Conditions and Stability Analysis for Saddle-Node Bifurcations of Solitary Waves in Generalized Nonlinear Schrödinger Equations

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Saddle-node bifurcations of solitary waves in generalized nonlinear Schrödinger equations with arbitrary forms of nonlinearity and external potentials in arbitrary spatial dimensions are analyzed. First, general conditions for these bifurcations are derived. Second, it is shown analytically that the linear stability of these solitary waves does not switch at saddle-node bifurcations, which is in stark contrast with finite-dimensional dynamical systems where stability switching takes place. Third, it is shown that this absence of stability switching does not contradict the Vakhitov–Kolokolov stability criterion or the results in finite-dimensional dynamical systems. Fourth, it is shown that this absence of stability switching holds not only for real potentials but also for complex potentials. Lastly, various numerical examples will be given to confirm these analytical findings. In particular, saddle-node bifurcations with both branches of solitary waves being stable will be presented.

1 Introduction

Saddle-node bifurcation is the generic and most common bifurcation in finite-dimensional dynamical systems [1]. In this bifurcation, there are two fixed-point branches on one side of the bifurcation point and no fixed points on the other side, and the stability of these two fixed-point branches switches at the bifurcation point (one branch stable and the other branch unstable). In nonlinear partial differential equations (which can be viewed as infinite-dimensional dynamical systems),

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this bifurcation occurs as well (it is also called fold bifurcation in the literature). For instance, solitary waves in nonlinear physical systems often exhibit this type of bifurcation. Examples include the Boussinesq equations and the fifth-order Korteweg-de Vries equation in water waves [2–4], the Swift–Hohenberg equation in pattern formation [5], the nonlinear Schrödinger (NLS) equations with localized or periodic potentials in nonlinear optics and Bose–Einstein condensates [6–9], and many others. Motivated by stability switching of saddle-node bifurcations in finite-dimensional dynamical systems, it is widely believed that in nonlinear partial differential equations, stability of solitary waves also always switches at saddle-node bifurcations (see [5–8] for examples). Even though it was claimed on numerical evidence that both branches of saddle-node bifurcations were stable for various solitons in a Kronig–Penney model with cubic-quintic nonlinearity [10], that numerical evidence was not reliable since many solitons which the authors claimed to be stable are actually unstable. Thus that work could not shake this pervasive belief of stability switching.

In this paper, we show that this belief of universal stability switching at saddle-node bifurcations in nonlinear partial differential equations is incorrect. Specifically, we show that in generalized nonlinear Schrödinger equations with arbitrary forms of nonlinearity and external real or complex potentials, stability of solitary waves actually does not switch at saddle-node bifurcations. This fact is proved analytically in two ways by using the general conditions of saddle-node bifurcations, eigenvalue-bifurcation analysis and the method of contradiction. It is also verified numerically by several examples, where both branches of solitary waves are stable at saddle-node bifurcations. In addition, we show that this absence of stability switching does not contradict the Vakhitov–Kolokolov stability criterion even though the lower and upper branches of the saddle-node bifurcation have opposite signs of power slopes. Reconciliation of our results with those in finite-dimensional dynamical systems is also provided.

2 Conditions for Saddle-Node Bifurcations

We consider generalized nonlinear Schrödinger equations with arbitrary forms of nonlinearity and external potentials in arbitrary spatial dimensions,

$$iU_t + \nabla^2 U + F(|U|^2, \mathbf{x})U = 0, \tag{1}$$

where ∇^2 is the Laplacian in the N-dimensional space $\mathbf{x} = (x_1, x_2, \cdots, x_N)$, and $F(\cdot,\cdot)$ is a general function which contains nonlinearity as well as external potentials. These equations include the Gross–Pitaevskii equation in Bose–Einstein condensates and nonlinear light-transmission equations in linear potentials and nonlinear lattices as special cases [11–14]. Below, we will first focus on the case where the function F is real-valued, which applies when the system (1) is conservative. Extension to the non-conservative case of complex functions of F will be considered in Sect. 6 later.

When the function F is real, Eq. (1) admits stationary solitary waves of the form

$$U(\mathbf{x}, t) = e^{i\mu t} u(\mathbf{x}), \tag{2}$$

where $u(\mathbf{x})$ is a localized real function satisfying

$$\nabla^2 u - \mu u + F(u^2, \mathbf{x}) u = 0, \tag{3}$$

and μ is a real propagation constant which is a free parameter. Under certain conditions, these solitary waves undergo saddle-node bifurcations at special values of μ [6–9]. A signature of these bifurcations is that on one side of the bifurcation point μ_0, there are no solitary wave solutions; but on the other side of μ_0, there are two distinct solitary-wave branches which merge with each other at $\mu = \mu_0$. To derive conditions for these bifurcations, we introduce the linearization operator of Eq. (3),

$$L_1 = \nabla^2 - \mu + \partial_u [F(u^2, \mathbf{x}) u]. \tag{4}$$

We also introduce the standard inner product of functions

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}, \tag{5}$$

where the superscript '*' stands for complex conjugation. Our analysis starts with the basic observation that, if a bifurcation occurs at $\mu = \mu_0$, by denoting the corresponding solitary wave and the linearization operator as

$$u_0(\mathbf{x}) = u(\mathbf{x}; \mu_0), \quad L_{10} = L_1|_{\mu=\mu_0,\ u=u_0}, \tag{6}$$

then the linear operator L_{10} should have a discrete zero eigenvalue. This is a necessary condition for all types of bifurcations (not just for saddle-node bifurcations). To derive sufficient conditions for saddle-node bifurcations, let us make the following assumption.

Assumption 1 It is assumed that this zero eigenvalue of L_{10} is discrete and simple.

This assumption holds for all bifurcations in one spatial dimension since L_{10} in this case is a Sturm–Liouville operator whose discrete eigenvalues are all simple. This assumption holds for many bifurcations in higher spatial dimensions as well. Under this assumption, we denote the unique discrete (localized) eigenfunction of L_{10} at the zero eigenvalue as $\psi(\mathbf{x})$, i.e.,

$$L_{10}\psi = 0. \tag{7}$$

Since L_{10} is a real operator, we can normalize the eigenfunction ψ to be a real function and hence require ψ real below. We also denote

$$G(u; \mathbf{x}) = F(u^2; \mathbf{x}) u, \quad G_k(\mathbf{x}) = \partial_u^k G|_{u=u_0}, \tag{8}$$

where $k = 1, 2, 3, \cdots$. Then the sufficient condition for saddle-node bifurcations of solitary waves is given by the following theorem.

Theorem 1 Under Assumption 1 and the above notations, if $\langle u_0, \psi \rangle \neq 0$ and $\langle G_2, \psi^3 \rangle \neq 0$, then a saddle-node bifurcation of solitary waves occurs at $\mu = \mu_0$ in Eq. (1).

Proof Solitary waves which exist near $\mu = \mu_0$ admit the following perturbation series expansions

$$u(\mathbf{x}; \mu) = \sum_{k=0}^{\infty} (\mu - \mu_0)^{k/2} u_k(\mathbf{x}). \tag{9}$$

Inserting this expansion into Eq. (3), we get the following equations for u_k at order $(\mu - \mu_0)^{k/2}$, $k = 0, 1, 2,$:

$$\nabla^2 u_0 - \mu_0 u_0 + F(u_0^2, \mathbf{x}) u_0 = 0, \tag{10}$$

$$L_{10} u_1 = 0, \tag{11}$$

$$L_{10} u_2 = u_0 - G_2 u_1^2 / 2!, \tag{12}$$

and so on. Equation (10) for u_0 is satisfied automatically since u_0 is a solitary wave at $\mu = \mu_0$. The u_1 solution to Eq. (11) is found from (7) as

$$u_1 = b_1 \psi, \tag{13}$$

where b_1 is a constant. The u_2 function satisfies the linear inhomogeneous equation (12). Due to the Fredholm Alternative Theorem and the fact that L_{10} is self-adjoint, Eq. (12) admits a localized solution for u_2 if and only if the homogeneous solution ψ is orthogonal to the inhomogeneous term, i.e.,

$$\langle \psi, u_0 - G_2 u_1^2 / 2 \rangle = 0. \tag{14}$$

Inserting the solution (13) into this orthogonality condition and recalling the conditions in Theorem 1, we find that

$$b_1 = \pm \beta, \quad \beta \equiv \sqrt{\frac{2\langle u_0, \psi \rangle}{\langle G_2, \psi^3 \rangle}}. \tag{15}$$

Thus, we get two b_1 values $\pm\beta$ which are opposite of each other. Inserting the corresponding u_1 solutions (13) into (9), we then get two perturbation-series solutions of solitary waves $u(\mathbf{x}; \mu)$ as

$$u^{\pm}(\mathbf{x}; \mu) = u_0(\mathbf{x}) \pm \beta (\mu - \mu_0)^{1/2} \psi(\mathbf{x}) + O(\mu - \mu_0). \tag{16}$$

If $\langle u_0, \psi \rangle$ and $\langle G_2, \psi^3 \rangle$ have the same sign, then β is real. Recalling that $u_0(\mathbf{x})$ and $\psi(\mathbf{x})$ are both real as well, we see that these perturbation-series solutions (16) give two real-valued (legitimate) solitary waves when $\mu > \mu_0$, but these solitary waves do not exist when $\mu < \mu_0$. On the other hand, if $\langle u_0, \psi \rangle$ and $\langle G_2, \psi^3 \rangle$ have the opposite sign, β is purely imaginary. In this case, the perturbation series (16) give two real-valued solitary waves when $\mu < \mu_0$ but not when $\mu > \mu_0$.

The above perturbation calculations can be continued to higher orders. We can show that the two real solitary-wave solutions (16), which exist on only one side of $\mu = \mu_0$, can be constructed to all orders of $(\mu - \mu_0)^{1/2}$. In addition, these two solitary waves $u^{\pm}(\mathbf{x}; \mu)$ merge with each other when $\mu \to \mu_0$. We can also show that except these two solitary-wave branches, there are no other solitary-wave solutions near the bifurcation point. Thus a saddle-node bifurcation occurs at $\mu = \mu_0$. This completes the proof of Theorem 1. □

Using the perturbation expansion (16) of solitary waves, we can also calculate the power function $P(\mu)$ of these waves near the saddle-node bifurcation point μ_0. The power P of a solitary wave $u(\mathbf{x})$ is defined as

$$P = \int_{-\infty}^{\infty} u^2(\mathbf{x}) d\mathbf{x}. \tag{17}$$

Using (16) and the condition in Theorem 1, we readily find that

$$P^{\pm}(\mu) = P_0 \pm (\mu - \mu_0)^{1/2} P_1 + O(\mu - \mu_0), \tag{18}$$

where coefficients P_0 and P_1 are

$$P_0 = \langle u_0, u_0 \rangle, \quad P_1 = 2\beta \langle u_0, \psi \rangle \neq 0,$$

and β is given in Eq. (15). It is seen that the power function is a horizontally-oriented parabola, which is not surprising for a saddle-node bifurcation.

3 Stability Analysis for Saddle-Node Bifurcations

Stability properties of solitary waves near saddle-node bifurcations is an important issue. In finite-dimensional dynamical systems, the stability of fixed points is known to switch at saddle-node bifurcations, and this switching is caused by a linear-stability eigenvalue of the fixed points crossing zero along the real axis [1]. For solitary waves in nonlinear partial differential equations (which can be viewed as fixed points in infinite-dimensional dynamical systems), it is widely believed that their stability also always switches at saddle-node bifurcations. We find that this belief is incorrect. Below, we show that for solitary waves (2) in the generalized NLS equations (1), there are no linear-stability eigenvalues crossing zero at a saddle-node bifurcation point, thus stability-switching does not occur.

To study the linear stability of solitary waves (2) in Eq. (1), we perturb them as [13]

$$U(\mathbf{x}, t) = e^{i\mu t}\left\{u(\mathbf{x}) + [v(\mathbf{x}) + w(\mathbf{x})]e^{\lambda t} + [v^*(\mathbf{x}) - w^*(\mathbf{x})]e^{\lambda^* t}\right\}, \tag{19}$$

where $v, w \ll 1$ are normal-mode perturbations, and λ is the mode's eigenvalue. Inserting this perturbed solution into (1) and linearizing, we obtain the following linear-stability eigenvalue problem

$$\mathcal{L}\Phi = -i\lambda\Phi, \tag{20}$$

where

$$\mathcal{L} = \begin{bmatrix} 0 & L_0 \\ L_1 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} v \\ w \end{bmatrix}, \tag{21}$$

$$L_0 = \nabla^2 - \mu + F(u^2, \mathbf{x}), \tag{22}$$

and L_1 has been given in Eq. (4).

At a saddle-node bifurcation point $\mu = \mu_0$, we denote

$$L_{00} = L_0|_{\mu=\mu_0,\ u=u_0}, \quad \mathcal{L}_0 = \mathcal{L}_{\mu=\mu_0,\ u=u_0}. \tag{23}$$

Then in view of Eq. (3), we have

$$L_{00}u_0 = 0, \tag{24}$$

thus zero is a discrete eigenvalue of L_{00}. From this equation as well as (7), we have

$$\mathcal{L}_0\begin{bmatrix} 0 \\ u_0 \end{bmatrix} = \mathcal{L}_0\begin{bmatrix} \psi \\ 0 \end{bmatrix} = 0, \tag{25}$$

thus zero is also a discrete eigenvalue of $\mathcal{L}_0$.

On the bifurcation of the zero eigenvalue in $\mathcal{L}_0$ when μ moves away from μ_0, we have the following main result.

Theorem 2 Assuming that zero is a simple discrete eigenvalue of L_{00} and L_{10}, then at a saddle-node bifurcation point μ_0, no eigenvalues of the linear-stability operator $\mathcal{L}$ cross zero, thus no stability switching occurs.

Proof The idea of the proof is to show that, when μ moves away from μ_0, the algebraic multiplicity of the zero eigenvalue in $\mathcal{L}$ does not decrease, thus the zero eigenvalue in $\mathcal{L}$ cannot bifurcate out to nonzero.

At the saddle-node bifurcation point $\mu = \mu_0$, $(0, u_0)^T$ and $(\psi, 0)^T$ are two linearly independent eigenfunctions of the zero eigenvalue in $\mathcal{L}_0$ in view of Eq. (25). Here the superscript 'T' represents the transpose of a vector. Under the assumption of Theorem 2, zero is a simple discrete eigenvalue of L_{00} and L_{10}. Thus it is easy to see that $\mathcal{L}_0$ does not admit any additional eigenfunctions at the zero eigenvalue, which means that the geometric multiplicity of the zero eigenvalue in $\mathcal{L}_0$ is two. To determine the algebraic multiplicity of the zero eigenvalue in $\mathcal{L}_0$, we need to examine the number of generalized eigenfunctions of this zero eigenvalue. The lowest-order generalized eigenfunction $(f_1, g_1)^T$ to the eigenfunction $(0, u_0)^T$ of this zero eigenvalue satisfies the equation

$$\mathcal{L}_0 \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} 0 \\ u_0 \end{bmatrix}, \tag{26}$$

so the equation for f_1 is

$$L_{10} f_1 = u_0. \tag{27}$$

From Eq. (7), we see that this inhomogeneous equation has a homogeneous localized solution ψ. In addition, from conditions of saddle-node bifurcations in Theorem 1, $\langle u_0, \psi \rangle \neq 0$. Furthermore, L_{10} is a self-adjoint operator. Thus, from the Fredholm Alternative Theorem, the inhomogeneous equation (27) does not admit any localized solution, which means that the eigenfunction $(0, u_0)^T$ of the zero eigenvalue in $\mathcal{L}_0$ does not have any generalized eigenfunctions. Similarly, we can show that the eigenfunction $(\psi, 0)^T$ of the zero eigenvalue in $\mathcal{L}_0$ does not have any generalized eigenfunctions either. Hence the algebraic multiplicity of the zero eigenvalue in $\mathcal{L}_0$ is equal to its geometric multiplicity and is two.

Away from the bifurcation point (i.e., $\mu \neq \mu_0$,) $\mathcal{L}$ always has a zero eigenmode

$$\mathcal{L} \begin{bmatrix} 0 \\ u \end{bmatrix} = 0 \tag{28}$$

in view of Eq. (1). In addition, by differentiating Eq. (1) with respect to μ, we also get

$$\mathcal{L} \begin{bmatrix} u_\mu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}, \tag{29}$$

thus $(u_\mu, 0)^T$ is a generalized eigenfunction of the zero eigenvalue in $\mathcal{L}$. This means that the algebraic multiplicity of the zero eigenvalue in $\mathcal{L}$ is at least two when $\mu \neq \mu_0$.

If nonzero eigenvalues bifurcate out from the zero eigenvalue in $\mathcal{L}$, the algebraic multiplicity of this zero eigenvalue must decrease. Our results above show that, when μ moves away from μ_0, the algebraic multiplicity of the zero eigenvalue in $\mathcal{L}$ does not decrease, thus there cannot be nonzero eigenvalues of $\mathcal{L}$ bifurcating out from zero. Consequently, no eigenvalues of $\mathcal{L}$ cross zero at the saddle-node bifurcation point, thus no stability switching occurs. This completes the proof of Theorem 2. □

We would like to add that under the assumption of Theorem 2, we can readily show that in the neighborhood of a saddle-node bifurcation point ($0 < |\mu - \mu_0| \ll 1$), $\mathcal{L}$ does not admit any additional eigenfunctions or generalized eigenfunctions at the zero eigenvalue, thus the algebraic multiplicity of the zero eigenvalue in $\mathcal{L}$ is exactly two. This means that the algebraic multiplicity of $\mathcal{L}$'s zero eigenvalue does not change when μ moves away from the bifurcation point μ_0, thus there is no eigenvalue bifurcation out of the origin at a saddle-node bifurcation.

Alternative Proof Theorem 2 can also be proved by the following alternative method of contradiction. For the eigenvalue problem (20), i.e.,

$$L_0 w = -i\lambda v, \qquad L_1 v = -i\lambda w, \tag{30}$$

by taking the inner product of the first equation with the solitary wave $u(\mathbf{x})$ and noticing that L_0 is a self-adjoint operator, we get

$$-i\lambda\langle u, v\rangle = \langle u, L_0 w\rangle = \langle L_0 u, w\rangle = 0.$$

Thus, for any non-zero eigenvalue λ, its eigenfunction v must be orthogonal to u, i.e.,

$$\langle u, v\rangle = 0. \tag{31}$$

Similarly, by taking the inner product of the second equation in (30) with u_μ and recalling $L_1 u_\mu = u$ from (29), we get

$$-i\lambda\langle u_\mu, w\rangle = \langle u_\mu, L_1 v\rangle = \langle L_1 u_\mu, v\rangle = \langle u, v\rangle.$$

Thus for any non-zero eigenvalue λ, in view of the orthogonality (31), we see that u_μ and w must be orthogonal as well, i.e.,

$$\langle u_\mu, w\rangle = 0. \tag{32}$$

Now suppose at a saddle-node bifurcation point μ_0, non-zero eigenvalues bifurcate out from the origin. Then when μ is very close to μ_0, these non-zero eigenvalues are very small. Thus, from the eigenvalue Eq. (30) and our assumption in Theorem 2, we see that

$$(v, w) \to (c_1\psi, c_2 u_0), \quad \text{when } \mu \to \mu_0, \tag{33}$$

where c_1 and c_2 are certain constants which cannot be zero simultaneously. In addition, we see from (16) that when $\mu \to \mu_0$, $u_\mu \propto \psi$. Upon substituting these expressions into the orthogonality conditions (31) and (32) and taking the limit of $\mu \to \mu_0$, we find that

$$c_1\langle u_0, \psi\rangle = 0, \quad c_2\langle \psi, u_0\rangle = 0. \tag{34}$$

Since at a saddle-node bifurcation, $\langle u_0, \psi\rangle \neq 0$ (see Theorem 1), the above relations then give $c_1 = c_2 = 0$, which contradicts our earlier requirement that c_1 and c_2 not being zero simultaneously. Thus there cannot be eigenvalues bifurcating out from the origin at a saddle-node bifurcation. This also proves Theorem 2. □

4 Consistency with the Generalized VK Criterion for Positive Solitary Waves

The above result of no stability switching at a saddle-node bifurcation in the generalized NLS equations (1) applies to general real-valued solitary waves, certainly including positive (or equivalently sign-definite) solitary waves.

For positive solitary waves in NLS-type equations, the stability result best known in the physical community is the Vakhitov–Kolokolov (VK) stability criterion, which says that the solitary wave $u(\mathbf{x};\mu)$ is linearly stable if its power slope $P'(\mu)$ is positive and linearly unstable if $P'(\mu)$ is negative [12, 13, 15]. Near a saddle-node bifurcation, we know from Eq. (18) that the power function is a horizontally-oriented parabola. Thus the lower and upper branches always have opposite signs of power slope. Then from this VK stability criterion, one might conclude that the lower and upper branches should have opposite stability as well, which would contradict our result in Theorem 2. The error in this reasoning is that the above (original) VK criterion, which derives linear stability exclusively from the power slope, applies only when Eq. (1) has no external potentials (i.e., $F = F(|U|^2)$ only) [13]. This no-potential condition for the use of the original VK criterion is not well recognized in the physical community. Saddle-node bifurcations, however, can only occur when F depends also on $\mathbf{x}$ explicitly, i.e., Eq. (1) has external potentials such as linear or nonlinear potentials (the latter means that the coefficients of nonlinear terms are spatially modulated [14]). In this case, the original VK criterion must be generalized. The generalized VK criterion derives linear stability not only from the power slope but also from the number of positive eigenvalues in the operator L_1 [13]. Thus even though the lower and upper branches of a saddle-node bifurcation have opposite signs of power slope, if the number of positive eigenvalues in L_1 also changes between the two branches, then both branches can still have the same linear stability. Below we will show that the number of positive eigenvalues in L_1 *does* change at a saddle-node bifurcation, thus there is no contradiction between our analytical result in Theorem 2 and the generalized VK criterion. In Example 1 of the next section, we will further see explicitly that Theorem 2 and the generalized VK criterion give exactly the same stability results.

Our result on positive eigenvalues in L_1 near a saddle-node bifurcation is given in the following theorem.

Theorem 3 Assume the conditions of Theorem 1 (for a saddle-node bifurcation) hold. Then across this saddle-node bifurcation point (from the lower branch to the upper one or vise versa), a simple real eigenvalue of L_1 crosses zero, thus the number of positive eigenvalues in L_1 changes by one.

Proof The eigenvalue problem for the linear Schrödinger operator L_1 is

$$L_1\Psi = \Lambda\Psi, \tag{35}$$

where Λ is a real eigenvalue and Ψ is the associated eigenfunction. At the saddle-node bifurcation point $\mu = \mu_0$, $L_{10}\psi = 0$ (see Eq. 7), thus $\Lambda = 0$ is a discrete eigenvalue of L_1. In addition, due Assumption 1, $\Lambda = 0$ is also a simple eigenvalue of L_1. Next we calculate how this simple zero eigenvalue of L_1 bifurcates out when μ is away from the bifurcation point μ_0.

Before we start, we first expand the operator L_1 as

$$L_1 = L_{10} + (\mu-\mu_0)^{1/2}L_{11} + (\mu-\mu_0)L_{12} + \cdots. \tag{36}$$

The solitary wave $u(\mathbf{x})$ is also expanded by Eq. (9). Upon substituting these two expansions into the relation $L_1 u_\mu = u$ out of (29), at order $(\mu - \mu_0)^{-1/2}$, we get

$$L_{10} u_1 = 0. \tag{37}$$

This is consistent with our earlier formula (13) for u_1. At $O(1)$, we get

$$L_{11} u_1 = 2(u_0 - L_{10} u_2). \tag{38}$$

This relation will be needed later.

Now we expand Λ and Ψ into the following perturbation series,

$$\Lambda = (\mu - \mu_0)^{1/2} \gamma_1 + (\mu - \mu_0)\gamma_2 + \cdots, \tag{39}$$

$$\Psi = \Psi_0 + (\mu - \mu_0)^{1/2} \Psi_1 + (\mu - \mu_0)\Psi_2 + \cdots. \tag{40}$$

Substituting these expansions as well as the expansion of L_1 above into the eigenvalue equation (35), at $O(1)$ we get

$$L_{10} \Psi_0 = 0. \tag{41}$$

For convenience, instead of taking the Ψ_0 solution as ψ, we take it as

$$\Psi_0 = u_1. \tag{42}$$

Note that u_1 is related to ψ by a non-zero constant [see Eqs. (13) and (15)].

At order $(\mu - \mu_0)^{1/2}$, we get the equation for Ψ_1 as

$$L_{10} \Psi = \gamma_1 u_1 - L_{11} u_1. \tag{43}$$

Inserting (38) into this equation and imposing the Fredholm solvability condition (which says that its right hand side must be orthogonal to the homogeneous solution u_1), we find that the eigenvalue coefficient γ_1 is obtained as

$$\gamma_1 = \frac{2\langle u_1, u_0 \rangle}{\langle u_1, u_1 \rangle}. \tag{44}$$

For the two branches of the saddle-node bifurcation, the solutions u_1 are given by Eqs. (13) and (15), i.e.,

$$u_1 = \pm \beta \psi. \tag{45}$$

Inserting this u_1 into the γ_1 formula (44) and then back into the eigenvalue expansion (39), we find that the leading-order term of the eigenvalues on the two solution branches are

$$\Lambda^{\pm} = \pm (\mu - \mu_0)^{1/2} \alpha, \tag{46}$$

where

$$\alpha \equiv \frac{2\langle u_0, \psi \rangle}{\beta \langle \psi, \psi \rangle} \neq 0. \tag{47}$$

This eigenvalue formula shows that as the solution moves from the lower to the upper branches (or vise versa), this eigenvalue crosses zero and changes sign. Thus the number of positive eigenvalues in L_1 changes by one. This completes the proof of Theorem 3. □

5 Numerical Examples

In this section, we use two examples to confirm the above analytical findings.

Example 1 Consider Eq. (1) with a symmetric double-well potential and cubic-quintic nonlinearity, i.e.,

$$iU_t + U_{xx} - V(x)U + |U|^2U - |U|^4U = 0, \tag{48}$$

where the double-well potential

$$V(x) = -3\left[\operatorname{sech}^2(x + 1.5) + \operatorname{sech}^2(x - 1.5)\right] \tag{49}$$

is shown in Fig. 1a, and the quintic nonlinearity has the opposite sign of the cubic nonlinearity. A similar model and its various solitary waves were considered in [16]. Solitary waves in this conservative system (48) are of the form (2), where $u(x)$ is real. We have computed these solitary waves by the Newton-conjugate-gradient method [13], and their power curve is plotted in Fig. 1b. It is seen that a saddle-node bifurcation occurs at $\mu_0 \approx 2.16$. Two solitary waves at $\mu = 2.1$ on the lower and upper branches near this bifurcation point are displayed in Fig. 1c, d. To determine the linear stability of these solitary waves, we have computed their linear-stability spectra by the Fourier collocation method [13], and these spectra are shown in Fig. 1e, f respectively. It is seen that none of the spectra contains unstable eigenvalues, indicating that these solitary waves on both lower and upper branches are linearly stable. We have also performed this spectrum computation for other solitary waves on the power curve of Fig. 1b, and found that they are all linearly stable. Thus there is no stability switching at the saddle-node bifurcation point, in agreement with our analytical result. Additionally, we have found numerically that the zero eigenvalue for all these solitons has algebraic multiplicity two, again in agreement with our analytical result.

In the spectra of Fig. 1e, f, one may notice that there is a pair of purely imaginary eigenvalues $\pm\lambda$ near the origin and may wonder whether that pair of eigenvalues ever collide at the origin at the saddle-node bifurcation point. Numerically we have tracked this pair of eigenvalues as the solitary wave crosses the bifurcation point, and the eigenvalue λ (with a negative imaginary part) versus μ is shown in Fig. 1g.

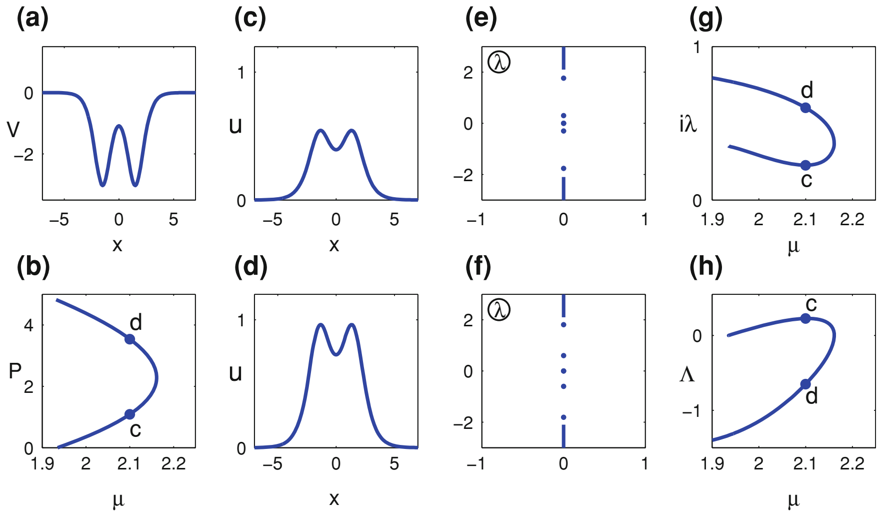

Fig. 1 Saddle-node bifurcation and linear-stability behaviors of solitary waves in Example 1. **a** Potential (49); **b** power curve of solitons; **c**, **d** soliton profiles at points marked by the same letters in (**b**); **e**, **f** stability spectra of solitons in(**c**, **d**); **g** the imaginary discrete eigenvalue λ (near the origin) versus μ ; **h** the largest real eigenvalue Λ in L_1 versus μ . The letters 'c, d' in (**g**, **h**) correspond to those on the power curve in (**b**) to show correspondence of solution branches

Here the plotted quantity is $i\lambda$, which is real positive. We find that as the solitary wave crosses the saddle-node bifurcation point, this pair of imaginary eigenvalues never collide at or cross the origin. This confirms our analytical result in Theorem 2 that no eigenvalues cross zero at the saddle-node bifurcation point.

The solitary waves in Fig. 1 are positive, thus the generalized VK stability criterion applies. Now we show that this generalized VK criterion predicts linear stability for both branches of these solitary waves as well. For this purpose, we have numerically computed the largest eigenvalue Λ in operator L_1 for each solitary wave, and this Λ versus μ is plotted in Fig. 1h. We see that this Λ crosses zero at the saddle-node bifurcation point, in agreement with Theorem 3. In addition, on the lower branch of the power curve (where the power slope is positive), $\Lambda > 0$, thus L_1 has a single positive eigenvalue; whereas on the upper branch of the power curve (where the power slope is negative), $\Lambda < 0$, thus L_1 has no positive eigenvalues. Then according to the generalized VK stability criterion [13], solitary waves on both branches of the power curve are linearly stable, in agreement with Theorem 2 and the numerical spectra in Fig. 1e, f.

Lastly, we have also checked numerically that zero is a simple discrete eigenvalue for both L_{10} and L_{00} (the zero eigenvalue being simple in L_{10} is already clear from Fig. 1h). In addition, $\langle u_0, \psi \rangle \neq 0$ and $\langle G_2, \psi^3 \rangle \neq 0$. Thus assumptions of Theorems 1–3 are all met, hence these theorems apply to this example.

In saddle-node bifurcations in Eq. (1), solitary waves sometimes possess a non-zero minimum power, and the power curve assumes a slanted U-shape [7, 13]. In some reports of such bifurcations, the two solution branches were marked with opposite stability, giving readers the impression that stability switching occurs at the saddle-node bifurcation point (see [7], Fig. 3). Those markings contradict our Theorem 2 and are incorrect. What happens is that the stability switching actually occurs at the power minimum point (see [13], Sect. 5.4). Since this power minima is often very close to the saddle-node bifurcation point, one might get the wrong impression of stability switching at the saddle-node bifurcation instead. In the next example, we will clarify this issue explicitly.

Example 2 In this example, we consider the NLS equation with an asymmetric double-well potential,

$$iU_t + U_{xx} - V(x)U + |U|^2U = 0, \tag{50}$$

where the asymmetric potential $V(x)$ is taken as

$$V(x) = -3.5\text{sech}^2(x + 1.5) - 3\text{sech}^2(x - 1.5) \tag{51}$$

and is shown in Fig. 2a. Solitary waves in this conservative system are of the form (2), where $u(x)$ is a real localized function. A family of solitary waves with more of their energy located at the right (shallower) potential well exists (see Fig. 2c), and their power curve is shown in Fig. 2b. It is seen that this power curve exhibits a slanted U-shape with a non-zero minimal power, and a saddle-node bifurcation occurs. We have determined the linear stability of these solitary waves and the results are indicated on the power curve of Fig. 2b (with solid blue for stable waves and dashed red for unstable ones). From first sight, one may see that the lower branch is stable and the upper one unstable, thus stability switching seems to occur at the saddle-node bifurcation point (as Fig. 3 of [7] conveys to the reader). However, when we amplify the bifurcation region of the power curve (see Fig. 2d), we find that stability switching actually occurs at the minimum-power point (as explained in [13]) rather than the saddle-node bifurcation point. Indeed, from the spectra for solitary waves on the lower and upper branches very close to the saddle-node bifurcation point (see Fig. 2e, f), we see that both waves are linearly unstable, thus there is no stability switching at the saddle-node bifurcation, in agreement with Theorem 2.

6 Extension to Complex Potentials

In this section, we extend the above results to complex potentials, i.e., the function $F(\cdot, \cdot)$ in (1) is complex-valued. In this case, if F admits parity-time (PT) symmetry

$$F^*(|U|^2, \mathbf{x}) = F(|U|^2, -\mathbf{x}),$$

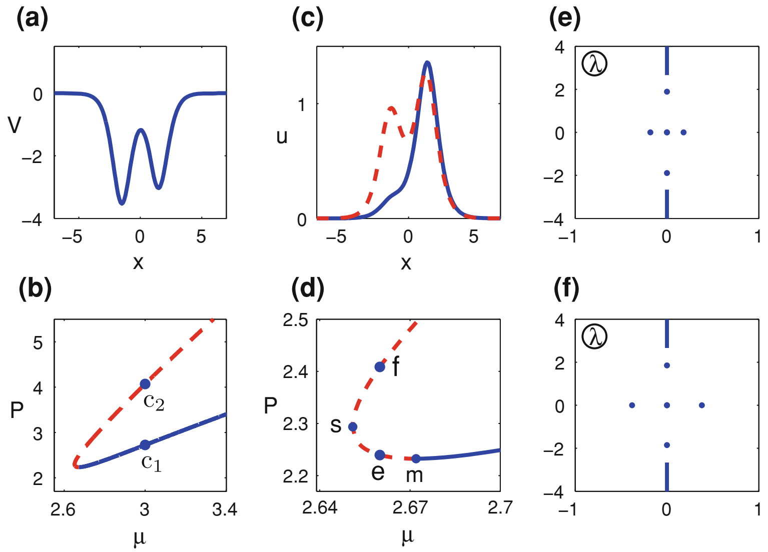

Fig. 2 Saddle-node bifurcation and linear-stability behaviors of solitary waves in Example 2. **a** Potential (51); **b** power curve of solitons (*solid blue* and *dashed red* represent stable and unstable solitons respectively); **c** profiles of solitons at points 'c_1, c_2 ' of (**b**), with *solid blue* for point c_1 and *dashed red* for point c_2; **d** amplification of the power curve in (**b**) near the bifurcation point; point 'm' is the power minimum, and point 's' is the bifurcation point; **e**, **f** stability spectra of solitons at points marked by letters 'e, f' in (**d**)

then solitary waves (2) can still exist over a continuous range of real μ values [17, 18], and saddle-node bifurcations can also occur. By slightly modifying the previous analysis, we can show that there is no stability switching at saddle-node bifurcations in these nonconservative systems either. Details of these slight modifications will not be presented here. For our purpose, it would suffice to present a numerical example.

Example 3 We still consider Eq. (48) in Example 1 but now with a complex PT-symmetric localized potential

$$V(x) = -3\left[\operatorname{sech}^2(x+1.5) + \operatorname{sech}^2(x-1.5)\right] + 0.25i\left[\operatorname{sech}^2(x+1.5) - \operatorname{sech}^2(x-1.5)\right], \tag{52}$$

see Fig. 3a. This nonconservative system still admits solitary waves (2) for continuous real ranges of μ, but $u(x)$ is complex-valued now. We have numerically obtained a family of these solitons by the Newton-conjugate-gradient method applied to a normal equation [13], and the power curve of these solitons is plotted in Fig. 3b. Again a saddle-node bifurcation can be seen at $\mu_0 \approx 2.02$. For solitary waves on the lower and upper branches near this bifurcation point (see Fig. 3c, d), their stability spectra lie entirely on the imaginary axis (see Fig. 3e, f), indicating that they are all linearly stable. Hence no stability switching occurs at saddle-node

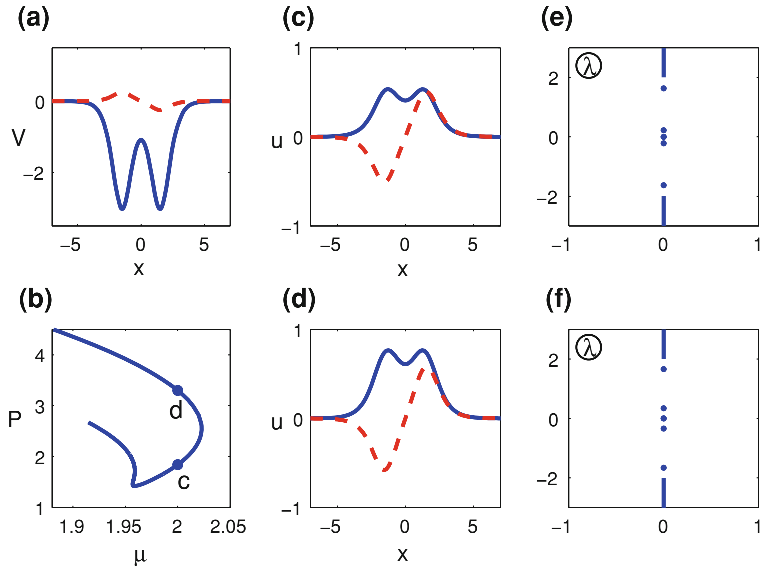

Fig. 3 Saddle-node bifurcation and linear stability of solitons in PT-symmetric potentials of Example 3. **a** PT potential (52); *solid blue* is the real part of the potential and *dashed red* is the imaginary part; **b** power curve of solitons; **c**, **d** soliton profiles at points 'c, d' in (**b**); *solid blue* is the real part of u and *dashed red* is the imaginary part; **e**, **f** stability spectra of solitons in (**c**, **d**)

bifurcations in this nonconservative system either. For this system, the zero eigenvalue of all solitons on the power curve also has algebraic multiplicity two as in the previous two examples. This explains why no zero-eigenvalue bifurcation occurs in this case either.

7 Summary and Discussion

In summary, we have derived analytical conditions for saddle-node bifurcations of solitary waves in generalized NLS equations (1) with arbitrary nonlinearities and potentials. More importantly, we have shown, both analytically and numerically, that for real as well as complex potentials, stability of solitary waves does not switch at saddle-node bifurcations. This disproves a wide-spread belief that such stability switching should always occur in nonlinear partial differential equations. We have also shown that for positive solitary waves in Eq. (1), this absence of stability switching at a saddle-node bifurcation is consistent with the generalized Vakhitov–Kolokolov stability criterion. Since the generalized NLS equations (1) arise frequently in nonlinear optics, Bose–Einstein condensates and other physical disciplines, our finding could have broad impact.

How can one reconcile our result of no-stability-switching in the generalized NLS equations (1) with the widely accepted result of stability switching in finite-dimensional dynamical systems? One might argue that Eq. (1) is an infinite-dimensional dynamical system, thus results from finite-dimensional dynamical systems do not apply. While this explanation sounds reasonable, it is not the key reason in our opinion. We believe the key reason is that, when the result of stability switching is derived in finite-dimensional dynamical systems, it is always assumed that zero is a simple eigenvalue of the Jacobian (linearization) matrix of the system at a saddle-node bifurcation point (see Ref. [1], Theorem 3.4.1, Hypothesis SN1). For the generalized NLS equations (1), the counterpart of this Jacobian matrix is the linearization operator $\mathcal{L}_0$ defined in Eq. (23), but zero is *not* a simple eigenvalue of $\mathcal{L}_0$ in view of Eq. (25). Thus the assumption for stability switching is not met in our system, hence our result of no stability switching does not contradict that in finite-dimensional dynamical systems.

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