Internal Oscillations and Radiation Damping of Vector Solitons

By Dmitry E. Pelinovsky and Jianke Yang

Internal modes of vector solitons and their radiation-induced damping are studied analytically and numerically in the framework of coupled nonlinear Schrödinger equations. Bifurcations of internal modes from the integrable systems are analyzed, and the region of their existence in the parameter space of vector solitons is determined. In addition, radiation-induced decay of internal oscillations is investigated. Both exponential and algebraic decay rates are identified.

1. Introduction

The coupled nonlinear Schrödinger (NLS) equations arise in many physical systems when two wave packets of different frequencies and wavenumbers strongly interact with each other. For example, the interaction of two wave packets on water surface is governed by these equations when certain geometric conditions are satisfied [1, 2]. More significantly, these equations describe evolution of pulse envelopes along the two orthogonal polarizations in birefringent nonlinear optical fibers [3], which are used in fiber communication systems [4–7] and all-optical switching devices [8]. The rapid advancement of all-optical communication networks in the past 10 years has generated

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great interest and progress in mathematical studies of pulse dynamics in the coupled NLS equations [9–18]. In particular, stable solitary wave solutions called *vector solitons* were found in this system [9, 10]. When they are perturbed, they will undergo complicated, long-lasting internal oscillations [15–18]. These oscillations are caused by the excitation of internal (shape) modes of vector solitons. The amplitude of internal oscillations gradually decays because of dispersive radiation that is induced by nonlinear interaction between these modes and continuous spectrum [17].

In a broader context, the existence of internal modes is fairly common in nonintegrable wave systems. For example, the properties of an internal mode of kinks have been known in the ϕ^4 model for a long time [19]. Recently, an analytical technique was developed to detect a birth of internal modes of solitary waves from the continuous spectrum of perturbed integrable systems [20]. This technique was applied to the double sine-Gordon and discrete NLS equations. The comprehensive studies of internal modes of envelope solitons in the cubic–quintic NLS equation [21] showed that a single internal mode exists for every solitary wave when the quintic coefficient is positive and does not exist otherwise. The existence of internal modes for solitary waves in $\chi^{(2)}$ medium has also been demonstrated in [22]. In many of these cases, the internal modes originate from the continuous spectrum of the associated linear operator.

The importance of internal modes is threefold. First, they induce internal oscillations to the underlying solitary wave. These oscillations may decay slower than typical dispersive radiation [21, 23]. Thus, they can dominate wave evolution for a long time. Second, internal modes are an integral part of the stability analysis of solitary waves in perturbed nonintegrable systems [24]. When a nonintegrable system is perturbed, the internal modes may bifurcate from the real axis and become unstable. In this case, the solitary wave may form a stable periodically oscillating state [24]. Chaotic dynamics and destruction of solitary waves are also possible under the instability bifurcations of internal modes [25, 26]. Last, when solitary waves collide with each other, the internal modes can temporarily store kinetic energy of colliding waves. This mechanism can lead to resonance structures in solitary wave collisions [14, 19].

When internal modes are superimposed on the underlying solitary waves, long-lasting internal (shape) oscillations would arise. These oscillations gradually decay because of nonlinear interactions of the internal mode with the wave's continuous spectrum. If the eigenvalue of the internal mode falls into the gap of the wave's continuous spectrum, this decay would be very slow (algebraically) in time [21, 27]. Moreover, when the internal oscillations of solitary waves are adiabatic; i.e., the soliton's shape remains self-similar, but the soliton parameters are varying in time, even slower (logarithmic) decay of oscillations can be found [23]. On the other hand, if the internal mode is embedded into the continuous spectrum, a generic perturbation to a wave system can destroy the existence of this mode, and the damping of the mode occurs faster (exponentially) in time. Within this context, the decay rate of the oscillatory state is known in theory of quantum resonances as the Fermat golden rule [28].

In this paper, we comprehensively investigate the internal modes and their radiation-induced damping in the coupled NLS equations. This problem has been examined previously in [17]. However, many important questions are still left open. For example, the region for existence of internal modes was not detected for the parameter space of vector solitons. The bifurcations of internal modes were studied only for the special case of vector solitons with equal amplitudes rather than for a general vector soliton. It was predicted that the radiation damping destroys the internal oscillations of vector solitons, but the decay rate was not found. In this paper, we address these questions and present a complete analysis of the problem of internal oscillations of vector solitons in the coupled NLS equations.

2. Vector solitons and their internal modes

The coupled NLS equations can be written in the form,

$$iA_t + A_{xx} + (|A|^2 + \beta|B|^2)A = 0, \tag{1}$$

$$iB_t + B_{xx} + (|B|^2 + \beta|A|^2)B = 0, \tag{2}$$

when normalizations of the variables have been made. The variables A and B represent the complex amplitudes of two wave packets, and β is a real parameter. We assume throughout our paper that $\beta > 0$. Vector solitons in equations (1) and (2) are the special solitary wave solutions of the form,

$$A = e^{i\omega_1^2 t}\, r_1(x), \tag{3}$$

$$B = e^{i\omega_2^2 t}\, r_2(x), \tag{4}$$

where r_1 and r_2 are real functions satisfying the following equations,

$$r_{1xx} - \omega_1^2 r_1 + (r_1^2 + \beta r_2^2) r_1 = 0, \tag{5}$$

$$r_{2xx} - \omega_2^2 r_2 + (r_2^2 + \beta r_1^2) r_2 = 0. \tag{6}$$

Without loss of generality, we rescale $\omega_1 = 1$, and denote $\omega_2 = \omega\,(> 0)$. There is an infinite set of solutions of equations (5) and (6) that represent multihumped vector solitons [29, 30]. However, we consider only the single-humped vector solitons because of evidences that only they are stable [13].

The single-humped vector solitons exist when ω is in the interval (ω_I, ω_{II}), where

$$\omega_I(\beta) = \frac{\sqrt{1+8\beta}-1}{2}, \quad \omega_{II}(\beta) = \frac{2}{\sqrt{1+8\beta}-1}. \tag{7}$$

Graphs of these boundary curves are shown in Figure 2(a). In the particular case $\omega = 1$, both the components r_1 and r_2 of a vector soliton have equal amplitudes. This special solution is

$$r_1 = r_2 = \sqrt{\frac{2}{1+\beta}}\,\mathrm{sech}x. \tag{8}$$

When $\omega \to \omega_I$ or $\omega \to \omega_{II}$, the vector soliton reduces to the degenerate vector soliton,

$$r_1 = \sqrt{2}\,\mathrm{sech}x, \qquad r_2 = 0, \tag{9}$$

or

$$r_1 = 0, \qquad r_2 = \sqrt{2}\omega\,\mathrm{sech}\omega x, \tag{10}$$

respectively. Degenerate vector solitons also exist for any values of ω and β as reductions of the coupled system (1) and (2) into a single NLS equation.

When vector solitons (3) and (4) are perturbed, we write

$$A = e^{it}\{r_1(x) + \widetilde{A}(x,t)\}, \tag{11}$$

$$B = e^{i\omega^2 t}\{r_2(x) + \widetilde{B}(x,t)\}, \tag{12}$$

where $\widetilde{A}$ and $\widetilde{B}$ are small disturbances. The linearized equations for the disturbances can be readily obtained by substituting (11) and (12) into (1) and (2) and neglecting terms of second and higher orders. We write these linearized equations in the matrix form,

$$i\mathscr{R}_t + \mathscr{L}\mathscr{R} = 0, \tag{13}$$

where $\mathscr{R} \equiv (\widetilde{A}, \widetilde{A}^*, \widetilde{B}, \widetilde{B}^*)^{\mathrm{T}}$ and the operator $\mathscr{L}$ is

$$\mathscr{L} = \begin{pmatrix} (\partial_{xx} - 1 + 2r_1^2 + \beta r_2^2)\sigma_3 + ir_1^2\sigma_2 & \beta r_1 r_2(\sigma_3 + i\sigma_2) \\ \beta r_1 r_2(\sigma_3 + i\sigma_2) & (\partial_{xx} - \omega^2 + 2r_2^2 + \beta r_1^2)\sigma_3 + ir_2^2\sigma_2 \end{pmatrix}. \tag{14}$$

Here

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

are Pauli spin matrices. The eigenstates of $\mathscr{R}$ are of the form

$$\mathscr{R} = e^{i\lambda t}\Psi(x), \tag{16}$$

where $\Psi = [\psi_1, \psi_2, \psi_3, \psi_4]^{\mathrm{T}}$ and

$$\mathscr{L}\Psi = \lambda\Psi. \tag{17}$$

The internal modes of vector solitons (3) and (4) are simply the discrete eigenfunctions Ψ of the operator $\mathscr{L}$ with nonzero real eigenvalues λ.

3. Internal modes of vector solitons with equal amplitudes

When $\omega = 1$, the vector solitons are given by (8), where r_1 and r_2 have the same amplitudes. To analyze the internal modes of such vector solitons, it is convenient to employ the following change of variables [13, 17]:

$$\phi_1 = \psi_1 + \psi_3, \qquad \phi_2 = \psi_2 + \psi_4, \qquad \phi_3 = \psi_1 - \psi_3, \qquad \phi_4 = \psi_2 - \psi_4. \quad (18)$$

Then (17) becomes

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_\beta \end{pmatrix} \Phi = \lambda \Phi, \quad (19)$$

where $\Phi = [\phi_1, \phi_2, \phi_3, \phi_4]^{\mathrm{T}}$,

$$L_0 = (\partial_{xx} - 1)\sigma_3 + 2\,\mathrm{sech}^2 x\,(2\sigma_3 + i\sigma_2), \quad (20)$$

and

$$L_\beta = (\partial_{xx} - 1)\sigma_3 + 2\,\mathrm{sech}^2 x \left(\frac{2}{1+\beta}\sigma_3 + \frac{1-\beta}{1+\beta} i\sigma_2 \right). \quad (21)$$

The operator L_0 arises in the linearization of the NLS equation around a NLS soliton. The discrete spectrum of L_0 consists of a single eigenvalue $\lambda = 0$, whose multiplicity is four. The continuous spectrum is located for $\{\lambda : |\lambda| \geq 1\}$ [31]. We suppose that the eigenvalues $\lambda = \pm\lambda_d$ for an internal mode, if exist, are located for $0 < \lambda_d < 1$. This eigenvalue may only appear as an eigenvalue of the operator L_β; i.e., it is determined by the problem,

$$\phi_{3xx} - \phi_3 + 2\mathrm{sech}^2 x \left(\frac{2}{1+\beta}\phi_3 + \frac{1-\beta}{1+\beta}\phi_4 \right) = \lambda\phi_3, \quad (22)$$

$$\phi_{4xx} - \phi_4 + 2\mathrm{sech}^2 x \left(\frac{2}{1+\beta}\phi_4 + \frac{1-\beta}{1+\beta}\phi_3 \right) = -\lambda\phi_4. \quad (23)$$

This eigenvalue problem was solved numerically in [17], where the eigenvalues $\lambda = \pm\lambda_d$ were found to exist for $0 < \beta < 1$. The numerical values for $\lambda_d = \lambda_d(\beta)$ are presented in Figure 1(a). It is clear from this figure that the eigenvalue undertakes two different bifurcations as $\beta \to 0$ and $\beta \to 1$. In the first case; i.e., when $0 < \beta \ll 1$, a pair of the discrete eigenvalues $\lambda = \pm\lambda_d$ (internal modes) bifurcate from the zero degenerate eigenvalue of the operator $L_{\beta=0}$. In the second case; i.e., when $0 < 1 - \beta \ll 1$, the eigenvalue $\lambda = \lambda_d$ merges into the edge of the continuous spectrum of $L_{\beta=1}$ at $\lambda = 1$.

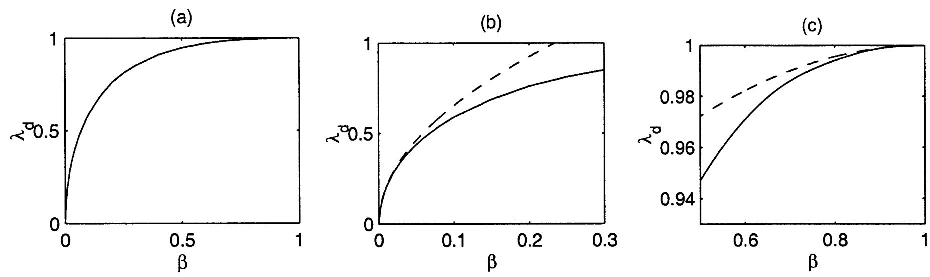

Figure 1. Eigenvalues λ_d of the internal modes for vector solitons with equal amplitudes (8); solid curves: numerical values; dashed curves: analytical approximation: (b) formula (24) for $\beta \to 0$; (c) formula (27) for $\beta \to 1$.

The first bifurcation has been analyzed perturbatively in [13, 15], where it was found that, to the leading order,

$$\lambda_d \approx \sqrt{\frac{64\beta}{15}} \text{ when } \beta \approx 0. \tag{24}$$

The other bifurcation at $\beta \approx 1$ can be analyzed according to the following heuristic approach. A more general and rigorous analysis is presented in the next section. The value $\lambda = 1$ is an edge point of the continuous spectrum of the operator $L_{\beta=1}$, and the corresponding eigenfunction is

$$\phi_3 = 0, \qquad \phi_4 = \tanh x. \tag{25}$$

Suppose that $\beta = 1 - \epsilon$, where $|\epsilon| \ll 1$, and $1 - \lambda_d \sim \mathrm{O}(\epsilon^2)$. It follows from the asymptotic balance for (22) and (23) in the limit $\epsilon \to 0$ that $\phi_3 = O(\epsilon)$ and ϕ_4 satisfies the equation

$$\phi_{4xx} + (2+\epsilon)\mathrm{sech}^2 x\phi_4 + O(\epsilon^2) = (1-\lambda_d)\phi_4. \tag{26}$$

Here we recall that, in the bifurcation of discrete eigenvalues from the continuous spectrum for linear Schrödinger operators, $O(\epsilon)$ perturbations in the potential cause $O(\epsilon^2)$ change in the discrete eigenvalues [20]. The $O(\epsilon^2)$ perturbations in the potential lead to higher-order changes of λ_d, which are negligible up to order ϵ^2. Therefore, we neglect the $O(\epsilon^2)$ term in (26). The reduced equation is simply a linear Schrödinger equation with a solvable potential [13]. A localized solution ϕ_4 exists for $\epsilon > 0$; i.e., for $\beta < 1$, and approaches the eigenmode (25) in the limit $\epsilon \to 0$. The asymptotic approximation of the eigenvalue λ_d is

$$\lambda_d \approx 1 - \frac{1}{9}(1-\beta)^2 \text{ when } \beta \approx 1. \tag{27}$$

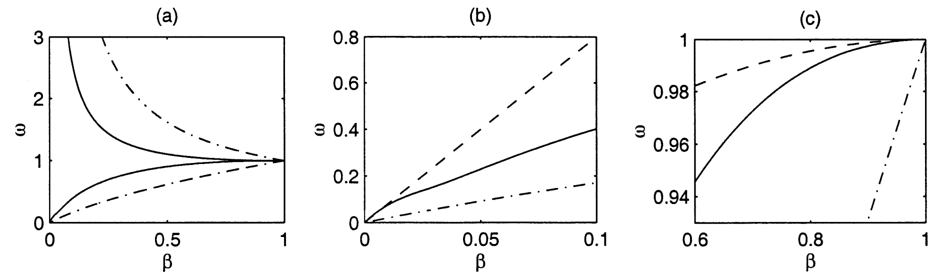

Figure 2. Existence region of internal modes and its comparison with analytical predictions for $\beta \sim 0$ and $\beta \sim 1$. (a) The numerically determined existence region of internal modes, bounded by solid lines. The dash–dotted lines are the boundaries (7) for nondegenerate vector solitons. (b) Comparison of the lower boundary of internal modes to the analytical approximation (62) (dashed curve) when $\beta \sim 0$. (c) Comparison of the lower boundary of internal modes to the analytical approximation (99) (dashed curve) when $\beta \sim 1$.

The comparison of the asymptotic results (24) and (27) with the numerical data is shown in Figure 1(b) and (c) respectively. The agreement can be seen to be quite satisfactory.

4. Internal modes of nondegenerate vector solitons

In this section, we study the internal modes of nondegenerate vector solitons (3) and (4) with an arbitrary ratio between the components r_1 and r_2. These solitons exist in the region of the (β, ω) plane bounded by the curves $\omega = \omega_I(\beta)$ and $\omega = \omega_{II}(\beta)$, where ω_I and ω_{II} are given by (7) (see Figure 2(a)). We show that the internal modes exist only in a smaller region of the parameter plane of vector solitons, where $0 < \beta < 1$ and $\omega_I < \omega_{thr}^-(\beta) < \omega < \omega_{thr}^+(\beta) < \omega_{II}$. In other words, internal modes exist only for vector solitons with moderate amplitude ratio $r_2(0)/r_1(0)$.

First, we study analytically the bifurcations of internal modes as $\beta \to 0$ and $\beta \to 1$. Then, we construct the region of existence of internal modes numerically and compare it to the analytical predictions. For convenience, we employ the following change of variables [21]:

$$u_1 = \psi_2 + \psi_1, \qquad v_1 = \psi_2 - \psi_1, \qquad u_2 = \psi_4 + \psi_3, \qquad v_2 = \psi_4 - \psi_3. \quad (28)$$

In these new variables, the eigenvalue problem (17) becomes

$$\hat{\mathscr{L}} Y = \lambda Y, \qquad (29)$$

where $Y = (u_1, v_1, u_2, v_2)^{\mathrm{T}}$, and

$$\hat{\mathscr{L}} = \begin{bmatrix} 0 & -\partial_{xx} + 1 - r_1^2 - \beta r_2^2 & 0 & 0 \\ -\partial_{xx} + 1 - 3r_1^2 - \beta r_2^2 & 0 & -2\beta r_1 r_2 & 0 \\ 0 & 0 & 0 & -\partial_{xx} + \omega^2 - r_2^2 - \beta r_1^2 \\ -2\beta r_1 r_2 & 0 & -\partial_{xx} + \omega^2 - 3r_2^2 - \beta r_1^2 & 0 \end{bmatrix}. \tag{30}$$

4.1. *Bifurcation of internal modes in the limit $\beta \to 0$*

When $\beta = 0$, the operator $\hat{\mathscr{L}}$ has a single discrete eigenvalue $\lambda = 0$, whose algebraic multiplicity is eight. When β moves away from zero, two of the multiplied zero eigenvalues bifurcate into a pair of discrete nonzero eigenvalues $\lambda = \pm\lambda_d$. The special case of this bifurcation for $\omega = 1$ was discussed in Section 3. Here, we study this bifurcation for nondegenerate vector solitons with arbitrary ω. The vector soliton (r_1, r_2) can be expanded for $\beta \ll 1$ as follows:

$$r_1 = r_{10}(x) + \beta r_{11}(x) + O(\beta^2), \tag{31}$$

$$r_2 = r_{20}(x) + \beta r_{21}(x) + O(\beta^2), \tag{32}$$

where

$$r_{10} = \sqrt{2}\,\mathrm{sech}x, \qquad r_{20} = \sqrt{2}\,\omega\,\mathrm{sech}\omega x. \tag{33}$$

When this expansion is substituted into (5) and (6), we find that the functions r_{11} and r_{21} satisfy the equations

$$r_{11xx} - r_{11} + 3r_{10}^2 r_{11} = -r_{10} r_{20}^2, \tag{34}$$

$$r_{21xx} - \omega^2 r_{21} + 3r_{20}^2 r_{21} = -r_{10}^2 r_{20}. \tag{35}$$

The localized solutions of the homogeneous equations of (34) and (35) are r_{10x} and r_{20x}, respectively. These functions are odd in x and, therefore, they are orthogonal to the inhomogeneous terms of (34) and (35). As a result, solutions for r_{11} and r_{21} defined by (34) and (35) are localized in x. For our purpose, it is not necessary to calculate their functional forms explicitly.

Next, we expand the discrete eigenvalue λ_d and its eigenfunction Y as follows:

$$\lambda_d = \sqrt{\beta}\lambda_{d1} + O(\beta\sqrt{\beta}), \tag{36}$$

and

$$u_1 = u_{10}(x) + \beta u_{12}(x) + O(\beta^2), \qquad v_1 = \sqrt{\beta} v_{11}(x) + O(\beta\sqrt{\beta}), \quad (37)$$

$$u_2 = u_{20}(x) + \beta u_{22}(x) + O(\beta^2), \qquad v_2 = \sqrt{\beta} v_{21}(x) + O(\beta\sqrt{\beta}). \quad (38)$$

When this expansion is substituted into the eigenvalue problem (29), we get at the zero-th order,

$$u_{10xx} - u_{10} + 3r_{10}^2 u_{10} = 0, \quad (39)$$

$$u_{20xx} - \omega^2 u_{20} + 3r_{20}^2 u_{20} = 0. \quad (40)$$

The localized solutions for u_{10} and u_{20} are

$$u_{10} = a_1 r_{10x}, \qquad u_{20} = a_2 r_{20x}, \quad (41)$$

where a_1 and a_2 are constant coefficients. The linear dependence between these coefficients will be revealed in the higher orders (see (50) below). At order $O(\sqrt{\beta})$, the equations for v_{11} and v_{21} are:

$$v_{11xx} - v_{11} + r_{10}^2 v_{11} = -\lambda_{d1} u_{10}, \quad (42)$$

$$v_{21xx} - \omega^2 v_{21} + r_{20}^2 v_{21} = -\lambda_{d1} u_{20}, \quad (43)$$

whose localized solutions are found to be

$$v_{11} = -\frac{1}{2} a_1 \lambda_{d1} x r_{10}, \qquad v_{21} = -\frac{1}{2} a_2 \lambda_{d1} x r_{20}. \quad (44)$$

The homogeneous solutions of (42) and (43), which are proportional to r_{10} and r_{20}, respectively, are not included in the above v_{11} and v_{21}. These homogeneous solutions have different symmetry with respect to x, and they correspond to those (zero) eigenvalues of $\hat{\mathscr{L}}$ that are not affected by the bifurcation under consideration.

At order $O(\beta)$, the equations for u_{12} and u_{22} are

$$u_{12xx} - u_{12} + 3r_{10}^2 u_{12} = F_1, \quad (45)$$

$$u_{22xx} - \omega^2 u_{22} + 3r_{20}^2 u_{22} = F_2, \quad (46)$$

where

$$F_1 = -\lambda_{d1} v_{11} - (6r_{10} r_{11} u_{10} + r_{20}^2 u_{10} + 2r_{10} r_{20} u_{20}), \quad (47)$$

and

$$F_2 = -\lambda_{d1} v_{21} - (6r_{20} r_{21} u_{20} + r_{10}^2 u_{20} + 2r_{10} r_{20} u_{10}). \quad (48)$$

These equations have localized solutions provided the two solvability conditions are satisfied,

$$\int_{-\infty}^{\infty} F_1 r_{10x}\, dx = \int_{-\infty}^{\infty} F_2 r_{20x}\, dx = 0. \tag{49}$$

These conditions reduce to the linear algebraic system,

$$\begin{pmatrix} f_{11} + \lambda_{d1}^2 & f_{12} \\ f_{21} & f_{22} + \lambda_{d1}^2 \omega \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0, \tag{50}$$

where

$$f_{11} = \int_{-\infty}^{\infty} (r_{10x})^2 (6 r_{10} r_{11} + r_{20}^2)\, dx, \tag{51}$$

$$f_{12} = f_{21} = 2 \int_{-\infty}^{\infty} r_{10} r_{10x} r_{20} r_{20x}\, dx, \tag{52}$$

and

$$f_{22} = \int_{-\infty}^{\infty} (r_{20x})^2 (6 r_{20} r_{21} + r_{10}^2)\, dx. \tag{53}$$

Because (50) should have nontrivial solutions, the determinant of its coefficient matrix must vanish. Thus, we get

$$\omega \lambda_{d1}^4 + \lambda_{d1}^2 (f_{22} + \omega f_{11}) + (f_{11} f_{22} - f_{12} f_{21}) = 0. \tag{54}$$

It turns out that the parameters $f_{ij} (i, j = 1, 2)$ are simply related. In fact, if we differentiate (34) and (35) with respect to x, then multiply them by r_{10x} and r_{20x}, respectively, and finally integrate them over the x-axis, we readily find that

$$f_{11} = -f_{12} = -f_{21} = f_{22}. \tag{55}$$

These relations can also be derived if we notice that $\lambda_{d1} = 0$ and $a_1 = a_2$ correspond to a neutral mode of $\hat{\mathscr{L}}$, which exists for all β values. When this mode is substituted into (50), relations (55) naturally follow.

After relations (55) are utilized, (54) becomes

$$\lambda_{d1}^2 = f_{12} \left(1 + \frac{1}{\omega} \right), \tag{56}$$

where

$$f_{12} = 8\omega^3 \int_{-\infty}^{\infty} \operatorname{sech}^2 x \ \ \tanh x \ \ \operatorname{sech}^2 \omega x \ \ \tanh \omega x \ dx > 0. \tag{57}$$

This equation determines the eigenvalues of the internal modes when they are bifurcated from the zero eigenvalue of the operator $\hat{\mathscr{L}}$ for $\beta \ll 1$. In the special case where $\omega = 1$, we find that $f_{12} = 32/15$. Thus $\lambda_{d1} = \sqrt{64/15}$, and

$$\lambda_d = \sqrt{\frac{64\beta}{15}} + O(\beta\sqrt{\beta}). \tag{58}$$

This reproduces the result (24) in Section 3.

Next we examine the region in the parameter plane (β, ω) where the internal modes exist in the limit $\beta \to 0$. The internal modes are localized in x only if the eigenvalues $\lambda = \pm\lambda_d$ are located in the gap of the continuous spectrum of the operator $\hat{\mathscr{L}}$; i.e., they satisfy the condition

$$\lambda_d < \min\{1, \omega^2\}. \tag{59}$$

Using the results (36) and (56), we find that this condition is met for the region in the (β, ω) plane, where

$$\beta < \beta_{thr}(\omega) = \frac{(\min\{1, \omega^2\})^2}{f_{12}(1 + \omega^{-1})}. \tag{60}$$

The function $\beta_{thr}(\omega)$ is positive for $\omega > 0$. It increases for ω from 0 to 1, then decreases for ω from 1 to ∞. In addition, for $\omega \ll 1$ and $\omega \gg 1$, we have the following asymptotic expressions:

$$\beta_{thr}(\omega) \longrightarrow \begin{cases} \frac{\omega}{8}, & \omega \ll 1, \\ \frac{1}{8\omega}, & \omega \gg 1. \end{cases} \tag{61}$$

Because condition (60) is valid asymptotically for $\beta \ll 1$, we substitute (61) for this limit and find the domain of existence of the internal mode, $\omega_{thr}^- < \omega < \omega_{thr}^+$, in the form

$$\omega_{thr}^- = 8\beta + O(\beta^2), \qquad \omega_{thr}^+ = \frac{1}{8\beta} + O(1), \qquad \beta \ll 1. \tag{62}$$

This result represents the asymptotic approximation for the region of existence of an internal mode, valid for small β. To compare, the parameter region of existence of vector solitons is given for small β by

$$2\beta \le \omega \le \frac{1}{2\beta}, \qquad \beta \ll 1. \tag{63}$$

Thus, we conclude that internal modes exist in a smaller region as compared to the region of existence of vector solitons.

4.2. Bifurcation of internal modes in the limit $\beta \to 1$

When $\beta = 1$, four branches of the continuous spectrum of the operator $\hat{\mathscr{L}}$ are located for real λ such that $|\lambda| \geq 1$. As β moves away from 1, the edge points $\lambda = \pm 1$ may bifurcate out of the continuous spectrum and become discrete eigenvalues. The special case of this bifurcation for solitons of equal amplitudes was analyzed in Section 3. Here, we study the nondegenerate vector solitons for arbitrary amplitude ratio by developing the method of [20, 25].

When $\beta \neq 1$, denote $\epsilon = 1 - \beta$ and suppose $\omega^2 = 1 - \epsilon\Delta$, where Δ is a parameter. Then, the vector soliton r_1 and r_2 can be expanded in terms of ϵ as

$$r_1 = R_1(x) + \epsilon R_{11}(x) + O(\epsilon^2), \tag{64}$$

$$r_2 = R_2(x) + \epsilon R_{21}(x) + O(\epsilon^2), \tag{65}$$

where

$$R_1 = \sqrt{2}\cos\theta \operatorname{sech} x, \qquad R_2 = \sqrt{2}\sin\theta \operatorname{sech} x, \tag{66}$$

are the vector soliton of the Manakov equations, and θ is its polarization. Substituting this expansion into (5) and (6), at order ϵ, we find that the equations for R_{11} and R_{21} are

$$R_{11xx} - R_{11} + 2(3\cos^2\theta + \sin^2\theta)\operatorname{sech}^2 x\, R_{11} + 4\sin\theta\cos\theta\operatorname{sech}^2 x\, R_{21} = N_1, \tag{67}$$

$$R_{21xx} - R_{21} + 2(3\sin^2\theta + \cos^2\theta)\operatorname{sech}^2 x\, R_{21} + 4\sin\theta\cos\theta\operatorname{sech}^2 x\, R_{11} = N_2, \tag{68}$$

where

$$N_1 = R_1 R_2^2, \qquad N_2 = R_1^2 R_2 - \Delta R_2. \tag{69}$$

The two localized homogeneous solutions of (67) and (68) are (R_{1x}, R_{2x}) and $(R_2, -R_1)$, which correspond to the position and polarization shifts of the Manakov soliton. For localized solutions R_{11} and R_{21} to exist, these homogeneous solutions must be orthogonal to the inhomogeneous terms (N_1, N_2). The first orthogonality is satisfied, because (R_{1x}, R_{2x}) is odd, and (N_1, N_2) is even. The second orthogonality is satisfied only when

$$\Delta = \frac{4}{3}\cos(2\theta). \tag{70}$$

This relation determines the polarization θ of the vector soliton when ω is specified.

Next, we expand the linear operator $\hat{\mathscr{L}}$ as

$$\hat{\mathscr{L}} = \hat{\mathscr{L}_0} + \epsilon(\hat{\mathscr{L}_1} - \Delta\hat{\mathscr{M}_1}) + O(\epsilon^2), \tag{71}$$

where

$$\hat{\mathscr{L}_0} = \begin{bmatrix} 0 & -\partial_{xx} + 1 - R_1^2 - R_2^2 & 0 & 0 \\ -\partial_{xx} + 1 - 3R_1^2 - R_2^2 & 0 & -2R_1R_2 & 0 \\ 0 & 0 & 0 & -\partial_{xx} + 1 - R_2^2 - R_1^2 \\ -2R_1R_2 & 0 & -\partial_{xx} + 1 - 3R_2^2 - R_1^2 & 0 \end{bmatrix} \tag{72}$$

is the linear operator for the Manakov system,

$$\hat{\mathscr{L}_1} = \begin{bmatrix} 0 & -2R_1R_{11} - 2R_2R_{21} + R_2^2 & 0 & 0 \\ -6R_1R_{11} - 2R_2R_{21} + R_2^2 & 0 & -2R_1R_{11} - 2R_2R_{21} + 2R_1R_2 & 0 \\ 0 & 0 & 0 & -2R_1R_{11} - 2R_2R_{21} + R_1^2 \\ -2R_1R_{11} - 2R_2R_{21} + 2R_1R_2 & 0 & -6R_2R_{21} - 2R_1R_{11} + R_1^2 & 0 \end{bmatrix}, \tag{73}$$

and

$$\hat{\mathscr{M}_1} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_1 \end{bmatrix}. \tag{74}$$

At $\epsilon = 0$, the Manakov system is exactly integrable, and we can find a complete set of eigenfunctions for the operator $\hat{\mathscr{L}_0}$ [32]. For $\epsilon \neq 0$, we can expand the localized eigenfunction of the linear system (29) into the complete set,

$$Y(x) = \sum_{j=1}^{4} \int_{-\infty}^{\infty} \zeta_j(k) Y_j(x, k) dk + \sum_{j=1}^{6} \zeta_{d_j} Y_{d_j}(x), \tag{75}$$

where $Y_{d_j}(j = 1, \ldots, 6)$ are the six eigenfunctions and generalized eigenfunctions of the zero eigenvalue for the operator $\hat{\mathscr{L}_0}$. The continuous eigenfunctions $Y_j(j = 1, \ldots, 4)$ satisfy the following boundary conditions as $x \to +\infty$,

$$Y_1(x, k) \longrightarrow [1, 1, 0, 0]^{\mathrm{T}} e^{ikx}, \qquad \lambda_1 = \lambda_0(k), \tag{76}$$

$$Y_2(x, k) \longrightarrow [0, 0, 1, 1]^{\mathrm{T}} e^{ikx}, \qquad \lambda_2 = \lambda_0(k), \tag{77}$$

$$Y_3(x, k) \longrightarrow [1, -1, 0, 0]^{\mathrm{T}} e^{ikx}, \qquad \lambda_3 = -\lambda_0(k), \tag{78}$$

$$Y_4(x, k) \longrightarrow [0, 0, 1, -1]^{\mathrm{T}} e^{ikx}, \qquad \lambda_4 = -\lambda_0(k), \tag{79}$$

where $\lambda_0(k) = 1 + k^2$. For each $Y_n(x, k) = (u_{1n}, v_{1n}, u_{2n}, v_{2n})^{\mathrm{T}}$ we define the adjoint state $Z_n(x, k)$,

$$Z_n(x, k) = (v_{1n}, u_{1n}, v_{2n}, u_{2n})^{\mathrm{T}}, \tag{80}$$

and the regular inner product,

$$\langle Z_n(k'), Y_m(k)\rangle = \int_{-\infty}^{\infty} Z_n^{*\mathrm{T}}(x, k') \cdot Y_m(x, k)\, dx. \tag{81}$$

It was verified in [32] that the only nonzero inner products of the continuous eigenfunctions are

$$\langle Z_1(k'), Y_1(k)\rangle = \langle Z_2(k'), Y_2(k)\rangle = 4\pi\delta(k - k'), \tag{82}$$

and

$$\langle Z_3(k'), Y_3(k)\rangle = \langle Z_4(k'), Y_4(k)\rangle = -4\pi\delta(k - k'). \tag{83}$$

In addition, we define the following scalar products that are used in our analysis,

$$\langle Z_m(k), \hat{\mathscr{M}_1} Y_n(k')\rangle = 4\pi a_{mn}(k)\delta(k - k') + P_{mn}(k, k'), \quad 1 \le m, n \le 4, \tag{84}$$

where $a_{mn}(k)$ and $P_{mn}(k, k')$ are, respectively, singular and nonsingular contributions of the scalar products. These coefficients can be found explicitly, but we only need the expressions for $a_{mn}(k)$ at $k = 0$. Using (75), we transform the linear eigenvalue problem (29) and (71) at $\lambda = \lambda_d$ into a system of integral equations for the coefficients $\zeta_j(k)$, $1 \le j \le 4$. Because only the first two branches (76) and (77) are responsible for the bifurcation of $\lambda = \lambda_d$ from the edge point $\lambda = 1$, we truncate the system of integral equations at the first two branches and find the system

$$\alpha_i(k) + \frac{\epsilon \Delta a_{ij}(k)}{\lambda_d - \lambda_0(k) + \epsilon \Delta a_{jj}(k)} \alpha_j(k) = \frac{\epsilon}{4\pi} \times \sum_{m=1}^{2} \int_{-\infty}^{\infty} \frac{K_{im}(k, k') - \Delta P_{im}(k, k')}{\lambda_d - \lambda_0(k') + \epsilon \Delta a_{mm}(k')} \alpha_m(k')\, dk' + \mathrm{O}(\epsilon^2), \tag{85}$$

where $i, j = 1, 2,\ i \neq j$,

$$K_{ij}(k, k') = \langle Z_i(k), \hat{\mathscr{L}}_1 Y_j(k')\rangle. \tag{86}$$

and the coefficients $\alpha_j(k)$ are related to $\zeta_i(k)$,

$$\zeta_i(k) = \frac{\alpha_i(k)}{\lambda_d - \lambda_0(k) + \epsilon\Delta a_{ii}(k)}. \tag{87}$$

Suppose that $\omega^2 < 1$ (i.e., $\epsilon\Delta > 0$) and a discrete eigenvalue $\lambda = \lambda_d$ bifurcates as $|\epsilon| \ll 1$ from the point $\lambda = \omega^2$, which is the edge of the continuous spectrum of the operator $\hat{\mathscr{L}}$ at $\beta \neq 1$. Furthermore, we denote

$$\lambda_d = \omega^2 - \epsilon^2\tau^2 = 1 - \epsilon^2(\tau^2 + \epsilon^{-1}\Delta). \tag{88}$$

Then, the dominant contribution to the integral in (85) is from a simple pole, and we get the asymptotic approximation as $\epsilon \to 0$,

$$\alpha_i(k) + \frac{\epsilon\Delta a_{ij}(k)}{\lambda_d - \lambda_0(k) + \epsilon\Delta a_{jj}(k)}\alpha_j(k) = -\frac{\operatorname{sign}(\epsilon)}{4} \times \sum_{m=1}^{2} \frac{K_{im}(k,0) - \Delta P_{im}(k,0)}{\sqrt{\tau^2 + \epsilon^{-1}\Delta(1 - a_{mm}(0))}}\alpha_m(0) + O(\epsilon), \tag{89}$$

where $i, j = 1, 2$ and $i \neq j$. Setting $k = 0$, we transform equations (89) to a homogeneous linear system that has nonzero solutions for $\alpha_i(0)$, $i = 1, 2$, if a certain determinant condition is met. To simplify this determinant condition, we first derive the explicit expressions for $Y_j(x, 0)$, $j = 1, 2$, satisfying (29) for $\hat{\mathscr{L}} = \hat{\mathscr{L}}_0$ and $\lambda = 1$,

$$Y_1(x,0) = \begin{bmatrix} \cos^2\theta + \sin^2\theta\,\tanh x - 2\cos^2\theta\,\operatorname{sech}^2 x, \\ \cos^2\theta + \sin^2\theta\,\tanh x \\ \sin\theta\cos\theta\,(1 - \tanh x - 2\,\operatorname{sech}^2 x) \\ \sin\theta\cos\theta\,(1 - \tanh x) \end{bmatrix}, \tag{90}$$

$$Y_2(x,0) = \begin{bmatrix} \sin\theta\cos\theta\,(1 - \tanh x - 2\,\operatorname{sech}^2 x) \\ \sin\theta\cos\theta\,(1 - \tanh x) \\ \sin^2\theta + \cos^2\theta\,\tanh x - 2\sin^2\theta\,\operatorname{sech}^2 x, \\ \sin^2\theta + \cos^2\theta\,\tanh x \end{bmatrix}. \tag{91}$$

Using the expressions for $Y_j(x, 0)$ and $Z_i(x, 0)$ derived above, we find explicitly the coefficients $a_{ij}(0)$ for the singular terms in (84),

$$a_{11}(0) = \frac{1}{2}\sin^2 2\theta, \qquad a_{12}(0) = a_{21}(0) = -\frac{1}{2}\sin 2\theta\cos 2\theta, \tag{92}$$
$$a_{22}(0) = \frac{1}{2}(1 + \cos^2 2\theta).$$

Now, we notice that the leading order $\epsilon \to 0$ in (89) requires $\Delta = O(\epsilon)$; i.e., $\theta = \pi/4 + O(\epsilon)$ according to (70). Therefore, we approximate all expressions involved in the system of algebraic equations (89) at the special case $\theta = \pi/4$. In this limit, it follows from (67) and (68) that

$$R_{11} = R_{21} = \frac{1}{4}\,\mathrm{sech} x. \tag{93}$$

Then, the operator $\hat{\mathscr{L}_1}$ becomes

$$\hat{\mathscr{L}_1} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\mathrm{sech}^2 x & 0 & \mathrm{sech}^2 x & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{sech}^2 x & 0 & -\mathrm{sech}^2 x & 0 \end{bmatrix}, \tag{94}$$

so that the coefficients $K_{ij}(0,0)$ and $a_{ij}(0)$ can be evaluated explicitly as

$$K_{11}(0,0) = K_{22}(0,0) = -\frac{2}{3}, \qquad K_{12}(0,0) = K_{21}(0,0) = \frac{2}{3}, \tag{95}$$

$$a_{11}(0) = a_{22}(0) = \frac{1}{2}, \qquad a_{12}(0) = a_{21}(0) = 0. \tag{96}$$

As a result, the determinant equation for the algebraic system (89) takes the following form at $\theta = \pi/4$,

$$\sqrt{\tau^2 + \frac{1}{2}\epsilon^{-1}\Delta} = \frac{1}{3}\mathrm{sign}(\epsilon) + O(\epsilon). \tag{97}$$

If $\epsilon < 0$; i.e., $\beta > 1$, the bifurcation of an internal mode does not occur, because (97) has no solution for real τ. On the other hand, for $\epsilon > 0$; i.e., $\beta < 1$, the internal mode exists, and the discrete eigenvalue $\lambda = \lambda_d$ can be approximated from (88) and (97) as

$$\lambda_d = \frac{1}{2}(1+\omega^2) - \frac{1}{9}(1-\beta)^2 + O(1-\beta)^3. \tag{98}$$

We notice that this formula works also for the case $\omega > 1$; i.e., $\epsilon\Delta < 0$. In the special case $\omega = 1$; i.e., $\Delta = 0$, the expansion (98) reproduces the result (27).

The internal mode exists if the discrete eigenvalue fits in the gap of the continuous spectrum of the operator $\hat{\mathscr{L}}$; i.e., $\lambda_d < \min(1, \omega^2)$. Using (98), we find that the region of existence of the internal mode is bounded as $\omega_{thr}^- < \omega < \omega_{thr}^+$, where

$$\omega_{thr}^{\pm} = 1 \pm \frac{1}{9}(1-\beta)^2 + O(1-\beta)^3, \qquad \beta \lesssim 1. \tag{99}$$

At the boundary, the polarization θ of the vector soliton is obtained from (70) in the form,

$$\theta = \frac{\pi}{4} \mp \frac{1}{12}(1-\beta) + O(1-\beta)^2. \tag{100}$$

4.3. Region of existence of internal modes

In the previous subsections, we analytically studied the bifurcation of internal modes at $\beta \to 0$ and $\beta \to 1$, and asymptotically obtained the region of existence of internal modes in these two limits. Here, we determine numerically the region of existence of internal modes for arbitrary β values, and compare it with the analytical results (62) and (99). Our numeric scheme is based on the observation that, at the lower boundary of this region $\omega = \omega^-_{thr}(\beta)$, the discrete eigenvalue λ_d of the problem (17) merges with the value ω^2, which is the edge point of the continuous spectrum of the operator $\mathscr{L}$. Then, the upper boundary $\omega = \omega^+_{thr}(\beta)$ can be found as $\omega^+_{thr} = 1/\omega^-_{thr}$.

Now, we solve the linear system (17) for $\lambda = \omega^2$ and look for a nontrivial solution $\Psi = \Psi(x)$ bounded at infinity. Such a solution exists only for a special value $\omega = \omega^-_{thr}$ that lies on the lower boundary of the existence region of internal modes. We used the shooting method to find this boundary value. The results are displayed in Figure 2(a) for $0 < \beta < 1$, where solid curves bound the region of existence of the internal modes of vector solitons. The boundaries of the existence region of vector solitons are also plotted in Figure 2(a) by dash-dotted curves (see (7)). Clearly, the internal modes exist only for those nondegenerate vector solitons whose amplitude ratios $r_2(0)/r_1(0)$ are neither too large nor too small.

Next, we compare the numerically obtained boundaries of internal modes with the asymptotic approximations (62) and (99) for $\beta \sim 0$ and $\beta \sim 1$. Only the comparison of the lower boundary is made and presented in Figure 2(b) and (c). We see that the analytical approximations at the leading order do capture the essential features of the numerical curve. In particular, when $\beta = 2/3$, which is a significant value in nonlinear optics, we find that the numeric and analytic boundary values are

$$\omega^-_{thr}(2/3) = 0.965\,\text{(numeric)}, \qquad \omega^-_{thr}(2/3) = 0.9877\,\text{(analytic)}, \tag{101}$$

which are reasonably close. On this boundary, the polarization of the vector soliton, defined as $\arctan[r_2(0)/r_1(0)]$, is numerically found to be 41.12°. This agrees relatively well with the analytical formula (100), which gives $\theta = 43.41^\circ$.

When $\beta > 1$, we did not find numerically the internal modes for nondegenerate vector solitons. This is consistent with the bifurcation result derived above, which shows that the internal modes do not appear when $\beta \gtrsim 1$. Thus, we conclude that for nondegenerate vector solitons, internal modes exist only for $\beta < 1$ and in the region shown in Figure 2(a).

5. Internal modes of degenerate vector solitons

For degenerate vector solitons (9) and (10), the operator $\mathscr{L}$ is greatly simplified, and the spectrum of internal oscillations can be readily determined. We consider only the degenerate soliton (9). The results for the other degenerate soliton (10) can be derived in a similar manner.

For the degenerate soliton (9) where $r_2 = 0$, the eigenvalue system (17) reduces to the degenerate form,

$$\mathscr{L}\Psi = \begin{pmatrix} L_0 & 0 \\ 0 & L_{lin} \end{pmatrix} \Psi = \lambda_d \Psi, \tag{102}$$

where L_0 is given by (20) and L_{lin} is the linear Schrödinger operator,

$$L_{lin} = (\partial_{xx} - \omega^2 + 2\beta \operatorname{sech}^2 x)\sigma_3. \tag{103}$$

An internal mode corresponds to the eigenvalue $\lambda_d \neq 0$, which is not included in the spectrum of L_0. Therefore, we set

$$\psi_1 = \psi_2 = 0. \tag{104}$$

The eigenvalue problem (102) for the components ψ_3 and ψ_4 is decoupled. We only consider the ψ_3 equation for $\lambda_d = \lambda - \omega^2$, which removes the parameter ω from the problem. The function ψ_3 satisfies the linear Schrödinger equation,

$$\psi_{3xx} + 2\beta \operatorname{sech}^2 x\, \psi_3 = \lambda \psi_3. \tag{105}$$

It can be solved exactly by transforming it into a hypergeometric equation [33]. The discrete eigenvalues are given by

$$\lambda = \lambda_n = \frac{1}{4}[\sqrt{1+8\beta} - (2n-1)]^2, \tag{106}$$

where n is a positive integer such that $n < (\sqrt{1+8\beta} + 1)/2$. Thus, a finite number of discrete eigenstates can be obtained. The first eigenstate $\lambda = \lambda_1$ is a fundamental mode that exists for any value of $\beta > 0$,

$$\lambda_1 = \omega_I^2 = \frac{1}{4}[\sqrt{1+8\beta} - 1)]^2, \qquad \psi_3^{(1)} = b_1(x) \equiv \operatorname{sech}^{\omega_I} x. \tag{107}$$

This fundamental mode is not an internal mode. In fact, it corresponds to a nondegenerate vector soliton with a small r_2 component, where $r_2 \propto b_1(x)$. Such a vector soliton is called a *wave-daughter-wave* soliton. It exists near the boundary $\omega = \omega_I(\beta)$ of the existence region. When $\beta \leq 1$, the fundamental mode (107) is the only one that exists in (105). Thus, we conclude that, for $0 < \beta < 1$, degenerate vector solitons have no internal modes responsible for internal oscillations. This conclusion coincides with the analysis of Section 4 (see Figure 2(a)).

When $1 < \beta \leq 3$, one more discrete eigenstate $\lambda = \lambda_2$ can be found in (105), where

$$\lambda_2 = \frac{1}{4}[\sqrt{1+8\beta} - 3)]^2, \qquad \psi_3^{(2)} = b_2(x) \equiv \operatorname{sech}^{\omega_I} x \tanh x. \tag{108}$$

This mode corresponds to a nondegenerate multihumped vector soliton with a small r_2 component ($r_2 \propto b_2(x)$), which exists near the boundary $\omega = \sqrt{\lambda_2}$ [29]. We do not consider multihumped vector soliton here. However, the discrete eigenstate (108) also has another meaning for the dynamics of the single-humped vector soliton. When the mode (108) is superimposed on the fundamental mode (107), the resulting combination oscillates at frequency $\lambda_d^{(1)} = \lambda_2 - \lambda_1$. Therefore, the eigenmode (108) can be also regarded as an internal mode of a wave-daughter-wave soliton with a small r_2 component when $\omega \approx \omega_I(\beta)$. However, we check that $\lambda_d^{(1)} = 2 - \sqrt{1+8\beta} < -1$. Therefore, this internal mode is embedded in the continuous spectrum of the operator L_0. It implies that this mode is not localized for the A component; i.e., for (ψ_1, ψ_2), in the linear problem (17) associated with the wave-daughter-wave soliton when $\omega \approx \omega_I(\beta)$. When the daughter-wave component r_2 goes to zero ($\omega \to \omega_I$), then $\psi_k \to 0$ ($k = 1, 2$), and this mode becomes localized.

When $\beta > 3$, more discrete eigenstates (106) can be found. However, all these modes are embedded in the continuous spectrum of L_0, because $\lambda_d^{(n-1)} = \lambda_n - \lambda_1 < -1$ for $n \geq 2$.

6. Radiative decay of internal oscillations of degenerate solitons

Here we study the time evolution of internal oscillations around a degenerate vector soliton (9). We follow the analysis of [17, 21, 26] and expand the solution of the coupled NLS system (1) and (2) into the following perturbation series:

$$A = e^{it}[A_0 + \epsilon^2 A_2 + O(\epsilon^4)], \tag{109}$$

$$B = \epsilon B_1 + \epsilon^3 B_3 + O(\epsilon^5), \tag{110}$$

where $A_0 = \sqrt{2}\operatorname{sech} x$ and ϵ is the amplitude parameter, $\epsilon \ll 1$. The function B_1 is superposed as a linear combination of the eigenmodes $b_n(x)$ corresponding to the eigenvalues λ_n (106),

$$B_1 = \sum_{n=1}^{N} c_n(T) b_n(x) e^{i\lambda_n t}. \tag{111}$$

The amplitude c_n of the n-th mode is slowly varying on the time scale $T = \epsilon^2 t$. As we discussed in Section 5, the fundamental eigenmode (107) is not an internal mode and does not induce internal oscillations. Therefore, we assume $N \geq 2$ throughout this section. It implies that we consider the degenerate vector soliton for the case $\beta > 1$.

6.1. *Radiative decay of a single internal mode*

When $\beta > 1$, the two lowest modes (107) and (108) always exist, and the second one is an internal mode embedded in the continuous spectrum of $\mathscr{L}$. Hence, we impose the perturbation B_1 as

$$B_1 = c_1(T)b_1(x)e^{i\lambda_1 t} + c_2(T)b_2(x)e^{i\lambda_2 t}. \tag{112}$$

When the expansion (109) and (110) are substituted into (1) and (2), the resulting equations at the zero-th and first orders of ϵ are automatically satisfied because of our choices of A_0 and B_1. The equation for $W = (A_2, A_2^*)^{\mathrm{T}}$ is

$$(i\partial_t + L_0)W = \eta(1, -1)^{\mathrm{T}}, \tag{113}$$

where

$$\eta = -\beta A_0[|c_1|^2 b_1^2 + |c_2|^2 b_2^2 + (c_1 c_2^* e^{i(\lambda_1-\lambda_2)t} + c_1^* c_2 e^{-i(\lambda_1-\lambda_2)t})b_1 b_2]. \tag{114}$$

The solution for W is

$$W = \left[|c_1|^2 w_1(x) + |c_2|^2 w_2(x)\right](1, 1)^{\mathrm{T}} + c_1 c_2^* w_3(x) e^{i(\lambda_1-\lambda_2)t} + c_1^* c_2 w_4(x) e^{-i(\lambda_1-\lambda_2)t}. \tag{115}$$

Here, the real and scalar functions w_1 and w_2 are localized solutions of the equations

$$w_{nxx} - w_n + 3A_0^2 w_n = -\beta A_0 b_n^2, \qquad n = 1, 2. \tag{116}$$

Note that such solutions exist, because the solvability conditions of these equations are satisfied. The complex vector functions w_3 and w_4 satisfy the equations

$$[L_0 - (\lambda_1 - \lambda_2)]w_3 = -\beta A_0 b_1 b_2 (1, -1)^{\mathrm{T}}, \tag{117}$$

and

$$[L_0 + (\lambda_1 - \lambda_2)]w_4 = -\beta A_0 b_1 b_2 (1, -1)^{\mathrm{T}}. \tag{118}$$

It is easy to check that the values $\pm(\lambda_1 - \lambda_2)$ are in the continuous spectrum of L_0. The solutions of w_3 and w_4 are not localized, because the solvability conditions are not satisfied [17]. At infinity, the nonvanishing tails of these solutions must travel outward from the degenerate vector soliton. Thus, we impose the following boundary conditions as $x \to \pm\infty$,

$$w_3 \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} g_\pm e^{-ik|x|}, \tag{119}$$

$$w_4 \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} h_\pm e^{ik|x|}, \tag{120}$$

where $k = \sqrt{\lambda_1 - \lambda_2 - 1} > 0$, and $g_\pm$ and $h_\pm$ are the amplitudes of nonvanishing tails of w_3 and w_4. These amplitudes can be determined as follows.

We define a new inner product of two vector functions $f(x)$ and $g(x)$ as

$$\langle \mathrm{f}, \mathrm{g} \rangle_2 = \int_{-\infty}^{\infty} \mathrm{f}^{\mathrm{T}} \sigma_3 \mathrm{g}\, dx. \tag{121}$$

Then, it is easy to show that the operator L_0 is self-adjoint under this inner product. We also introduce the eigenfunction $\varphi(x, k)$ of L_0 with eigenvalue $\lambda_1 - \lambda_2 = 1 + k^2$ [31]:

$$\varphi(x, k) = e^{ikx}\left(1 - \frac{2ik}{(k+i)^2} e^{-x}\operatorname{sech} x\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{e^{ikx}\operatorname{sech}^2 x}{(k+i)^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{122}$$

At infinity, $\varphi(x, k)$ satisfies the boundary conditions,

$$\varphi(x, k) \longrightarrow \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, & x \to \infty, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{(k-i)^2}{(k+i)^2} e^{ikx}, & x \to -\infty. \end{cases} \tag{123}$$

We further define the function

$$I(\beta) = -\beta \langle \varphi, A_0 b_1 b_2 (1, -1)^{\mathrm{T}} \rangle_2. \tag{124}$$

Then, in view of (117), (119), and (123), we find that

$$I(\beta) = \langle \varphi, [L_0 - (\lambda_1 - \lambda_2)] w_3 \rangle_2 = -2ikg_+. \tag{125}$$

Therefore,

$$g_+ = -\frac{I(\beta)}{2ik}. \tag{126}$$

If we replace $\varphi(x, k)$ by $\varphi(x, -k) = \varphi^*(x, k)$, which is the other eigenfunction of L_0 for eigenvalue $\lambda_1 - \lambda_2$, then repeating the above algebra leads to

$$g_- = -\frac{(k-i)^2}{(k+i)^2} \frac{I^*(\beta)}{2ik}. \tag{127}$$

Comparing (117) and (118) we find the relation,

$$w_4 = \sigma_1 w_3^*, \tag{128}$$

where σ_1 is defined in (15). As a result, the amplitudes of w_3 and w_4's tails at infinity are related as,

$$h_\pm = g_\pm^*. \tag{129}$$

It has been established in [17] that $I(\beta)$ never vanishes. Thus, the nonvanishing tails in w_3 and w_4 are always excited because of interaction of the internal mode with the continuous spectrum.

With the second-order solution W fully determined, we proceed to order $O(\epsilon^3)$. At this order, suppression of secular terms in B_3 leads to the following evolution equations for $c_1(T)$ and $c_2(T)$:

$$i\alpha_1 \frac{dc_1}{dT} = (\delta_{11}|c_1|^2 + \delta_{12}|c_2|^2)c_1, \tag{130}$$

$$i\alpha_2 \frac{dc_2}{dT} = (\delta_{21}|c_1|^2 + \delta_{22}|c_2|^2)c_2, \tag{131}$$

where

$$\alpha_n = \int_{-\infty}^{\infty} b_n^2 dx \; (> 0), \qquad n = 1, 2, \tag{132}$$

$$\delta_{nn} = -\int_{-\infty}^{\infty} (b_n^4 + 2\beta A_0 b_n^2 w_n)\, dx, \qquad n = 1, 2, \tag{133}$$

$$\delta_{12} = -\int_{-\infty}^{\infty} [2b_1^2 b_2^2 + 2\beta A_0 b_1^2 w_2 + \beta A_0 b_1 b_2 (w_3[1] + w_3[2])]\, dx, \tag{134}$$

$$\delta_{21} = -\int_{-\infty}^{\infty} [2b_1^2 b_2^2 + 2\beta A_0 b_2^2 w_1 + \beta A_0 b_1 b_2 (w_4[1] + w_4[2])]\, dx, \tag{135}$$

and $w_n[1]$ and $w_n[2]$ are the first and second elements of the vector function $w_n(x)$ for $n = 3, 4$. Notice that α_n and δ_{nn} $(n = 1, 2)$ are real quantities; whereas, δ_{12} and δ_{21} are complex. In addition,

$$\mathrm{Im}(\delta_{12}) = -\mathrm{Im}(\delta_{21}) \equiv \gamma, \tag{136}$$

which follows from (128). It is easy to show from (134) that

$$\gamma = -\frac{\beta}{2i} \langle A_0 b_1 b_2 (1, -1)^{\mathrm{T}}, w_3 - w_3^* \rangle. \tag{137}$$

Then we obtain from (117) and (137) that

$$\gamma = k(|g_+|^2 + |g_-|^2) = \frac{|I(\beta)|^2}{2k} \; (> 0). \tag{138}$$

The equations (130) and (131) can be easily solved for the squared amplitudes,

$$Q_n = |c_n|^2, \qquad n = 1, 2, \tag{139}$$

which solve the equations:

$$\alpha_1 \frac{dQ_1}{dT} = 2\gamma Q_1 Q_2, \tag{140}$$

$$\alpha_2 \frac{dQ_2}{dT} = -2\gamma Q_1 Q_2. \tag{141}$$

These equations exhibit a conserved quantity,

$$\alpha_1 Q_1 + \alpha_2 Q_2 = \alpha_1 Q_0 \ (> 0). \tag{142}$$

Making use of it, we solve (140) and (141) for $Q_1(T)$ and $Q_2(T)$ as $T \geq 0$,

$$Q_1 = \frac{Q_0 Q_1(0)}{Q_1(0) + [Q_0 - Q_1(0)]e^{-\mu T}}, \tag{143}$$

$$Q_2 = \frac{Q_0 Q_2(0)}{Q_0 - Q_1(0) + Q_1(0)e^{\mu T}}, \tag{144}$$

where $Q_1(0)$ and $Q_2(0)$ are initial data and

$$\mu = \frac{2\gamma Q_0}{\alpha_2} \ (> 0). \tag{145}$$

We see that the internal mode (second eigenmode) exponentially decays at rate $O(\epsilon^2)$. Meanwhile, the first mode asymptotically reaches the equilibrium state with a fixed amplitude,

$$c_{1\infty} = \sqrt{Q_0}\, e^{-\frac{i\delta_{11} Q_0}{\alpha_1} T}.$$

This equilibrium state, together with the degenerate vector soliton (9), forms a new nondegenerate vector soliton (wave-daughter-wave soliton), whose component amplitudes are approximately $\sqrt{2}$ and $\sqrt{Q_0}\epsilon$, respectively. The corresponding parameter ω is approximately given by

$$\omega^2 \approx \omega_I^2 - \frac{\delta_{11} Q_0}{\alpha_1} \epsilon^2.$$

This result is consistent with (3.8) in [29] for a wave-daughter-wave soliton. As time goes to infinity, the internally oscillational state (109) and (110) relaxes into this vector soliton, while the dispersive radiation eventually dies out. We emphasize here that the decay of this type of internal oscillations is exponentially fast. This very strong radiation damping is explained by the fact that the internal mode of the partial system (103) has the eigenvalue $\lambda_d^{(1)} = \lambda_2 - \lambda_1$ embedded in the continuous spectrum of the linear operator $\mathscr{L}$ of the full system (17). As a result, the internal mode of the full system (17) is not localized for the A component and the eigenvalue λ_d is complex-valued as

$$\lambda_d = \lambda_d^{(1)} - \frac{\mathrm{Re}(\delta_{21})Q_0}{\alpha_2}\epsilon^2 + i\frac{\gamma Q_0}{\alpha_2}\epsilon^2.$$

In this interpretation, it is assumed that $|c_2| \ll |c_1|$, so that this internal mode can be regarded as one for a wave-daughter-wave soliton. The exponential decay of the embedded eigenvalues is known in theory of quantum resonances as the Fermat golden rule [28].

To verify the above analytical results on radiation damping, we solve (1) and (2) numerically subject to the initial condition,

$$A(x,0) = \sqrt{2}\,\mathrm{sech}x, \tag{146}$$

$$B(x,0) = \epsilon[c_1 b_1(x) + c_2 b_2(x)]. \tag{147}$$

We choose the following values of parameters,

$$\beta = 2,\ \epsilon = 0.2,\ c_1(0) = c_2(0) = 1. \tag{148}$$

The contour plots of $|A|$ and $|B|$ are shown in Figure 3(a) and (b). We can see that the internal oscillations, indeed, quickly dampens out as predicted, and the solutions approach a nondegenerate vector soliton state. In Figure 3(c), the evolution of $|B(0,t)|$ is plotted. This quantity exponentially approaches 0.28, accompanied by weak oscillations. Notice from (110) and (112) that $|B(0,t)|$ is approximately equal to $\epsilon c_1(T)$ with error of order $O(\epsilon^3)$. Thus,

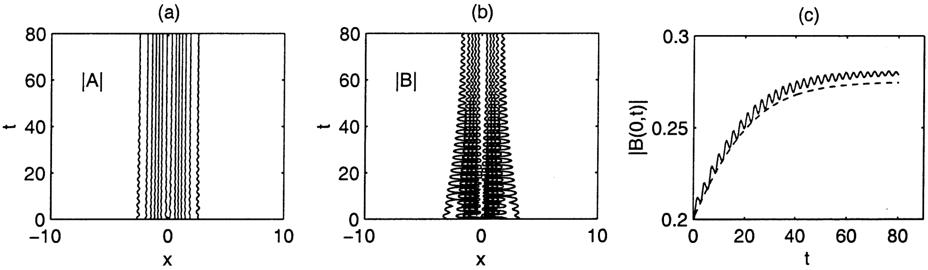

Figure 3. Numerical simulations of internal oscillations in the degenerate vector soliton (9) and comparison with the analytical predictions. The initial condition is given by (146), (147), and (148). (a) The contour plot of $|A|$ at levels 0.2:0.2:1.4; (b) the contour plot of $|B|$ at levels 0.05:0.05:0.25; (c) comparison of $|B(0,t)|$ (solid curve) with the analytical prediction $\epsilon c_1(T)$ (see (139) and (143)) (dashed curve).

we can compare it to $\epsilon c_1(T)$, where $c_1(T)$ is given analytically by (139) and (143). For $\beta = 2$, we find that

$$\alpha_1 = 1.5347,\ \alpha_2 = 1.3665,\ \gamma = 0.6320. \tag{149}$$

The other parameters in (143) can be obtained from (148). The resulting function $\epsilon c_1(T)$ is then plotted in Figure 3(c) and compared with $|B(0, t)|$. The good agreement between these data testifies that our perturbation analysis does capture the essential dynamics of radiation damping in the internal oscillations of the degenerate vector solitons. We attribute the weak oscillations in $|B(0, t)|$ to its higher-order effects and to dispersive radiation, which decays much slower, $O(t^{-1/2})$.

6.2. *Radiative decay of several internal modes*

When $N(N-1)/2 < \beta \leq N(N+1)/2$, where N is an integer, there are N eigenmodes (106) that correspond to $(N-1)$ internal modes embedded in the continuous spectrum of $\mathscr{L}$. If all of them are excited, their radiation damping can be analyzed according to a generalization of the approach in Section 6.1. In this case, the function B_1 in the perturbation expansion (109) and (110) is given by (111). At order $O(\epsilon^2)$, the equation for $W = (A_2, A_2^*)^{\mathrm{T}}$ is still (113), but now η is given by

$$\eta = -\beta A_0 \sum_{n=1}^{N} \sum_{m=1}^{N} c_n c_m^* b_n(x) b_m(x) e^{i(\lambda_n - \lambda_m)t}. \tag{150}$$

The solution for W is

$$W = \left[\sum_{n=1}^{N} |c_n|^2 w_n(x)\right](1, 1)^T + \sum_{n \neq m} c_n c_m^* w_{nm} e^{i(\lambda_n - \lambda_m)t}, \tag{151}$$

where the scalar functions w_n are real and localized solutions of the equations (116) for $1 \leq n \leq N$ and the complex vector functions w_{nm} satisfy the equations

$$[L_0 - (\lambda_n - \lambda_m)] w_{nm} = -\beta A_0 b_n b_m (1, -1)^{\mathrm{T}}, \qquad n \neq m. \tag{152}$$

It is easy to check that $|\lambda_n - \lambda_m| > 1$ for $m \neq n$. Thus $\lambda_n - \lambda_m$ falls in the continuous spectrum of L_0 for any $n \neq m$. Therefore, the solution w_{nm} has nonvanishing tails as $x \to \infty$. We impose the following boundary conditions,

$$w_{nm} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} g_{nm}^{\pm} e^{-ik_{nm}|x|}, \qquad x \to \pm\infty, \tag{153}$$

$$w_{mn} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} g_{mn}^{\pm} e^{ik_{mn}|x|}, \qquad x \to \pm\infty, \tag{154}$$

where $n < m$, and $k_{nm} = \sqrt{\lambda_n - \lambda_m - 1} > 0$. It is easy to show that

$$w_{ij} = \sigma_1 w_{ji}^*, \qquad g_{ij}^{\pm} = g_{ji}^{\pm *}. \tag{155}$$

At order $O(\epsilon^3)$, slow evolution equations for c_n will be obtained when secular terms in B_3 are suppressed. Here, we assume that there are no four-wave resonances; i.e.,

$$\lambda_n + \lambda_m = \lambda_l + \lambda_k, \tag{156}$$

only if $n = l$ and $m = k$. Then, evolution equations for c_n can be found in the form

$$i\alpha_n \frac{dc_n}{dT} = \left(\sum_{m=1}^{N} \delta_{nm} |c_m|^2 \right) c_n, \qquad 1 \le n \le N, \tag{157}$$

where

$$\alpha_n = \int_{-\infty}^{\infty} b_n^2(x)\, dx, \tag{158}$$

$$\delta_{nn} = -\int_{-\infty}^{\infty} (b_n^4 + 2\beta A_0 b_n^2 w_n)\, dx, \tag{159}$$

and

$$\delta_{nm} = -\int_{-\infty}^{\infty} \{2b_n^2 b_m^2 + 2\beta A_0 b_n^2 w_m + \beta A_0 b_n b_m (w_{nm}[1] + w_{nm}[2])\} dx. \tag{160}$$

Here, α_n and δ_{nn} are real, and $\delta_{nm} (n \ne m)$ complex. In addition, we define

$$\gamma_{nm} \equiv \text{Im}(\delta_{nm}) = -\text{Im}(\delta_{mn}) = -\gamma_{mn}, \qquad 1 \le n, m \le N. \tag{161}$$

Similar to the previous analysis, we derive explicitly,

$$\gamma_{nm} = k_{nm}(|g_{nm}^+|^2 + |g_{nm}^-|^2)\,(> 0), \qquad n < m. \tag{162}$$

Now, we analyze (157) in more detail. Denoting

$$Q_n = |c_n|^2, \qquad 1 \le n \le N, \tag{163}$$

we transform equations for Q_i to a real N-dimensional system,

$$\alpha_n \frac{dQ_n}{dT} = 2Q_n \sum_{n \ne m} \gamma_{nm} Q_m, \qquad 1 \le n, m \le N. \tag{164}$$

This system has a conserved quantity

$$\sum_{n=1}^{N} \alpha_n Q_n = \alpha_1 Q_0\,(> 0). \tag{165}$$

The fixed points of (164) are determined as

$$Q_k^{(0)} = \frac{\alpha_1}{\alpha_k} Q_0, \qquad Q_n = 0, \qquad n \neq k. \tag{166}$$

Stability of these fixed points can be analyzed by adding the perturbations,

$$Q_k = Q_k^{(0)} + \widetilde{Q}_k, \qquad Q_n = \widetilde{Q}_n, \qquad n \neq k. \tag{167}$$

The linearized equations for $\widetilde{Q}_n$ are

$$\alpha_n \frac{d\widetilde{Q}_n}{dT} = 2Q_k^{(0)} \gamma_{nk} \widetilde{Q}_n, \qquad n \neq k, \tag{168}$$

$$\alpha_k \frac{d\widetilde{Q}_k}{dT} = 2Q_k^{(0)} \sum_{n \neq k} \gamma_{kn} \widetilde{Q}_n. \tag{169}$$

The stability criterion of the fixed point is that

$$\gamma_{nk} < 0 \qquad \text{for any} \qquad n \neq k. \tag{170}$$

This criterion is satisfied only for $k = 1$ according to (162). It implies that only the fundamental mode $b_1(x)$ at $\lambda = \lambda_1$ is stable and attracts asymptotically the development of an initial perturbation. All the other eigenmodes are exponentially unstable and decay because of radiation damping. Thus, we conclude that any internal oscillations of degenerate vector solitons would decay exponentially, and the solutions would relax into a nondegenerate vector soliton.

7. Radiation damping of internal modes in nondegenerate solitons

The internal mode of a nondegenerate vector soliton exists for $0 < \beta < 1$, and ω is in the region $\omega_{thr}^{-}(\beta) < \omega < \omega_{thr}^{+}(\beta)$ shown in Figure 2(a). Radiation-induced damping of the internal mode has been addressed in [17], where it was shown qualitatively that this mode would decay because of energy loss through dispersive radiation. Here, we study this problem quantitatively and show that the internal mode decays algebraically because of nonlinear mechanism of emission of radiation. For simplicity, we consider the radiation damping of internal oscillations for vector solitons of equal amplitudes (8). The damping for the general nondegenerate vector solitons can be studied in a similar manner.

It is helpful here to transform (1) and (2) to the form,

$$ip_t + p_{xx} + \left(\frac{1+\beta}{4}|p|^2 + \frac{1}{2}|q|^2\right)p + \frac{1-\beta}{4} q^2 p^* = 0, \tag{171}$$

$$iq_t + q_{xx} + \left(\frac{1+\beta}{4}|q|^2 + \frac{1}{2}|p|^2\right)q + \frac{1-\beta}{4}p^2 q^* = 0, \tag{172}$$

where

$$p = A + B, \qquad q = A - B. \tag{173}$$

The vector soliton of equal amplitudes (8) is given by

$$p = p_0(x)e^{it}, \qquad q = 0, \tag{174}$$

where

$$p_0 = \sqrt{\frac{8}{1+\beta}}\,\mathrm{sech}x.$$

When it is perturbed, we express the internal mode (see Section 3) as

$$q = q_1(x,t)e^{it}, \tag{175}$$

where

$$q_1 = C\,h_1(x)e^{i\lambda_d t} + C^* h_2(x)e^{-i\lambda_d t}. \tag{176}$$

Here, the real function $(h_1, h_2)^{\mathrm{T}}$ represents the discrete eigenmode of the operator L_β with eigenvalue $\lambda = \lambda_d > 0$ (see Figure 1(a)), and C is an arbitrary complex constant (amplitude of the mode).

When the internal mode (176) is initially imposed on the vector soliton (174), the solution can be expanded into the following perturbation series,

$$p = e^{it}\{p_0 + \epsilon^2 p_2 + O(\epsilon^4)\}, \tag{177}$$

$$q = e^{it}\{\epsilon q_1 + \epsilon^3 q_3 + O(\epsilon^5)\}. \tag{178}$$

Now the amplitude C of the internal mode evolves slowly on the time scale $T = \epsilon^2 t$. At order $O(\epsilon^2)$, $\mathscr{D} \equiv (p_2, p_2^*)^T$ satisfies the equation

$$i\mathscr{D}_t + L_0\mathscr{D} = -\frac{1}{2}p_0\left[|C|^2\eta_0\begin{pmatrix}1\\-1\end{pmatrix} + C^2 e^{2i\lambda_d t}\begin{pmatrix}\eta_1\\-\eta_2\end{pmatrix}\right. \tag{179}$$
$$\left. + C^{*2}e^{-2i\lambda_d t}\begin{pmatrix}\eta_2\\-\eta_1\end{pmatrix}\right],$$

where L_0 is given by (20), and the scalar functions η_0, η_1, and η_2 are

$$\eta_0 = h_1^2 + (1-\beta)h_1h_2 + h_2^2, \tag{180}$$

$$\eta_1 = h_1h_2 + \frac{1}{2}(1-\beta)h_1^2, \tag{181}$$

$$\eta_2 = h_1h_2 + \frac{1}{2}(1-\beta)h_2^2. \tag{182}$$

The solution $\mathscr{D}$ can be presented in the form

$$\mathscr{D}(x,t) = \left[|C|^2 d_0(x)(1,1)^{\mathrm{T}} + C^2 d_1(x) e^{2i\lambda_d t} + C^{*2} d_2(x) e^{-2i\lambda_d t}\right], \tag{183}$$

where the scalar real function d_0 is the localized solution of equation

$$d_{0xx} - d_0 + 6\,\mathrm{sech}^2 x\, d_0 = -\frac{1}{2} p_0 \eta_0, \tag{184}$$

and vector complex functions d_1 and d_2 satisfy the equations

$$(L_0 - 2\lambda_d) d_1 = -\frac{1}{2} p_0 (\eta_1, -\eta_2)^{\mathrm{T}}, \tag{185}$$

$$(L_0 - 2\lambda_d) d_2 = -\frac{1}{2} p_0 (\eta_2, -\eta_1)^{\mathrm{T}}. \tag{186}$$

If $2\lambda_d > 1$; i.e., $0.07 \lesssim \beta < 1$ (see Figure 1(b)), then the eigenvalue $2\lambda_d$ lies in the continuous spectrum of L_0. In addition, the compatibility condition for the existence of localized solutions for $d_1(x)$, and $d_2(x)$ is not satisfied [17]. Therefore, the functions $d_1(x)$ and $d_2(x)$ have nonvanishing tails at infinity that must travel outward from the vector soliton. Thus, we impose the following boundary conditions for d_1 and d_2 as $x \to \pm\infty$,

$$d_1 \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} a_{\pm} e^{-ik|x|}, \tag{187}$$

$$d_2 \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} b_{\pm} e^{ik|x|}, \tag{188}$$

where $k = \sqrt{2\lambda_d - 1}$. As in Section 6.1, we can show that

$$d_2 = \sigma_1 d_1^*, \qquad b_{\pm} = a_{\pm}^*. \tag{189}$$

At order $O(\epsilon^3)$, suppression of secular growth in the q_3 solution results in the evolution equation for the amplitude $C = C(T)$ of the internal mode,

$$i\alpha \frac{dC}{dT} + \delta |C|^2 C = 0, \tag{190}$$

where

$$\alpha = \int_{\infty}^{\infty} (h_2^2 - h_1^2)\, dx, \tag{191}$$

and

$$\delta = -\int_{\infty}^{\infty} dx \left[\frac{\beta+1}{4}(h_1^4 + 4h_1^2 h_2^2 + h_2^4) + (h_1^2 + h_2^2 \right. \tag{192}$$

$$\left. + (1-\beta) h_1 h_2) p_0 d_0 + p_0(\eta_1 d_1[1] + \eta_2 d_1[2]) \right].$$

By using the same technique as in Section 6.1, we can show that

$$\gamma \equiv \mathrm{Im}(\delta) = 2k(|a_+|^2 + |a_-|^2) > 0 \tag{193}$$

and

$$\alpha = \frac{1}{\lambda_d} \langle (h_1 - h_2), \hat{L}_{lin}(h_1 - h_2) \rangle > 0, \tag{194}$$

where $\hat{L}_{lin} = -\partial_{xx} + 1 - 2\,\mathrm{sech}^2 x$ is the positive definite Schrödinger operator, and the inner product is given by (81). Defining $Q = |C|^2$, we find the solution for Q as

$$Q = \frac{\alpha Q(0)}{\alpha + 2\gamma Q(0) T}. \tag{195}$$

Thus, the internal mode decays as $O(t^{-1/2})$ when time is large. This decay rate is the same as that of typical dispersive radiation. If $0 < \beta \lesssim 0.07$, the decay rate of the internal modes will be even slower. In fact, if

$$\frac{1}{n} < \lambda_d < \frac{1}{n-1}, \tag{196}$$

the amplitude of the internal mode decays as $O(t^{-1/2(n-1)})$ [21]. Finally, in the limit $\lambda_d \to 0$ ($\beta \to 0$), the decay rate of the mode amplitude becomes logarithmic, $C \sim (\ln t)^{-2}$ [23].

8. Concluding remarks

In this paper, we have studied internal modes and their radiation damping for vector solitons of coupled NLS equations. We found that, for nondegenerate vector solitons (3) and (4), internal modes exist only for $0 < \beta < 1$ and $\omega_{thr}^- < \omega < \omega_{thr}^+$ (see Figure 2(a)). For degenerate vector solitons, internal oscillations can be excited for $\beta > 1$. We have also showed that radiation-induced decay of internal oscillations is algebraic for nondegenerate vector solitons, with decay rate $O(t^{-1/2})$ or slower. For degenerate vector solitons, the decay rate of internal oscillations is exponential, which is much faster than that of typical dispersive radiation, $O(t^{-1/2})$. In both cases, the internally oscillating states relax asymptotically into a stationary vector soliton.

The results in this paper shed light on other related problems in the coupled NLS equations. One problem is the stability of vector solitons in perturbed coupled NLS equations. These equations describe pulse propagation along birefringent nonlinear optical fibers when fiber loss and amplification are taken into account [4, 5] and also when the fiber birefringence is weak [3, 8]. The internal modes detailed in this paper will play an important role in the analysis of this problem [24]. Another problem is the collision of vector solitons in the coupled NLS equations. The existence of internal modes of vector solitons suggests that a resonance structure similar to that in the ϕ^4 model may also arise in this collision [19]. These important questions are left for future studies.

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References

1. D. J. Benney and A. C. Newell, The propagation of nonlinear wave envelopes, *J. Math. Phys.* 46:133–139 (1967).
2. G. J. Roskes, Some nonlinear multiphase interactions, *Stud. Appl. Math.* 55:231–238 (1976).
3. C. R. Menyuk, Nonlinear pulse propagation in birefringent optical fibers, *IEEE J. Quantum Electron*, QE-23:174 (1987).
4. G. P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, San Diego, CA, 1995.
5. A. Hasegawa and Y. Kodama, *Solitons in Optical Communications*, Clarendon Press, Oxford, UK, 1995.
6. P. K. A. Wai and C. R. Menyuk, Polarization mode dispersion, decorrelation, and diffusion in optical fibers with randomly varying birefringence, *J. Lightwave Tech.* 14:148 (1996).
7. S. G. Evangelides, L. F. Mollenauer, J. P. Gordon, and N. S. Bergano, Polarization multiplexing with solitons, *J. Lightwave Tech.* 10:28 (1992).
8. M. N. Islam, *Ultrafast fiber switching devices and systems*, Cambridge University Press, New York, 1992.
9. C. R. Menyuk, Stability of solitons in birefringent optical fibers. I. Equal propagation amplitudes, *Opt. Lett.* 12:614 (1987).
10. C. R. Menyuk, Stability of solitons in birefringent optical fibers. II. Arbitrary amplitudes, *J. Opt. Soc. Am. B* 5:392 (1988).
11. B. A. Malomed and S. Wabnitz, Soliton annihilation and fusion from resonant inelastic collisions in birefringent optical fibers, *Opt. Lett.* 16:1388 (1991).
12. C. Sophocleous and D. F. Parker, Pulse collisions and polarization conversion for optical fibers, *Opt. Comm.* 112:214 (1994).
13. J. Yang and D. J. Benney, Some properties of nonlinear wave systems, *Stud. Appl. Math.* 96:111 (1996).
14. J. Yang, Multisoliton perturbation theory for the Manakov equations and its applications to nonlinear optics, *Phys. Rev. E.* 59:2393 (1999).
15. T. Ueda and W. L. Kath, Dynamics of coupled solitons in nonlinear optical fibers, *Phys. Rev. A* 42:563 (1990).

16. D. J. KAUP, B. A. MALOMED, and R. S. TASGAL, Internal dynamics of a vector soliton in a nonlinear optical fiber, *Phys. Rev. E* 48:3049 (1993).

17. J. YANG, Vector solitons and their internal oscillations in birefringent nonlinear optical fibers, *Stud. Appl. Math.* 98:61 (1997).

18. B. A. MALOMED and R. S. TASGAL, Internal vibrations of a vector soliton in the coupled nonlinear Schrödinger equations, *Phys. Rev. E* 58:2564 (1998).

19. D. K. CAMPBELL, J. F. SCHONFELD, and C. A. WINGATE, Resonance structure in kink–antikink interactions in ϕ^4 theory, *Physica* 9D:1 (1983).

20. Y. S. KIVSHAR, D. E. PELINOVSKY, T. CRETEGNY, and M. PEYRARD, Internal modes of solitary waves, *Phys. Rev. Lett.* 80:5032 (1998).

21. D. PELINOVSKY, Y. KIVSHAR, and V. AFANASJEV, Internal modes of envelope solitons, *Physica D* 116:121 (1998).

22. C. ETRICH, U. PESCHEL, F. LEDERER, B. A. MALOMED, and Y. S. KIVSHAR, Origin of the persistent oscillations of solitary waves in nonlinear quadratic media, *Phys. Rev. E* 54:4321 (1996).

23. D. PELINOVSKY, Radiative effects to the adiabatic dynamics of envelope-wave solitons, *Physica D* 119:301 (1998).

24. J. YANG and D. J. KAUP, Stability and evolution of solitary waves in perturbed generalized nonlinear Schrödinger equations, *SIAM J. Appl. Math.* 60:967 (2000).

25. I. V. BARASHENKOV, D. E. PELINOVSKY, and E. V. ZEMLYANAYA, Vibrations and oscillatory instabilities of gap solitons, *Phys. Rev. Lett.* 80:5117 (1998).

26. N. A. ALEXEEVA, I. V. BARASHENKOV, and D. E. PELINOVSKY, Dynamics of the parametrically driven NLS solitons beyond the onset of the oscillatory instability, *Nonlinearity* 12:103 (1999).

27. A. SOFFER and M. I. WEINSTEIN, Resonances, radiation damping, and instability in Hamiltonian nonlinear wave equations, *Invent. Math.* 136:9 (1999).

28. A. SOFFER and M. I. WEINSTEIN, Time dependent resonance theory, *Geom. Funct. Anal.* 8:1086 (1999).

29. J. YANG, Classification of the solitary waves in coupled nonlinear Schrödinger equations, *Physica D.* 108:92–112 (1997).

30. J. YANG, Multiple permanent-wave trains in nonlinear systems, *Stud. Appl. Math.* 100:127 (1998).

31. D. J. KAUP, Perturbation theory for solitons in optical fibers, *Phys. Rev. A* 42:5689 (1990).

32. T. I. LAKOBA and D. J. KAUP, Perturbation theory for the Manakov soliton and its applications to pulse propagation in randomly birefringent fibers, *Phys. Rev. E.* 56:6147 (1997).

33. L. D. LANDAU and E. M. LIFSHITZ, *Quantum mechanics: Nonrelativistic theory*, Pergamon, New York, 1977.

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