{(iJ - L)a_K}{a_K^*} \qquad \text{and} \qquad P_4 = \frac{a_K}{a_K^*} P_3. \qquad (3.18)
$$

From (3.4) and (3.17), it is readily shown that

$$
P_1 + P_1^* = -2L + 4\delta_r a_K a_K^* \qquad (< 0),
$$
$$
P_1 P_1^* - P_2 P_2^* = 4a_K a_K^* \sqrt{S_0^2 |\delta + i\gamma|^2 - [L(\delta_i + \gamma) + (J + KC_g)\delta_r]^2}
$$
$$
(> 0),
$$
$$
P_1 P_3 + (P_1 P_3)^*
$$
$$
+ P_2 P_4^* + P_2^* P_4 = -4[L\delta_r a_K a_K^* + (J + KC_g + (\delta_i + \gamma)a_K a_K^*)
$$
$$
(J + KC_g + 2\gamma a_K a_K^*)]. \qquad (3.19)
$$

In the next section, we use these equations to determine the stability of steady standing edge waves.

3.3. Stability results

A steady edge-wave mode (3.3) is possible if $\bar{J}_{\min} < J + KC_g < \bar{J}_{\max}$, with $\bar{J}_{\min}$ and $\bar{J}_{\max}$ given by (3.7) or (3.8). It is unstable if some normal-mode disturbances are unstable, and vice versa. A normal-mode disturbance is unstable if its eigenvalue σ has a positive real part; it is stable otherwise. In the present situation, σ is a root of the quartic equation (3.16). To solve (3.16) for general values of

K and m looks complicated, but actually it is not difficult. For our purpose, it is unnecessary to solve (3.16). The signs of $\mathrm{Re}(\sigma)$ can be determined by simply examining the coefficients of the equation (3.16). The results are given below.

I. If the edge-wave mode (3.3) is such that $P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4 < 0$, it is unstable. The unstable disturbance wavenumbers m are confined to the interval

$$0 < m^2 C_g^2 < m_1^2 C_g^2 \equiv -(P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4). \quad (3.20)$$

These unstable disturbances are standing waves of growing amplitudes because they have positive real eigenvalues σ.

II. If the edge-wave mode (3.3) is such that $0 < P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4 < P_1 P_1^* - P_2 P_2^* - 2L(P_1 + P_2)$, it is stable.

III. If the edge-wave mode (3.3) is such that $P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4 > P_1 P_1^* - P_2 P_2^* - 2L(P_1 + P_2)$, it is unstable. The unstable disturbances have wavenumbers m such that

$$m^2 C_g^2 > m_2^2 C_g^2, \quad (3.21)$$

where

$$m_2^2 C_g^2 \equiv -\frac{2L(P_1 P_1^* - P_2 P_2^*)(P_1 + P_2)(\frac{P_1 P_1^* - P_2 P_2^*}{P_1 + P_2 - 2L} - 2L)}{(P_1 + P_2 - 2L)\{(P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4) - (P_1 P_1^* - P_2 P_2^* - 2L(P_1 + P_2))\}}. \quad (3.22)$$

These unstable disturbances are traveling waves of growing amplitudes since their eigenvalues σ are complex.

We further show from the equation (3.16) that the eigenvalue

$$\sigma \longrightarrow \pm m C_g i + \sigma^{(0)} \qquad \text{as } m C_g \longrightarrow \infty, \quad (3.23)$$

where $\sigma^{(0)}$ is the root of the quadratic equation

$$\sigma^{(0)2} - \frac{P_1 + P_1^* - 2L}{2} \sigma^{(0)} - \frac{[P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4] - [P_1 P_1^* - P_2 P_2^* - 2L(P_1 + P_2)]}{4} = 0. \quad (3.24)$$

In this third case, because $P_1 P_3 + (P_1 P_3)^* + P_2 P_4^* + P_2^* P_4 > P_1 P_1^* - P_2 P_2^* - 2L(P_1 + P_2)$, one root of the equation (3.24) is real and positive and the other is

real and negative. The unstable disturbance takes the positive $\sigma^{(0)}$. Since $\sigma^{(0)}$ is independent of mC_g, very short disturbance waves tend to have the same growth rate $\sigma^{(0)}$.

Edge-wave modes of this kind are very unusual because they are unstable to small-scale disturbances. Interpretation of existence of these modes requires caution and is discussed later.

These stability results are not affected by a proportional change of γ and δ, and it is easy to show that

$$\sigma(\beta\gamma, \beta\delta) = \sigma(\gamma, \delta). \qquad (3.25)$$

A change in S_0 causes σ to change in a simple way:

$$\sigma(\beta m, \beta S_0, \beta L, \beta(J + KC_g)) = \beta\sigma(m, S_0, L, (J + KC_g)). \qquad (3.26)$$

These facts are helpful in determining the stability structure when different values of γ, δ, and S_0 are taken.

The general stability results immediately give the precise stability structure of steady state standing edge-wave modes on a given beach. As an example, we determine this structure on a mildly sloping beach.

On a mildly sloping beach,

$$\gamma = -\frac{1}{128}, \qquad \delta = -0.72 \times 10^{-2} + 0.19 \times 10^{-2}i. \qquad (3.27)$$

The damping coefficient L is small but hard to determine. It is usually set to be zero for simplicity.

In this case, since $\delta_i + \gamma < 0$, $\bar{J}_{\min}$ and $\bar{J}_{\max}$ are given by the equation (3.7) as

$$\bar{J}_{\min} = -\sqrt{S_0^2} = -S_0, \qquad \bar{J}_{\max} = \frac{|S_0(\delta + i\gamma)|}{|\delta_r|} = 1.29S_0. \qquad (3.28)$$

Steady edge-wave modes (3.3) are possible if

$$-S_0 < J + KC_g < 1.29S_0. \qquad (3.29)$$

The stability structure is shown in Figure 1.

Interval I.

$$-S_0 < J + KC_g < 0.82S_0.$$

The steady edge-wave modes in this interval are unstable. From the equations (3.19) and (3.20) we get

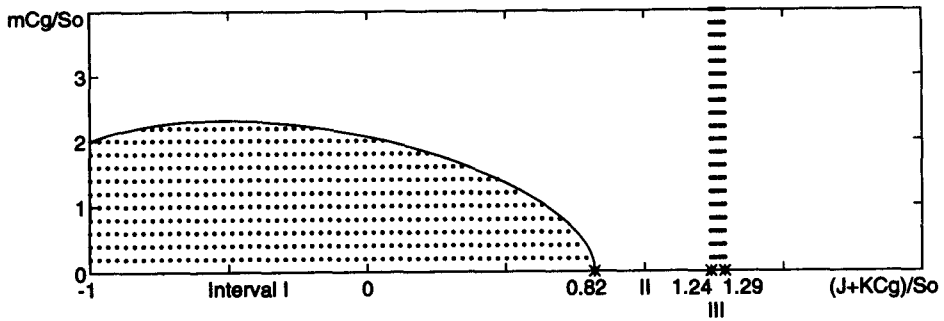$$m_1^2 C_g^2 = 4(J + KC_g + (\delta_i + \gamma)a_K a_K^*)(J + KC_g + 2\gamma a_K a_K^*), \qquad (3.30)$$

Figure 1.   Intervals of stable and unstable steady edge-wave modes (3.3) and regions of their stable and unstable disturbances on a mildly sloping beach. The unstable steady modes (3.3) are in the intervals I and III. The unstable disturbances are in the dotted area.

where

$$a_K a_K^* = \frac{-(J + KC_g)(\delta_i + \gamma) + \sqrt{S_0^2 |\delta + i\gamma|^2 - (J + KC_g)^2 \delta_r^2}}{|\delta + i\gamma|^2}. \qquad (3.31)$$

Values of $m_1 C_g$ are plotted against $J + KC_g$ in Figure 1. The unstable disturbance wavenumbers m are confined in the interval

$$0 < m^2 C_g^2 < m_1^2 C_g^2. \qquad (3.32)$$

These disturbances are standing waves of growing amplitudes.

Interval II.

$0.82 S_0 < J + KC_g < 1.24 S_0$.
The steady edge-wave modes in this interval are stable.

Interval III.

$1.24 S_0 < J + KC_g < 1.29 S_0$.
The steady edge-wave modes in this interval are unstable. $m_2^2 C_g^2$ is given by the equations (3.19) and (3.22). Since $L = 0$, $m_2 C_g = 0$. Any disturbances with $m^2 C_g^2 > 0$ are unstable, and they are traveling waves of growing amplitudes.

The unstable edge-wave mode Akylas [10] found on an open, mildly sloping beach corresponds to the one in Interval I with $J = 0$, $K = 0$, and $L = 0$. With

$S_0 = 1/4e^2$ taken, he numerically determined that

$$m_1 Cg \approx 0.052$$

and found that the eigenvalues σ of the unstable disturbances have very small imaginary parts.

Actually, for this mode, (3.30) and (3.31) give

$$a_0 a_0^* = \frac{S_0}{|\delta + i\gamma|}, \tag{3.33}$$

$$
\begin{aligned}
m_1^2 C_g^2 &= 8\gamma(\delta_i + \gamma)(a_K a_K^*)^2 \\
&= \frac{8\gamma(\delta_i + \gamma)S_0^2}{|\delta + i\gamma|^2} \\
&= 0.49 \times 10^{-2}.
\end{aligned} \tag{3.34}
$$

Therefore, the exact value for $m_1 C_g$ is

$$m_1 Cg = 0.070. \tag{3.35}$$

This value can be checked in Figure 1. Moreover, the eigenvalues σ of the unstable disturbances are exactly real and positive; thus they represent standing waves of growing amplitudes.

Different values of γ, δ, L slightly change the parameters in Figure 1 and $m_1 C_g$ and $m_2 C_g$, but the basic stability structure (as in Figure 1) does not change.

The above stability structure has two distinctive features:

1. Although many steady state standing edge-wave modes of the form (3.3) are unstable to large-scale modulations (Interval I, in Figure 1), some of them are stable (Interval II, in Figure 1). These stable ones are an attracting set and strongly affect the dynamics of edge-wave evolutions.
2. There exists a small interval of steady state standing edge-wave modes which are unstable to very short modulational disturbances (Interval III, in Figure 1). This seems to suggest a mechanism of short-wave excitation by a long wave. If it really occurs in edge waves, it will invalidate the edge-wave modulational equations we derived before. But we must be cautious here. For very short waves, surface tension and dispersion become important. But these effects are neglected in our formulation. Obviously more studies are needed to clarify the issue.

4. Nonlinear evolution of edge waves on a wide beach

My results on the stability of steady state standing edge waves provide some insight into the nonlinear evolution of edge waves. On a wide bounded beach, the steady edge-wave modes of the form (3.3) are discrete. The number of such stable and unstable modes depends on C_g and J in equations (2.25) and (2.26). Since stable steady modes form an attracting set, if they exist and are excited, the edge wave is likely to be attracted to one of these modes and settle down there. If they do not exist, the edge wave cannot settle down to any steady mode, and its evolution will be quite different. In this case, one possibility is that the energy will mostly exchange among a few adjacent discrete modes, and the edge wave will evolve into a limit cycle. This corresponds to the antinode–number alternation phenomenon observed in Guza and Inman's experiments [3].

To further investigate the edge-wave evolution on a wide bounded beach, numerical calculations are carried out for the governing equations (2.25), (2.26), and (2.27). To facilitate the computation, A, B are decomposed in the following form:

$$A = \sum_{K=-\infty}^{\infty} a_K(T)\, e^{-iKZ}, \qquad (4.1)$$

$$B = \sum_{K=-\infty}^{\infty} a_K(T)\, e^{iKZ}. \qquad (4.2)$$

When this decomposition is substituted into the equations (2.25), (2.26), and (2.27), an infinite-dimensional dynamical system for $a_K(T)$ is obtained. Truncation of this system is necessary for any numerical calculations. The choice for the number of a_K's depends on the accuracy required. A fourth-order Runge-Kutta scheme is used for this dynamical system in all our computations. Eleven a_K modes are chosen in most cases, including the two cases shown later in this paper.

For easy comparison with the previous stability results, we consider a beach of small angle α, where

$$\gamma = -\frac{1}{128}, \qquad \delta = -0.72 \times 10^{-2} + 0.19 \times 10^{-2}i. \qquad (4.3)$$

We also take

$$L = 0, \qquad S_0 = \frac{1}{4e^2}. \qquad (4.4)$$

Depending on the actual width of the beach, C_g and J may take different values.
For convenience, we normalize $S_0 = 1$, and other parameters change to

$$\gamma = -\frac{1}{128} \times 4e^2 = -0.23,$$

$$\delta = (-0.72 \times 10^{-2} + 0.19 \times 10^{-2}i) \times 4e^2 = -0.21 + 0.056i. \quad (4.5)$$

L is still zero. J and C_g are multiplied by $4e^2$ and still denoted as J and C_g. A stretch of the T coordinate is also needed.

Initially, the edge waves are very small. They are excited by the incident wave. In our computation we take small "white noise" initial conditions with

$$a_K(0) = 0.01 + 0.01i, \qquad K = 0, \pm 1, \pm 2, \ldots \qquad (4.6)$$

Many runs with different values of J and C_g have been carried out. When C_g is not very small, i.e., the beach is not very wide, the two solution behaviors for the following two sets of values of J and C_g are found to be typical.

1. $C_g = 1, J = 0$
    In this case, (3.29) shows that steady modes (3.3) exist for $K = 0$ and 1. The steady mode with $K = 0$ is unstable, and the one with $K = 1$ is stable. At the initial stage of edge-wave generation, since the $K = 0$ mode is at exact subharmonic resonance, it quickly grows and reaches its steady state amplitude. But its steady state is unstable. It then gradually loses its energy to its side-band modes with $K = \pm 1$ and excites them. Notice that the steady mode with $K = 1$ is stable. When it is excited, it absorbs energy and attracts the edge wave to reach its own steady state. The edge wave finally settles down to this steady mode. The time-evolution of a_K's is plotted in Figure 2.

2. $C_g = 1, J = 0.5$
    In this case, (3.29) shows that steady modes (3.3) exist for $K = -1$ and 0. These two steady modes are both unstable. Therefore, the edge wave cannot settle down to any steady mode. Instead, it evolves into a limit cycle, with energy largely confined to a few adjacent modes and exchanging among them, as illustrated in Figure 3. Notice that this behavior corresponds to the antinode–number alternation phenomenon observed in Guza and Inman's experiments [3].

These two types of edge-wave evolution are very distinctive. They are both possible on a wide bounded beach. The actual beach geometry and the incident wave dictate which one should occur.

For some runs, rapidly oscillating disturbances appear, which is expected from the previous stability results since some important effects, like surface tension and dispersion, are neglected. When this happens, such effects need to be included and more studies are needed. Nevertheless, the qualitative behaviors are still the same as those without the appearance of rapidly oscillating disturbances.

On a wider beach, C_g is smaller, and thus more stable and unstable steady modes (3.3) exist. Expectedly, in this situation, the dynamical system of a_K's shows richer behaviors. For instance, the edge wave not only may go to a steady (stable mode) state or a limit cycle but also may evolve into a quasi-periodic or even chaotic state. These aspects remain to be studied.
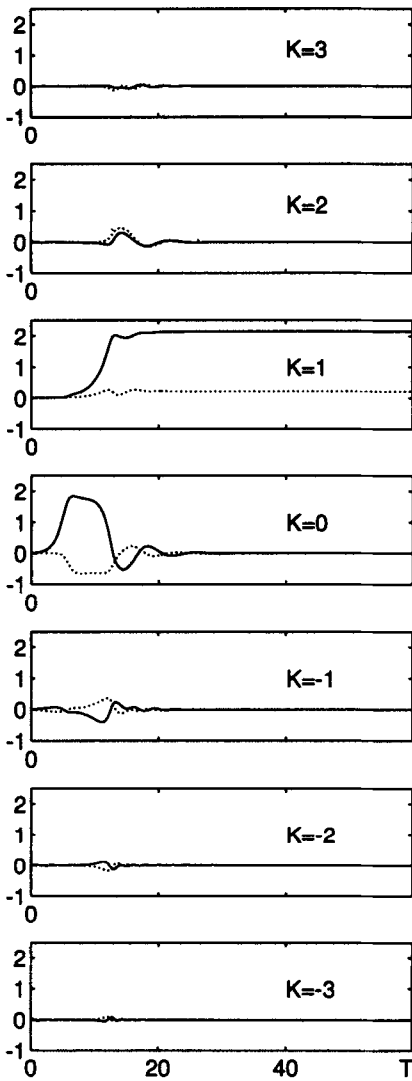
Figure 2. The time-evolution of a_K, K = 0, ±1, ±2, ±3. Parameters J = 0, C_g = 1. The solid line, Re(a_K); the dotted line, Im(a_K). This edge wave settles down to the stable steady mode (3.3) with K = 1.
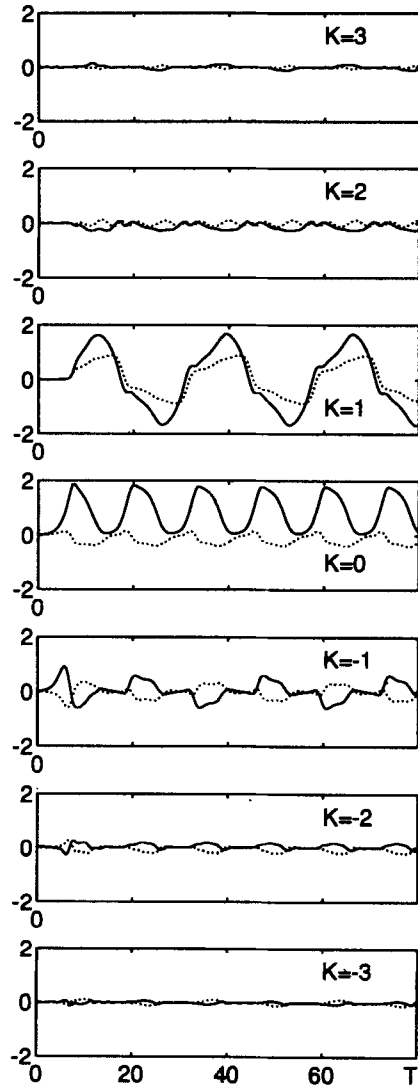
Figure 3. The time-evolution of a_K, K = 0, ±1, ±2, ±3. Parameters J = 0.5, C_g = 1. The solid line, Re(a_K); the dotted line, Im(a_K). This edge wave evolves into a limit cycle.

5. Summary

The stability of steady state standing edge waves to large-scale disturbances has been studied. Regions of stable and unstable edge-wave modes have been determined precisely and the stability structure obtained analytically. The nonlinear evolution of edge waves on a wide beach has also been considered. This evolution is strongly affected by the existance of stable edge-wave modes. An explanation has been found for the edge-wave antinode–number alternation phenomenon observed in Guza and Inman's experiments [3].

Acknowledgment

References

1. C. G. STOKES, In *Proc. 16th Br. Assoc. Adv. Sci.*, Southampton, pp. 1–20, John Murray, London, 1846.

2. F. URSELL, *Proc. Roy. Soc.* A 214:79–97 (1952).

3. R. T. GUZA and D. L. INMAN, *J. Geophys. Res.* 80:2997–3011 (1975).

4. A. J. BOWEN and D. L. INMAN, *J. Geophys. Res.* 74:5479–5490 (1969).

5. H. P. GREENSPAN, *J. Fluid Mech.* 1:574–592 (1956).

6. C. J. GALVIN, *Trans. Am. Geophys. Union* 46:112 (1965).

7. R. T. GUZA and R. E. DAVIS, *J. Geophys. Res.* 79:1285–1291 (1974).

8. G. B. WHITHAM, *J. Fluid Mech.* 74:353–368 (1976).

9. A. A. MINZONI and G. B. WHITHAM, *J. Fluid Mech.* 79:273–287 (1977).

10. T. R. AKYLAS, *J. Fluid Mech.* 132:197–208 (1983).

11. R. T. GUZA and A. J. BOWEN, *J. Mar. Res.* 34:269–293 (1976).