Dynamics of Embedded Solitons in the Extended Korteweg–de Vries Equations

By Jianke Yang

Embedded solitons are solitary waves residing inside the continuous spectrum of a wave system. They have been discovered in a wide array of physical situations recently. In this article, we present the first comprehensive theory on the dynamics of embedded solitons and nonlocal solitary waves in the framework of the perturbed fifth-order Korteweg–de Vries (KdV) hierarchy equation. Our method is based on the development of a soliton perturbation theory. By obtaining the analytical formula for the tail amplitudes of nonlocal solitary waves, we demonstrate the existence of single-hump embedded solitons for both Hamiltonian and non-Hamiltonian perturbations. These embedded solitons can be isolated (existing at a unique wave speed) or continuous (existing at all wave speeds). Under small wave speed limit, our results show that the tail amplitudes of nonlocal waves are exponentially small, and the product of the amplitude and cosine of the phase is a constant to leading order. This qualitatively reproduces the previous results on the fifth-order KdV equation obtained by exponential asymptotics techniques. We further study the dynamics of embedded solitons and prove that, under Hamiltonian perturbations, a localized wave initially moving faster than the embedded soliton will asymptotically approach this embedded soliton, whereas a localized wave moving slower than the embedded soliton will decay into radiation. Thus, the embedded soliton is semistable. Under non-Hamiltonian perturbations, stable embedded solitons are found for the first time.

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1. Introduction

The Korteweg–de Vries (KdV) equation arises as an approximate equation governing weakly nonlinear long waves in a shallow channel. When higher-order effects are included, the extended KdV equation

$$u_t + 6uu_x + u_{xxx} + \delta^2(u_{xxxxx} + c_1 uu_{xxx} + c_2 u_x u_{xx} + c_3 u^2 u_x) = 0 \qquad (1)$$

results. Here, δ is a small parameter that measures the wave amplitude. Without loss of generality, we rescale $\delta = 1$ throughout this article by employing the variable transform

$$\bar{x} = x/\delta, \qquad \bar{t} = t/\delta^3, \qquad \bar{u} = \delta^2 u, \qquad (2)$$

and dropping the bars. This scaling is important, because it makes the connection between our results in the present article and the previous results by exponential asymptotics techniques. The coefficients c_1, c_2, and c_3 in Equation (1) depend on the physical situation. For irrotational gravity waves, these coefficients have been derived by [1] and [2] using either the perturbation techniques or Lagrangian methods. The coefficients given there are both $(c_1, c_2, c_3) = (100/19, 230/19, -60/19)$. Equation (1) with these coefficients are not Hamiltonian, however (the irrotational water wave equations are Hamiltonian, see [3]). Hamiltonian equations have also been obtained by [4–6] using the Hamiltonian perturbation methods, but the coefficients they derived are not in agreement. For gravity–capillary waves on shallow water with Bond number close to 1/3, [7] have shown that the model equation is (1) with $c_1 = c_2 = c_3 = 0$. We call this equation the fifth-order KdV equation in this article. If $c_1 = 10$, $c_2 = 20$, $c_3 = 30$, Equation (1) is the fifth-order KdV hierarchy equation. We call it the KdV5 equation hereafter. If $c_1 = 15$, $c_2 = 15$, $c_3 = 45$, the equation is the Sawada–Kotera (also called Caudrey–Dodd–Gibbon) equation. Both these equations are exactly solvable by the inverse scattering method.

In the past 10 years, there has been an intensive effort on the study of the fifth-order KdV equation. The works of [8–15] have shown that this equation does not allow true solitary waves for any positive velocity. Instead, only nonlocal solitary waves with exponentially small tails can be found. The reason is partially that the fifth-order derivative perturbation brings all positive wave speeds into the continuous spectrum of the linear system. Thus, the solution at infinity is generally a continuous wave (cw) instead of a decaying tail. These results imply that an initially localized gravity–capillary wave on shallow water will shed exponentially small tail radiation and asymptotically decay. On the other hand, recent works [16–22] have shown that embedded solitons can exist inside the continuous spectrum for many wave systems, including the extended KdV Equation (1) for irrotational gravity waves [21].

These results suggest that gravity waves in the form of an embedded soliton *can* travel down a narrow canal without emission of tail radiation. This prediction is in striking contrast with the previous one from the fifth-order KdV equation for gravity–capillary waves. Of course, for this new prediction to be physically meaningful, the embedded solitons must be stable in some sense. This problem remains to be properly addressed. In this article, we investigate the dynamics of embedded solitons in the context of the perturbed KdV5 equation

$$u_t + 6uu_x + u_{xxx} + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = \epsilon F(u), \tag{3}$$

where

$$F(u) = -(auu_{xxx} + bu_x u_{xx} + cu^2 u_x) \tag{4}$$

is the perturbation term, $\epsilon \ll 1$, and a, b, c are constant coefficients. Equation (3) is Hamiltonian when $b = 2a$, and is non-Hamiltonian otherwise (see Section 3). This article has three motivations. First, we want to resolve the issue of gravity wave evolution on shallow water in light of the existence of embedded solitons. This is done by examining the evolution of KdV5 solitons under general perturbations. The second is more fundamental. As a generic object, embedded solitons have been identified in a number of wave systems [16–22]. However, for a heuristic semistability argument on single-hump embedded solitons we put forward for the second harmonic generation system [18], the analytical study on the dynamics of embedded solitons is still nonexistent for any wave system. This article presents the first rigorous dynamic theory for embedded solitons under perturbations in the perturbed KdV5 Equation (3). This article has obvious implications to the dynamics of embedded solitons in other wave systems. Our third motivation is to provide an alternative technique for calculating exponentially small tail amplitudes of nonlocal waves. Thus far, only the exponential asymptotics method as outlined in [8, 13] and the one based on the integral equations idea [15] are available, and they are quite complex.

The results of this article can be summarized as follows. Our analysis is based on the development of a soliton perturbation theory for the perturbed KdV5 Equation (3). By obtaining the leading order analytical expression for the tail amplitudes of nonlocal solitary waves in the underlying system, we establish that, under certain conditions on system parameters, single-hump embedded solitons can be found for both Hamiltonian and non-Hamiltonian perturbations. These embedded solitons can be either isolated (existing at a unique velocity) or continuous (existing at an arbitrary velocity). When taking the small wave speed limit, our results show that the tail amplitude is exponentially small, and the product of the tail amplitude and cosine of the phase is a constant. This qualitatively reproduces the previous results on the fifth-order KdV equation by exponential asymptotics techniques [8, 12, 13]. We also prove that, under Hamiltonian perturbations, asymmetric nonlocal solitary waves with small cw tails cannot exist at wave speeds different from the embedded soliton's. For non-Hamiltonian perturbations, asymmetric nonlocal waves do exist in certain parameter regions. Next, we present a comprehensive dynamic theory for embedded solitons. Our approach is to derive the evolution equation for the velocity of KdV5 solitons in the presence of general perturbations. In doing this, we are able to classify all possible soliton evolution scenarios under perturbations. We show that, for Hamiltonian perturbations, an initially localized wave moving faster than the embedded soliton will slow down and asymptotically approach the embedded soliton. However, a solitary wave initially moving slower than the embedded soliton will further slow down and gradually lose all its energy to radiation. Thus, the embedded soliton is semistable. This result is consistent with the heuristic semistability argument we presented in [18] for single-hump embedded solitons in a different Hamiltonian system. For non-Hamiltonian perturbations, we prove that single-hump embedded solitons are stable in certain parameter regimes. To the author's knowledge, this is the first report of stable embedded solitons in the literature. Another result we prove in this article is that, when a KdV5 soliton is under perturbations, the cw tail it sheds selects a phase that makes the tail amplitude minimal. A similar result was pointed out by [23] for the fifth-order KdV equation, but no proof was given there.

The remainder of this article is organized as follows. In Section 2, we examine nonlocal solitary waves in the perturbed KdV5 Equation (3). In Section 3, we study the dynamics of embedded solitons in Equation (3) under general perturbations. Section 4 presents some discussions.

2. Nonlocal solitary waves and embedded solitons in the perturbed KdV5 equation

In this section, we study nonlocal solitary waves in the perturbed KdV5 Equation (3) and analytically establish the existence of embedded solitons. For this purpose, we substitute the traveling wave solution

$$u(x,t) = u(x - Ct) \equiv u(\bar{x}) \tag{5}$$

into (3), where $C(>0)$ is the wave speed. Integrating once, and dropping the bars, we obtain the ordinary differential equation for $u(x)$:

$$u_{xxxx} + u_{xx} - Cu + 3u^2 + 10uu_{xx} + 5u_x^2 + 10u^3 = \epsilon G(u), \tag{6}$$

where

$$G(u) = -\left\{auu_{xx} + \frac{1}{2}(b-a)u_x^2 + \frac{1}{3}cu^3\right\}. \tag{7}$$

Note that without rescaling (2), the wave speed would be C/δ^2. Thus, the limit $\delta \to 0$ studied in previous works [8, 12] is equivalent to the limit $C \to 0$ here. This fact is used later in this paper. We also note that the continuous spectrum of the linear unperturbed equation of (6) is $C \in [-1/4, \infty)$. Thus, all positive wave speeds lie inside the continuous spectrum. One more remark is that $F(u) = G_x(u)$. When $\epsilon = 0$, Equation (3) has a one-parameter family of solitons

$$u_0(x) = \frac{1}{2}k^2\text{sech}^2\frac{kx}{2}, \tag{8}$$

where $C = k^2 + k^4$, and $k > 0$. Here, k (or C) is a free parameter. These solitons interact with each other elastically in the KdV5 model. When $0 \neq \epsilon \ll 1$, these solitons in general will bifurcate into nonlocal solitary waves. We study this bifurcation below and analytically determine the tail amplitudes of those nonlocal waves. On the basis of these results, the existence of embedded solitons will be established.

For $0 \neq \epsilon \ll 1$, we expand the solution u of Equation (6) into a regular perturbation series:

$$u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots. \tag{9}$$

At order ϵ, the equation for u_1 is

$$Lu_1 = G(u_0), \tag{10}$$

where the linear operator L is

$$L = \frac{d^4}{dx^4} + \frac{d^2}{dx^2} - C + 6u_0 + 10u_{0xx} + 30u_0^2 + 10\frac{d}{dx}\left(u_0\frac{d}{dx}\right), \tag{11}$$

which is self-adjoint. To solve the inhomogeneous Equation (10), we first determine the solutions to the homogeneous equation $L\psi = 0$. This is a fourth-order system, and thus, has four linearly independent solutions. The localized solution is trivial to get: it is $\psi_1 = u_{0x}$, which is antisymmetric. The other solutions are nonlocal and are more difficult to find. By inspection and with the help of Mathematica, we are able to obtain two other bounded solutions

$$\psi_2 = \frac{1}{\frac{p}{k}\left(1+\frac{p^2}{k^2}\right)}\left\{\cos px\left[\frac{2p}{k}\tanh^2\frac{kx}{2} - \left(\frac{p}{k} + \frac{p^3}{k^3}\right)\right]\right.$$
$$\left. - \sin px\left[\tanh^3\frac{kx}{2} - \left(1 + \frac{2p^2}{k^2}\right)\tanh\frac{kx}{2}\right]\right\}, \tag{12}$$

and

$$\psi_3 = -\frac{1}{\frac{p}{k}\left(1+\frac{p^2}{k^2}\right)}\left\{\cos px\left[\tanh^3\frac{kx}{2} - \left(1+\frac{2p^2}{k^2}\right)\tanh\frac{kx}{2}\right]\right.$$
$$\left. + \sin px\left[\frac{2p}{k}\tanh^2\frac{kx}{2} - \left(\frac{p}{k}+\frac{p^3}{k^3}\right)\right]\right\}. \tag{13}$$

Here, $p\,(>1)$ is related to the wave speed C by the equation $C = p^4 - p^2$. Interestingly, these linear modes are the same as those in the linearized KdV equation after rescaling [23]. Notice that ψ_2 is symmetric, and ψ_3 is antisymmetric. At infinity, the asymptotic behaviors of these solutions are

$$\psi_2 \longrightarrow \pm\sin(px \pm \phi_2), \qquad x \to \pm\infty, \tag{14}$$

$$\psi_3 \longrightarrow \sin(px \pm \phi_3), \qquad x \to \pm\infty, \tag{15}$$

where

$$\tan\phi_2 = \frac{1 - p^2/k^2}{2p/k}, \qquad \tan\phi_3 = \frac{2p/k}{1 - p^2/k^2}. \tag{16}$$

As $C \to 0^+$, $\phi_2 \to -\pi/2$, and $\phi_3 \to 0$. As $C \to \infty$, $\phi_2 \to 0$, and $\phi_3 \to \pi/2$. From Equation (16) and the above limiting values, it is easy to see that

$$\phi_3 - \phi_2 = \frac{\pi}{2}. \tag{17}$$

This relation is used in the later analysis. The fourth solution ψ_4 is unbounded and symmetric. We did not pursue its exact expression, because it is not needed in this article.

Now we are ready to solve the inhomogeneous Equation (10). If u_1 should be localized, the inhomogeneous term $G(u_0)$ must be orthogonal to the bounded solutions ψ_k $(k = 1, 2, 3)$ of the homogeneous equation, i.e.,

$$\langle G(u_0), \psi_1\rangle = \langle G(u_0), \psi_2\rangle = \langle G(u_0), \psi_3\rangle = 0. \tag{18}$$

And vice versa. Here, the inner product $\langle , \rangle$ is defined as

$$\langle f(x), g(x)\rangle \equiv \int_{-\infty}^{\infty} f(x)g(x)dx. \tag{19}$$

From Equations (7) and (8), we see that $G(u_0)$ is symmetric. Thus, it is automatically orthogonal to ψ_1 and ψ_3. However, it is not orthogonal to ψ_2 in general. In fact, using complex integration technique, and with the help of Mathematica, we have found that

$$\langle G(u_0), \psi_2\rangle = -\frac{\pi}{45}p(k^2+p^2)\{(14a+2b-4c)k^2$$
$$+ (6a+3b-c)p^2\}\frac{e^{-\pi p/k}}{1-e^{-2\pi p/k}}, \tag{20}$$

which is, indeed, nonzero in general. Thus, the u_1 solution is nonlocal, so is $u(x)$. In some special cases, $\langle G(u_0), \psi_2\rangle$ does vanish, then embedded solitons will be found. Next, we construct symmetric nonlocal solitary waves and embedded solitons, and prove the existence and nonexistence of asymmetric nonlocal waves.

2.1. *Symmetric nonlocal solitary waves*

In this section, we construct symmetric nonlocal solitary waves in Equation (6). At infinity, these waves have the following asymptotic behaviors:

$$u_1(x) \longrightarrow \pm R\sin(px \pm \phi), \qquad x \longrightarrow \pm\infty. \tag{21}$$

Here, R and ϕ are the tail amplitude and phase of the nonlocal wave. Notice that any multiple of ψ_2 can be added to u_1, and the new function still remains a symmetric nonlocal solution of Equation (10). Thus, there must be a free parameter in the u_1 solution. In (21), there are two parameters, amplitude R and phase ϕ. Below we show that only one of them is a free parameter. The other parameter is related to the free one by a simple equation.

For this purpose, we go back to Equation (10). Recall that the operator L is self-adjoint. Thus, we find that

$$\begin{aligned}\langle G(u_0), \psi_2\rangle = (&u_{1xxx}\psi_2 - u_{1xx}\psi_{2x} + u_{1x}\psi_{2xx}\\ &- u_1\psi_{2xxx} + u_{1x}\psi_2 - u_1\psi_{2x})|_{-\infty}^{\infty}.\end{aligned} \tag{22}$$

When the asymptotic behaviors of u_1 and ψ_2 in (14) and (21) are substituted into the above equation, we conclude that

$$R\sin(\phi - \phi_2) = \frac{\langle G(u_0), \psi_2\rangle}{2p(2p^2 - 1)}. \tag{23}$$

Here, the quantity $\langle G(u_0), \psi_2\rangle$ has been obtained in Equation (20) above. Formula (23) relates the tail amplitude and phase and is an important result of this article. We discuss its implications next.

When $\langle G(u_0), \psi_2\rangle \neq 0$, Equation (23) says that $R \neq 0$. In this case, the solution u_1, and u as well, is truly nonlocal. The minimum amplitude R occurs when $\phi - \phi_2 = \pm\pi/2$; i.e., u_1 and ψ_2 are $\pi/2$ out of phase at infinity. The maximum amplitude occurs when $\phi - \phi_2 = 0$ or $\pm\pi$, where R has a simple pole singularity. In the latter case, u_1 and ψ_2 are in phase or π-out of phase. This is exactly the resonance explanation for the infinite tail amplitude put forward by [10] on numerical grounds. Higher-order corrections to Equation (23) can be systematically constructed by taking the perturbation expansion (9) to higher orders.

To illustrate formula (23), we select a set of system parameters $\epsilon = 1$, $a = 0$, $b = 1$, and $c = 1.5$ for Equation (3). This ϵ is small compared with

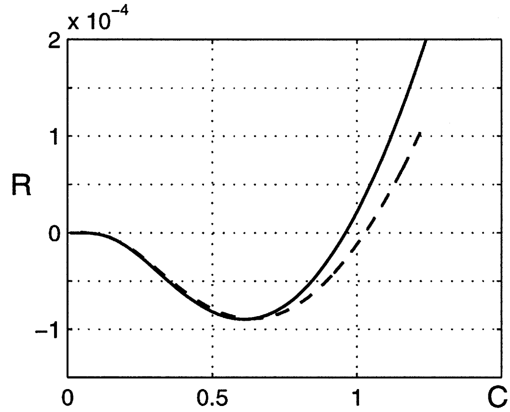

Figure 1. The amplitudes of nonlocal solitary waves in Equation (3) with $\epsilon = 1$, $a = 0$, $b = 1$, $c = 1.5$, and phase $\phi = 0$. The solid curve is obtained from the theoretical formula (23), and the dashed curve is the exact values obtained numerically.

the uu_{xxx}, $u_x u_{xx}$, and $u^2 u_x$ coefficients in the unperturbed Equation (3). We further choose the tail phase $\phi = 0$ in Equation (21). Then the tail amplitude R, as obtained from formula (23), is plotted in Figure 1 (solid curve). Note that because $\epsilon = 1$, this R curve is also the leading-order theoretical prediction for the actual tail amplitudes of nonlocal waves. We see that the magnitude of tail amplitudes is very small (on the order of 10^{-4}). This fact has important implications for the evolution of initially localized waves (see Section 3). A distinctive feature of this curve is that R vanishes at a single wave speed 0.96. This signals an isolated embedded soliton residing among nonlocal solitary waves. To check the accuracy of this theoretical R curve, we have numerically determined the tail amplitudes of nonlocal waves by the shooting method. The results are also shown in Figure 1 (dashed curve) for comparison. We see that the agreement is good for smaller C values and is reasonable for larger C values. The numerical curve crosses the $R = 0$ axis at wave speed 1.028. This is the true location of the embedded soliton. This wave speed differs from the theoretical prediction 0.96 by about 7%.

Formula (23) is obtained for arbitrary values of wave speed C. Interestingly, in the small wave speed C limit [i.e., $\delta \to 0$ in Equation (1)], Equation (23) qualitatively reproduces all major results on nonlocal solitary waves in the fifth-order KdV equation, which were obtained previously by exponential asymptotics techniques [8, 10, 12, 13] (see also [15]). To see this, we note that, as $C \to 0$, $k \to \sqrt{C}$, $p \to 1$, and $\phi_2 \to -\pi/2$. Thus, to leading order, Equation (23) reduces to

$$R \cos[\phi + O(\sqrt{C})] = \left[-\frac{\pi(6a + 3b - c)}{90} + O(\sqrt{C}) \right] e^{-\pi/\sqrt{C}}. \qquad (24)$$

This result indicates that, for small wave speed C, the tail amplitude R is proportional to $e^{-\pi/\sqrt{C}}$, which is exponentially small (see also Figure 1). This reproduces the same result obtained by [8]. Furthermore, the leading order phase-amplitude relation, $R\cos\phi = \text{const}$, is in complete agreement with that established in [12, 15]. Thus, the pole singularity in R occurs at phases $\phi = \pm\pi/2$ to leading order. Even the order of the next correction terms ($\sqrt{C}$) in (24) agrees with that given in [10, 12, 15]. However, some differences do exist. In the numerical work of Boyd [10], it was shown that the pole singularity in the tail amplitude R occurs at phases $\phi \neq \pm\pi/2$. This fact was not captured by [12]'s higher-order calculations using the exponential asymptotics method, because their next order corrections were only on the amplitude but not on the phase. In contrast, this fact is readily explained by our formulas (23) and (24), because these formulas clearly predict the $O(\sqrt{C})$ correction to the location of the pole singularity in R. In this respect, our perturbation theory is more advantageous (we note that this phase correction was also obtained by [15] using the integral equations approach). Another advantage of our theory is that C (or equivalently δ) does not need to be small, whereas it has to be in the other method. From the above comparisons, we conclude that our soliton perturbation method offers a simple and attractive alternative to the previous exponential asymptotics method. However, the exponential asymptotics technique can be applied to a wide variety of problems (see [24, 25]), whereas our method is restricted to perturbations of integrable equations supporting (embedded) solitons. Hence, our method is not a replacement of the other one. However, when our theory is applicable, it will generate much richer results. Last, we remark that the pole singularity in the tail amplitude R is obtained above from the leading order perturbation theory. When R becomes very large (to order ϵ^{-1}), the perturbation result would break down. The recent theoretical analysis by [26] and numerical calculations by [14] for the fifth-order KdV equation show that the actual tail amplitude does not go to infinity for any phase ϕ, although it does become very large at certain phase value if the coefficient of the fifth-order derivative term is small.

2.2. *Embedded solitons*

When $\langle G(u_0), \psi_2\rangle = 0$; i.e.,

$$(14a + 2b - 4c)k^2 + (6a + 3b - c)p^2 = 0, \tag{25}$$

the tail amplitude R in u_1 vanishes according to formula (23). In this case, u_1 is a localized solution, and $u(x)$ is a symmetric solitary wave (to order ϵ). Higher-order corrections in (9) can be systematically calculated. These corrections modify condition (25) slightly (see the end of this subsection) but still keep u localized and symmetric. The resulting solitary wave resides inside the continuous spectrum of Equation (6), and is, thus, an embedded soliton. The name *embedded soliton* was first proposed by the author and collaborators in [18] when we studied second-harmonic generating systems. Such objects have been identified in a number of other wave systems as well ([16–22]). There are two cases when condition (25) holds. The structures of embedded solitons are different in these two cases.

1. A one-parameter family of embedded solitons. This happens when $14a+2b-4c=6a+3b-c=0$, or $(a,b,c)\propto(1,-1,3)$. In this case, $R=0$ for all positive wave speeds C. Thus, an embedded soliton can be obtained for any C value. In principle, this result is valid only up to order ϵ, because formula (23) itself is derived at this order. However, closer examination indicates that, when $(a,b,c)\propto(1,-1,3)$, $G(u_0)$ vanishes for all wave speed C. Thus, the soliton family (8) satisfies Equation (6) for any value of ϵ. Consequently, a one-parameter family of embedded solitons (8) is found here. In the special case where $\epsilon(a,b,c)=(5,-5,15)$, Equation (6) becomes the Sawade–Kotera equation, which is known to possess the family of embedded solitons (8) as well.

2. A unique embedded soliton. When (a,b,c) is not proportional to $(1,-1,3)$, condition (25) can still be satisfied when

$$\frac{p^2}{k^2}=-\frac{14a+2b-4c}{6a+3b-c}, \tag{26}$$

or equivalently,

$$C=C_{\mathrm{ES}}\equiv-\frac{(14a+2b-4c)(6a+3b-c)}{(20a+5b-5c)^2}. \tag{27}$$

This is an isolated embedded soliton moving at a unique wave speed C_{ES} ("ES" in C_{ES} here is an abbreviation for embedded solitons). It exists only when

$$\frac{14a+2b-4c}{6a+3b-c}<-1, \tag{28}$$

or equivalently, c lies between $4a+b$ and $6a+3b$. For convenience, we write this condition as $c\in(4a+b,6a+3b)$. This notation does not imply that $6a+3b$ is larger than $4a+b$, however. Contrary to the first case, formula (27) is, indeed, valid only up to order ϵ (see also Figure 1). Higher-order corrections to C_{ES} can be found when the perturbation expansion (9) is pursued to higher orders (the exact C_{ES} value has been obtained by [16], see below). However, these corrections do not destroy the existence of embedded solitons, because C_{ES} is an isolated simple root of Equation (25) [or equivalently, Equation (20)].

It is important to emphasize that the above perturbation analysis exhausts all possible single-hump embedded solitons in the perturbed KdV5 Equation (3). Under condition $(a, b, c) \propto (1, -1, 3)$, embedded solitons (8) have the exact sech^2 shape. Under the other condition $c \in (4a + b, 6a + 3b)$, the above perturbation analysis indicates only that the shape of this isolated embedded soliton is close to sech^2. One question arises: In the latter case, does this embedded soliton also have the exact sech^2 shape? The answer is yes. Obviously, the reason cannot be found in the framework of the above perturbation theory. It comes when we compare the present result with those in [16], where special embedded solitons of exact sech^2 shape were sought and classified. In our notation, their results can be reformulated as follows.

The exact sech^2 solitary waves

$$u(x) = h\,\mathrm{sech}^2 \lambda x \tag{29}$$

exist in Equation (6) if the coefficients $\rho = 4\lambda^2$ and $\sigma = -4\lambda^2/h$ satisfy the following three polynomial equations:

$$\rho^2 + \rho - C = 0, \tag{30}$$

$$15\rho\sigma + (30 + \epsilon a + \epsilon b)\rho + 3\sigma + 6 = 0, \tag{31}$$

$$15\sigma^2 + (40 + 2\epsilon a + \epsilon b)\sigma + \frac{2}{3}(30 + \epsilon c) = 0. \tag{32}$$

It can be shown from the above equation that, if $(a, b, c) \propto (1, -1, 3)$, Equation (6) allows the family of embedded solitons (8) for arbitrary wave speed C. This agrees with our results above. More relevant to our question is the case when (a, b, c) is not proportional to $(1, -1, 3)$. In this case, we expand the parameters ρ and σ into regular perturbation series in ϵ. To leading order, we get from Equations (31) and (32) that,

$$\rho = -\frac{6a + 3b - c}{20a + 5b - 5c} + O(\epsilon), \quad \sigma = -2 - \frac{1}{30}(6a + 3b - c)\epsilon + O(\epsilon^2). \tag{33}$$

When these relations are substituted into Equation (30), we find that the exact sech^2 soliton exists only at a single wave speed

$$C = -\frac{(14a + 2b - 4c)(6a + 3b - c)}{(20a + 5b - 5c)^2} + O(\epsilon). \tag{34}$$

This is exactly the wave speed C_{ES} we found perturbatively before. This result is expected, because our perturbation analysis has exhausted all embedded solitons under small ϵ limit. Thus, the exact sech^2 solution studied in [16] must reduce to our perturbation result for $\epsilon \ll 1$, and vice versa. The conclusion is that, in the perturbed KdV5 Equation (3), all single-hump embedded solitons have exact sech^2 shape (29). Suggested by this result, we conjecture that all single-hump embedded solitons in the *general* extended KdV Equation (1) have exact sech^2 shape (29).

2.3. Asymmetric nonlocal solitary waves

Asymmetric nonlocal solitary waves with small cw tails attached to a single-hump core have been examined before by [10] and [27] for the fifth-order KdV equation. From energy flux viewpoint, [27] conjectured that such asymmetric nonlocal waves cannot exist. Boyd [10] numerically tried to obtain such asymmetric waves, but failed to find any. In this subsection, we prove analytically that, in the perturbed KdV5 Equation (3), if the perturbation is Hamiltonian, asymmetric single-hump nonlocal waves with small cw tails cannot exist at wave speeds different from the embedded soliton's. Under non-Hamiltonian perturbations, such asymmetric waves *can* exist in certain parameter regions. Note that the fifth-order KdV equation is Hamiltonian, and it does not support embedded solitons (see [10]). Thus, if we extrapolate our results to that equation, then the nonexistence of asymmetric nonlocal waves in that equation would be established theoretically.

We start by revisiting the first-order perturbation solution u_1 in Equation (10). In the previous two subsections, we focused on symmetric nonlocal solutions. In the general case, the u_1 solution is

$$u_1(x) = u_{1s}(x) + \alpha\psi_3(x), \tag{35}$$

where u_{1s} is the symmetric inhomogeneous solution studied above, ψ_3 is the antisymmetric homogeneous solution given in (13), and α is an arbitrary constant. Notice that u_1 now is asymmetric if $\alpha \neq 0$. Suppose the asymptotic behaviors of solution u_{1s} is given as in (21), then similar to Section 2.1, we can show that R and ϕ are related by Equation (23).

To determine whether the asymmetric solution $u(x)$ with u_1 given in (35) can exist, we calculate the higher-order corrections in the perturbation expansion (9). At order ϵ^2, the equation for u_2 is:

$$Lu_2 = W, \tag{36}$$

where

$$\begin{aligned} W = -\{ & a(u_0u_{1xx} + u_1u_{0xx}) + (b-a)u_{0x}u_{1x} \\ & + cu_0^2u_1 + 3u_1^2 + 10u_1u_{1xx} + 5u_{1x}^2 + 30u_0u_1^2\}. \end{aligned} \tag{37}$$

The solution u_2 must be bounded at infinity. Thus, the inhomogeneous term W in Equation (36) must be orthogonal to the localized homogeneous solution u_{0x}; i.e.,

$$\langle W, u_{0x}\rangle = 0. \tag{38}$$

The inner product $\langle W, u_{0x}\rangle$ can be calculated as follows. Recalling the form (11) of operator L, we have

$$\langle Lu_1, u_{1x}\rangle = \left[u_{1xxx}u_{1x} - \frac{1}{2}u_{1xx}^2 + \frac{1}{2}u_{1x}^2 - \frac{C}{2}u_1^2\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(3u_1^2 + 10u_1u_{1xx} + 5u_{1x}^2 + 30u_0u_1^2)u_{0x}dx. \quad (39)$$

On the other hand, recalling Equation (10), we also have

$$\langle Lu_1, u_{1x}\rangle = \int_{-\infty}^{\infty}\{a(u_0u_{1xx} + u_{0xx}u_1) + (b-a)u_{0x}u_{1x} + cu_0^2u_1 - 3(2a-b)u_{0xx}u_1\}u_{0x}dx. \quad (40)$$

When Equations (39) and (40) are equated, and the u_1 expression (35) and the asymptotic behaviors (15) and (21) of u_{1s} and ψ_3 utilized, we find that

$$\langle W, u_{0x}\rangle = \alpha\left\{2p^2(2p^2-1)R\sin(\phi-\phi_2) - 3(2a-b)\int_{-\infty}^{\infty}u_{0x}u_{0xx}\psi_3dx\right\}. \quad (41)$$

We have determined the integral in the above formula by complex integration and Mathematica. The result is

$$\int_{-\infty}^{\infty}u_{0x}u_{0xx}\psi_3dx = \frac{2\pi}{27}p^2(p^2+k^2)^2\frac{e^{-\pi p/k}}{1-e^{-2\pi p/k}}. \quad (42)$$

Equation (41) is an important result. We discuss its implications on the existence of asymmetric nonlocal solitary waves below.

1. For Hamiltonian perturbations, $b = 2a$. Under such perturbations, if condition (25) does not hold, we have $R\sin(\phi - \phi_2) \neq 0$ [see Equations (20) and (23)]. Then Equation (41) indicates that $\langle W, u_{0x}\rangle$ does not vanish unless $\alpha = 0$. Thus, no asymmetric nonlocal solitary waves will be obtained. Note that Equation (25) is the condition for the existence of embedded solitons. Thus, we conclude that, for Hamiltonian perturbations, at parameter values where embedded solitons do not exist, asymmetric nonlocal solitary waves with small cw tails cannot exist either. Recalling results from Section 2.2, the above statement can be made more specific. When $c \notin (4a + b, 6a + 3b)$, or $c \notin (6a, 12a)$ in view of the Hamiltonian condition $b = 2a$, asymmetric nonlocal solitary waves cannot exist for any wave speed C. When $c \in (6a, 12a)$, such asymmetric waves cannot exist for any $C \neq C_{ES}$, where C_{ES} is given in Equation (27). In the latter case, whether asymmetric waves exist or not at $C = C_{ES}$ needs further investigation.

2. For non-Hamiltonian perturbations, $b \neq 2a$. In this case, we substitute Equations (20), (23), and (42) into Equation (41), and find that

$$\langle W, u_{0x} \rangle = -\frac{\alpha\pi}{45} p^2(p^2+k^2)\{(34a-8b-4c)k^2 + (26a-7b-c)p^2\} \frac{e^{-\pi p/k}}{1-e^{-2\pi p/k}}. \tag{43}$$

To make $\langle W, u_{0x} \rangle$ vanish for $\alpha \neq 0$, we must have

$$(34a-8b-4c)k^2 + (26a-7b-c)p^2 = 0. \tag{44}$$

This condition is satisfied in two cases. First, $34a-8b-4c = 26a-7b-c = 0$; i.e., $(a, b, c) \propto (2, 7, 3)$. In this case, Equation (44) holds for all wave speeds C. Second, $(34a-8b-4c)(26a-7b-c) \neq 0$. In this case, condition (44) is met when

$$\frac{p^2}{k^2} = -\frac{34a-8b-4c}{26a-7b-c}, \tag{45}$$

or equivalently,

$$C = C_{\mathrm{AW}} \equiv -\frac{(34a-8b-4c)(26a-7b-c)}{(60a-15b-5c)^2}. \tag{46}$$

Here, wave speed C_{AW} does not correspond to an embedded soliton, because it does not satisfy condition (25). Instead, it corresponds to an asymmetric nonlocal wave (see below and Section 3). Here, "AW" in C_{AW} is an abbreviation for asymmetric waves. This isolated wave speed exists when

$$\frac{34a-8b-4c}{26a-7b-c} < -1; \tag{47}$$

that is, c lies between $12a-3b$ and $26a-7b$. As before, we denote this condition as $c \in (12a-3b, 26a-7b)$.

When condition (44) does not hold, for $\langle W, u_{0x} \rangle$ to vanish, α must be zero. Thus, no asymmetric nonlocal solitary waves will be found. If (44) does hold, our perturbation analysis above shows that asymmetric nonlocal waves exist up to order ϵ^2. Will they exist up to all orders? The answer is yes, at least in the case when $c \in (12a-3b, 26a-7b)$. The reason is the following. In Section 3, we show that the speed evolution of a KdV5 soliton under non-Hamiltonian perturbations is controlled by Equation (73) to leading order. Moreover, the wave speeds that satisfy condition (44) are the fixed points of that equation. The key observation we make is that, if a fixed point of Equation (73) is isolated and simple [i.e., the fixed point is a simple root of the dC/dT function in (73)], then higher-order corrections to this equation will not destroy this fixed point, but only slightly shift it. When $c \in (12a -$

$3b, 26a - 7b$), the dC/dT function is illustrated in Figure 3(a, b, e, f), where $C_{\rm AW}$ is an isolated and simple fixed point. Thus, this fixed point will persist to all orders of the perturbation expansion [the leading order value is given in (46)]. This point does not correspond to an embedded soliton. Rather, at this speed, the solitary wave will dynamically maintain its speed and core shape, meanwhile exciting upstream a low cw tail of constant amplitude. This cw tail extends to infinity at its group velocity, which is larger than the speed of the core. Thus, this fixed point corresponds to an asymmetric, one-sided nonlocal wave, which does exist under condition $c \in (12a - 3b, 26a - 7b)$. To the author's knowledge, this is the first demonstration of the existence of asymmetric nonlocal waves in the fifth-order KdV-type equations. If one wishes to perturbatively calculate this asymmetric wave, one needs to expand the wave speed $C_{\rm AW}$ into a perturbation series too, because this wave speed needs to be determined as well. In the other case where condition (44) holds, i.e., $(a, b, c) \propto (2, 7, 3)$, the evolution Equation (73) becomes $dC/dT = 0$ for all wave speeds. In this case, when higher-order effects are included, it is possible that the full dC/dT equation does not have fixed points any more. When that happens, we will not find asymmetric nonlocal waves. To settle the existence of asymmetric waves in this case, we need to pursue the perturbation expansion (9) to higher orders, which will not be done in this article.

3. Dynamics of embedded solitons

In this section, we study how the KdV5 solitons evolve under general perturbations [see Equation (3)]. We show that this evolution depends critically on whether or not embedded solitons exist, and whether or not the perturbation is Hamiltonian. Our method is to develop a dynamic soliton perturbation theory for Equation (3). Results from Section 2 are utilized extensively in this section.

When $\epsilon = 0$, Equation (3) is the integrable KdV5 equation, which supports a family of solitons given by (5) and (8). These solitons move at constant speed C and are stationary. When perturbations are imposed ($0 \neq \epsilon \ll 1$), the velocity C of the soliton will change on the slow time scale $T = \epsilon^2 t$. Meanwhile, energy radiation arises as well. In this case, it is appropriate to introduce the spatial coordinates

$$\bar{x} = x - \int_0^t C\, dt, \tag{48}$$

which moves with the soliton. In this frame, with the bars dropped, Equation (3) becomes

$$\begin{aligned} u_t - Cu_x + 6uu_x + u_{xxx} + u_{xxxxx} + 10uu_{xxx} \\ + 20u_x u_{xx} + 30u^2 u_x = \epsilon F(u). \end{aligned} \tag{49}$$

The solution of this equation can be expanded into a perturbation series:

$$u(x,t) = u_0(x,T) + \epsilon u_1(x,t,T) + \epsilon^2 u_2(x,t,T) + \cdots. \tag{50}$$

Here, u_0 is given in Equation (8). Note that $C = C(T)$, thus $k = k(T)$. This is why u_0 depends on slow time T as well. One of the primary objectives of this section is to derive the evolution equation for the wave speed C on time scale T. This will be achieved when we pursue the perturbation expansion (55) to order ϵ^2. We remark here that, in general, when perturbations are imposed, the position as well as the velocity of the soliton would evolve slowly (see [23, 28]). This position evolution can be worked out readily in the present perturbation theory, too. However, this evolution does not affect the velocity evolution [see Equation (62)], nor does it affect the amplitude (shape) of the soliton. Thus, it is not pursued here.

At order ϵ, the equation for u_1 is obtained:

$$u_{1t} + (Lu_1)_x = G_x(u_0). \tag{51}$$

Here, the fact $G_x(u) = F(u)$ has been utilized. In general, the inhomogeneous term in the above equation will continuously excite the cw tails that propagate into the far field. A key observation is that these cw tails appear only ahead of the soliton, not behind it. The reason is that, at infinity, the group velocity of the cw waves in u_1 relative to the moving frame is $d(pC)/dp - C = 2p^2(2p^2-1) > 0$ for all $C > 0$. Thus, at $t \gg 1$,

$$u_1(x,t) \longrightarrow \begin{cases} \widetilde{R}\sin(px + \tilde{\phi}), & x \gg 1, \\ 0, & x \ll -1. \end{cases} \tag{52}$$

The amplitude $\widetilde{R}$ and phase $\tilde{\phi}$ can be obtained as follows. As $t \to \infty$, the solution u_1 approaches a steady state with the same asymptotic behavior (52) at large distances. In this case, the time derivative in Equation (51) can be dropped. Integrating once with respect to x, and utilizing the asymptotic behavior (52), of u_1 as $x \to -\infty$, we find that the steady-state solution satisfies Equation (10) and the boundary condition (52). Recall that L is self-adjoint. Then utilizing Equations (17) and (22) and the asymptotic behaviors (14), (15), and (52), we obtain

$$\langle G(u_0), \psi_2 \rangle = p(2p^2-1)\widetilde{R}\sin(\tilde{\phi} - \phi_2), \tag{53}$$

$$\langle G(u_0), \psi_3 \rangle = p(2p^2-1)\widetilde{R}\sin(\tilde{\phi} - \phi_3) = 0. \tag{54}$$

Thus, the amplitude $\widetilde{R}$ and phase $\tilde{\phi}$ are found to be:

$$\widetilde{R} = \frac{\langle G(u_0), \psi_2 \rangle}{p(2p^2-1)}, \qquad \tilde{\phi} = \phi_3. \tag{55}$$

Here $\langle G(u_0), \psi_2 \rangle$ has been given in Equation (20) before. Note that the other solution to Equations (53) and (54) is $\tilde{\phi} = \phi_3 \pm \pi$, and $\widetilde{R}$ is negative of that given in (55). However, this solution results in the same tail behavior (52) as (55) does, and thus can be dropped.

Equation (56) is another important result of this article. First, $\tilde{\phi} = \phi_3$ says that the phase of the cw tail ahead of the soliton is the same as the phase of the antisymmetric homogeneous solution at infinity. At this phase, $\widetilde{R}$ is minimal [see Equations (17) and (53)]. Thus, the soliton, when perturbed, sheds minimum cw tail radiation possible. This is a nontrivial and stimulating result. It has been observed by [29] before for the fifth-order KdV equation, but no proof was given there. Second, the tail amplitude $\widetilde{R}$ of the one-sided nonlocal solitary wave is twice that of the symmetric nonlocal wave with the same phase [see Equations (23) and (53)]. This fact is more or less obvious (see also [8, 29]). The reason is that we can decompose the steady-state solution u_1 as

$$u_1(x) = u_{1s}(x) + \frac{1}{2}\widetilde{R}\psi_3(x), \tag{56}$$

where u_{1s} is the symmetric component and has the asymptotic behavior

$$u_{1s}(x) \longrightarrow \pm\frac{1}{2}\widetilde{R}\sin(px \pm \phi_3), \qquad x \longrightarrow \pm\infty. \tag{57}$$

Thus, the tail amplitude $\widetilde{R}$ of u_1 is, indeed, twice that of the symmetric nonlocal wave u_{1s}. The u_1 decomposition (56) is used later in this article.

With the first-order solution u_1 fully determined, we proceed to order ϵ^2. The equation for u_2 is

$$u_{2t} + (Lu_2)_x = W_x - u_{0T}, \tag{58}$$

where the expression for W is given in Equation (37). From this equation, we find that

$$\langle u_2, u_0 \rangle_t = \langle W_x - u_{0T}, u_0 \rangle. \tag{59}$$

To suppress secular growth in u_2, we must require that

$$\langle W_x - u_{0T}, u_0 \rangle = 0. \tag{60}$$

This equation leads directly to the evolution equation for the soliton wavespeed C:

$$\frac{dC}{dT} = -\frac{2(2k^2+1)}{k}\langle W, u_{0x} \rangle. \tag{61}$$

Here, the relation $C = k^2 + k^4$ has been utilized. The inner product $\langle W, u_{0x} \rangle$ can be found from Equations (39), (40), (52), and (56). When the result is substituted into Equation (61), we finally obtain the C evolution equation

$$\frac{dC}{dT} = -\frac{2k^2+1}{k}\widetilde{R}\left\{p^2(2p^2-1)\widetilde{R} - 3(2a-b)\int_{-\infty}^{\infty} u_{0x}u_{0xx}\psi_3 dx\right\}. \quad (62)$$

Here, $\widetilde{R}$ and the integral are given by Equations (20), (42), and (55). Equation (62) is the key result of this article. It controls the dynamic evolution of the soliton under general perturbations in Equation (3).

Another way to derive the dynamic Equation (62) is to use conservation laws after the first-order solution u_1 and its frontal cw tail amplitude $\widetilde{R}$ have been obtained. Equation (3) always conserves the quantity

$$\int_{-\infty}^{\infty} u\, dx,$$

which can be interpreted as mass. When $b = 2a$, Equation (3) is Hamiltonian. It can be cast in the form

$$u_t = -D_x \delta H, \quad (63)$$

where the Hamiltonian functional H is

$$H(u,\epsilon) = \int_{-\infty}^{\infty}\left\{u^3 - \frac{1}{2}u_x^2 + \frac{1}{2}u_{xx}^2 - \frac{10+\epsilon a}{2}uu_x^2 + \frac{30+\epsilon c}{12}u^4\right\}dx, \quad (64)$$

and δ represents the functional derivative. We obtained this Hamiltonian with the aid of the Mathematica program developed by Goktas and Hereman [30] for the computation of conserved densities. The Hamiltonian can be interpreted as energy, and is also conserved. When the system (3) is Hamiltonian, it conserves one more quantity

$$\int_{-\infty}^{\infty} u^2\, dx,$$

which can be interpreted as momentum. If Equation (3) is not Hamiltonian ($b \neq 2a$), the corresponding momentum and energy conservation laws become

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2 dx = 2(2a-b)\epsilon\int_{-\infty}^{\infty} uu_x u_{xx}\, dx, \quad (65)$$

and

$$\frac{dH}{dt} = (2a-b)\epsilon\int_{-\infty}^{\infty}\left\{u_{xxxx} + u_{xx} + 3u^2 + (10+\epsilon a)uu_{xx} + \frac{10+\epsilon(b-a)}{2}u_x^2 + \frac{30+\epsilon c}{3}u^3\right\}u_x u_{xx} dx. \quad (66)$$

The mass conservation law does not lead to any interesting results on soliton evolution. However, either of the momentum and energy conservation laws (65) and (66) yields the identic dynamic Equation (62) for wave speed C. This is shown below.

We first start from the momentum conservation law (66). When the perturbation expansion (50) is substituted into Equation (65) and terms up to order ϵ^2 retained, we get

$$\frac{d}{dt}\int_{-\infty}^{\infty}\{u_0^2 + 2\epsilon u_0(u_1+\epsilon u_2) + \epsilon^2 u_1^2\}\,dx \\ = 6(2a-b)\epsilon^2\int_{-\infty}^{\infty} u_{0x}u_{0xx}u_1\,dx. \tag{67}$$

Now the following facts need to be utilized. First, u_0^2 changes on time scale $T = \epsilon^2 t$. This can be seen directly from Equation (67). Second, u_1 and u_2 solutions at the soliton core quickly become stationary. Thus, the middle term in the left-hand side of Equation (67) can be dropped. Third, u_1 solution develops at its front side a cw tail whose amplitude $\widetilde{R}$ has been given in Equation (55). This tail, with wavenumber p, moves at its group velocity $C_g = (Cp)_p = 5p^4 - 3p^2$, which is larger than the wave velocity $C\,(=p^4 - p^2)$. From this fact, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_1^2\,dx = \frac{1}{2}\widetilde{R}^2(C_g - C) = p^2(2p^2-1)\widetilde{R}^2. \tag{68}$$

When Equation (68) and the u_1 decomposition (56) are substituted into Equation (67), Equation (62) is then reproduced.

The derivation of Equation (62) from the energy conservation law (66) is quite similar. When the perturbation expansion (50) is substituted into (66), and terms of order ϵ^3 and higher neglected, we get

$$\frac{d}{dt}\left\{H(u_0,0) + \frac{1}{2}\epsilon^2\int_{-\infty}^{\infty}(u_{1xx}^2 - u_{1x}^2)dx\right\} \\ = 3\epsilon^2(2a-b)C\int_{-\infty}^{\infty} u_{0x}u_{0xx}u_1\,dx. \tag{69}$$

Notice that

$$\frac{d}{dt}\int_{-\infty}^{\infty}(u_{1xx}^2 - u_{1x}^2)\,dx = \frac{1}{2}\widetilde{R}^2(p^4-p^2)(C_g - C) = Cp^2(2p^2-1)\widetilde{R}^2, \tag{70}$$

and

$$H(u_0,0) = \frac{1}{35}(5k^7 + 7k^5). \tag{71}$$

When these relations and u_1 decomposition (56) are substituted into Equation (69), Equation (62) is reproduced again.

Last, we study the dynamics of Equation (62), and classify all possible scenarios of KdV5 soliton evolution under general perturbations. The evolution behaviors for Hamiltonian and non-Hamiltonian perturbations are very different, and are treated separately below.

1. Evolution under Hamiltonian perturbations. In this case, $b = 2a$. Thus, when the $\widetilde{R}$ formulas (55) and (20) are substituted into Equation (62), we find that

$$\frac{dC}{dT} = -\frac{2k^2+1}{k(2p^2-1)}\left[\frac{\pi}{45}p(p^2+k^2)\frac{e^{-\pi p/k}}{1-e^{-2\pi p/k}}\right]^2 \times \left[(14a+2b-4c)k^2+(6a+3b-c)p^2\right]^2. \tag{72}$$

This equation has a unique fixed point $C = C_{\mathrm{ES}}$ when condition (25) is satisfied; i.e., $c \in (6a, 12a)$. At this wave speed, an embedded soliton can be found. In this case, the dC/dT function in (72) is illustrated in Figure 2a. We see that, for initial wave speed $C_0 > C_{\mathrm{ES}}$, C decreases and approaches C_{ES}^+ as $t \to \infty$. Thus, the soliton slows down and asymptotically approaches this unique embedded soliton. However, when $C_0 < C_{\mathrm{ES}}$, C decreases to zero, and the soliton asymptotically decays into radiation. Thus, the fixed point C_{ES} is semistable, as is the embedded soliton. The semistability of single-hump embedded solitons in a Hamiltonian system was first established heuristically by the author and collaborators in the context of the second-harmonic-generating system [18]. There, our argument was based on the fact that the embedded soliton was an isolated member of a family of nonlocal solitary waves. Thus, a generic small perturbation tends to move any embedded soliton over into an adjacent state, which is always a nonlocal solitary wave with a nonzero infinite tail. If the perturbed embedded soliton state has energy less than that of the embedded soliton, the energy lost in an attempt to build

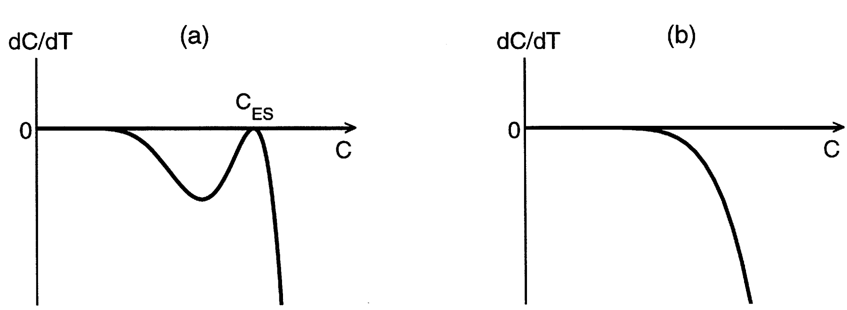

Figure 2. Classification of soliton evolution under Hamiltonian perturbations ($b = 2a$); (a) $c \in (6a, 12a)$; (b) $c \notin (6a, 12a)$.

the infinite tail drives the solution further away from the initial state. As a result, the perturbed embedded soliton can be expected to decay. On the other hand, if the perturbed state has higher energy than the embedded soliton, the energy lost through tail generation drags the pulse back toward the unperturbed embedded soliton. Thus, the embedded soliton under such perturbations would be stable. This heuristic semistability argument makes sense for single-hump embedded solitons, and this was backed up by our numerical simulations [18]. However, it is not a mathematical proof. Our analysis detailed above gives the first rigorous justification of that semistability result in the Hamiltonian Equation (3). This justification puts the heuristic argument of [18] on solid ground. Recall that the heuristic argument of [18] works equally well for any Hamiltonian system, thus the justification of that argument in this article for Equation (3) suggests that semistability of single-hump embedded solitons is a universal property in all Hamiltonian systems. If $C \notin (6a, 12a)$, the dC/dT function is sketched in Figure 2b. In this case, $dC/dT < 0$ for all wave speeds. Thus, for any initial wave speed C_0, $C \to 0$ as $t \to \infty$. In other words, the soliton always decays into radiation. From these results, we see that the existence of an embedded soliton changes the dynamics of soliton evolution drastically. Higher-order corrections to Equation (72) can be systematically determined by carrying out the perturbation expansion (50) to higher orders. However, these corrections will not cause qualitative changes to the soliton evolution behavior. The reason is that, for one thing, these corrections will not destroy the existence of embedded solitons (see Section 2.2). Furthermore, derivation of dC/dT Equation (62) from conservation laws approach indicates that, with higher-order corrections, dC/dT will still be proportional to R^2, where R is the modified tail amplitude and $R = 0$ gives the embedded soliton. Thus, semistability of embedded solitons still holds.

2. Evolution under non-Hamiltonian perturbations. Non-Hamiltonian perturbations also strongly influence the soliton evolution. In this case, $b \neq 2a$. After substituting the $\widetilde{R}$ and integral formulas (55) and (42) into Equation (62), we obtain

$$\begin{aligned}\frac{dC}{dT} = -\frac{2k^2+1}{k(2p^2-1)}&\left[\frac{\pi}{45}p(p^2+k^2)\frac{e^{-\pi p/k}}{1-e^{-2\pi p/k}}\right]^2\\ &\times\left[(14a+2b-4c)k^2+(6a+3b-c)p^2\right]\\ &\times\left[(34a-8b-4c)k^2+(26a-7b-c)p^2\right].\end{aligned} \tag{73}$$

This equation has fixed points when either condition (25) or (44) is satisfied. In the former case, the fixed point corresponds to an embedded soliton. In the latter case, the fixed point does not correspond to an embedded soliton, as with such parameters, the tail amplitude $\widetilde{R} \neq 0$. Instead, the fixed point represents an asymmetric nonlocal wave that consists of a stationary soliton core plus a cw tail of amplitude $\widetilde{R}$ extending to infinity at the front side of the core. From energy point of view, this steady nonlocal asymmetric wave is possible, because the perturbation is non-Hamiltonian. Thus, energy can be injected into the system by the perturbations. Higher-order corrections to Equation (73) can be determined systematically. However, if a fixed point of Equation (73) is isolated and simple, then its existence will not be affected by those higher-order corrections. Only its location will be shifted slightly.

Next, we classify all possible evolution scenarios of KdV5 solitons under non-Hamiltonian perturbations. Here, the conditions (25) and (44) play a critical role.

If $(a, b, c) \propto (1, -1, 3)$, condition (25) is satisfied for all wave speeds. Thus, $dC/dT = 0$. This is not surprising. As we mentioned in Section 2.2, when $(a, b, c) \propto (1, -1, 3)$, the KdV5 solitons (8) remain exact solitary wave solutions of the perturbed system (3). Thus, the wave speed C does not change under perturbations. In this case, one gets a continuous family of embedded solitons. Dynamically, this family of embedded solitons can be viewed as being neutrally stable.

If $(a, b, c) \propto (2, 7, 3)$, condition (44) is satisfied for all wave speeds, and we also have $dC/dT = 0$. This means that a KdV5 soliton of arbitrary initial velocity can still keep its velocity unchanged. Here, the soliton does not remain localized any longer. Instead, the soliton core continuously sheds a cw tail of constant amplitude $\widetilde{R}$ at its front side. This evolution scenario holds up to order ϵ^2, however. At this order, every wave speed can be viewed as a neutral fixed point. When higher-order effects are included, these fixed points may all be destroyed; i.e., the full dC/dT equation may not have any fixed points. In such cases, the asymptotic evolution of KdV5 solitons beyond slow time scale T will be changed qualitatively. To capture this asymptotic evolution, the perturbation calculations need to be carried out to higher orders. This lies beyond the scope of the present article.

When (a, b, c) is not proportional to either $(1, -1, 3)$ or $(2, 7, 3)$, the soliton evolution can be classified into eight more categories.

When $c \in (4a + b, 6a + 3b)$ and $(12a - 3b, 26a - 7b)$, there is a unique root C_{ES} and C_{AW} to Equations (25) and (44), respectively. The expressions for C_{ES} and C_{AW} can be found in (27) and (46). Here, we must have $(26a - 7b - c)(6a + 3b - c) > 0$. The reason is as follows. Because C_{ES} exists, $(14a + 2b - 4c)/(6a + 3b - c) < 0$. We rewrite Equation (44) as

$$\frac{26a - 7b - c}{6a + 3b - c}\frac{p^2}{k^2} + \frac{26a - 7b - c}{6a + 3b - c} + \frac{14a + 2b - 4c}{6a + 3b - c} - 1 = 0. \tag{74}$$

If $(26a-7b-c)(6a+3b-c) < 0$, every term in the above equation is negative. This contradicts our assumption that Equation (44) has a root C_{AW}. Note that both fixed points C_{ES} and C_{AW} here are isolated and simple, thus they will persist to all orders of the perturbation expansion. At the first fixed point, an isolated embedded soliton exists; at the second fixed point, asymmetric nonlocal solitary waves can be found.

Under the above conditions, we have two different cases.

- $(2a-b)(6a+3b-c) > 0$: In this case, we can easily show that $C_{\mathrm{ES}} < C_{\mathrm{AW}}$. The shape of the function dC/dT is shown in Figure 3a. Clearly, the fixed point C_{ES} of Equation (73) is unstable, and the fixed point C_{AW} is stable. If the initial velocity of a soliton $C_0 > C_{\mathrm{ES}}$, then $C \to C_{\mathrm{AW}}$. Otherwise, $C \to 0$. In the former case, the solution approaches a unique asymmetric (one-sided) nonlocal wave moving at speed C_{AW}.
- $(2a-b)(6a+3b-c) < 0$: Now $C_{\mathrm{ES}} > C_{\mathrm{AW}}$, and the dC/dT is shown in Figure 3b. This C_{ES} is then a stable fixed point and C_{AW} an unstable one. In this case, we find an isolated stable embedded soliton at wave speed C_{ES}. Under perturbations, a KdV5 soliton with $C_0 > C_{\mathrm{AW}}$ asymptotically approaches this embedded soliton, and that soliton with $C_0 < C_{\mathrm{AW}}$ decays into radiation. The existence of this stable embedded soliton under non-Hamiltonian perturbations is a novel discovery of this article.

The other six cases for $c \in (4a+b, 6a+3b)$ but $c \notin (12a-3b, 26a-7b)$, $c \notin (4a+b, 6a+3b)$ but $c \in (12a-3b, 26a-7b)$, and $c \notin$ either of $(4a+b, 6a+3b)$ and $(12a-3b, 26a-7b)$ can be classified similarly. Instead of listing them one by one, we summarize all possible evolution scenarios, under both Hamiltonian and non-Hamiltonian perturbations, in Tables 1 and 2 and Figures 2 and 3. Here, we want to remind the reader that $c \in (,)$ in these tables simply means that c lies between the two numbers inside the parentheses. It does not imply that the second number in the parentheses must be larger than the first one. We also remind the reader that the results in these tables and figures are the leading order results in the perturbation theory. When higher-order effects are included, the claims for the $(a, b, c) \propto (1, -1, 3)$ case remain the same; the statements for the $(a, b, c) \propto (2, 7, 3)$ case may be changed qualitatively, and the results for the other cases are qualitatively the same but quantitatively modified slightly.

Although the analysis above gives the correct asymptotic evolution of KdV5 solitons under perturbations, the time scale of the evolution is also important. From Figure 1, we know that the amplitude of the cw tail shed from a KdV5 soliton is very small. This implies that the evolution of a KdV5 soliton into its asymptotic state is very slow. This fact can be seen more clearly from Equation (62): smaller $\widetilde{R}$ leads to slower C change. For the system parameters $\epsilon = 1, a = 0, b = 1$, and $c = 1.5$ (as in Figure 1), we have found from Equation (62) or, equivalently (73), that the magnitude of the dC/dT function is on the order of 10^{-5} (the graph is qualitatively the same as Figure 3d and is not shown here). Thus, although any KdV5 soliton would

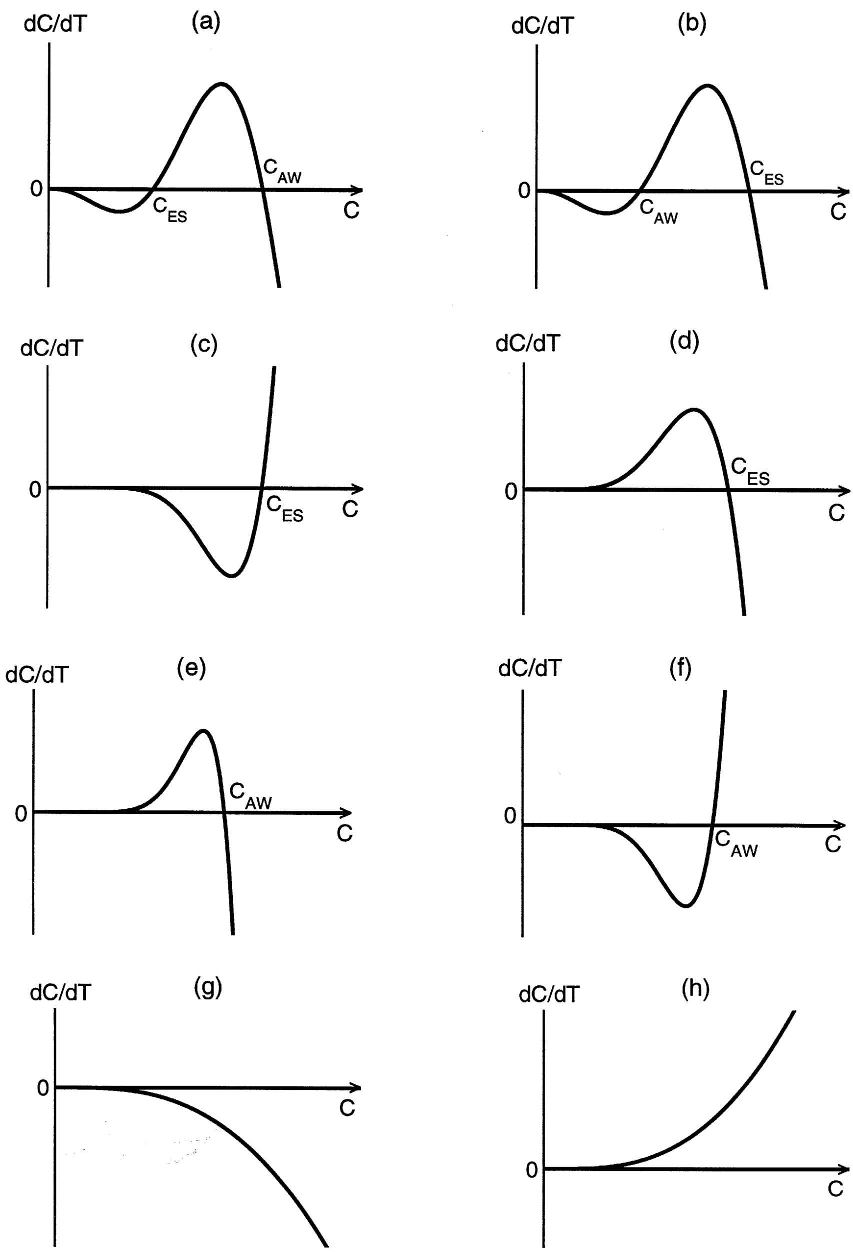

Figure 3. Classification of soliton evolution under non-Hamiltonian perturbations ($b \neq 2a$). The parameter conditions for each graph are listed in Table 2.

Table 1
Classification of Soliton Evolution Under Hamiltonian Perturbations ($b = 2a$)

Parameter Conditions	$\frac{dC}{dT}$ Graph	C_{ES}
$c \in (6a, 12a)$	Figure 2a	Semistable
$c \notin (6a, 12a)$	Figure 2b	

asymptotically approach the unique stable embedded soliton, the time scale of this evolution is on the order of 10^5, which is extremely slow. Qualitatively, this is true for other evolution scenarios as well.

4. Discussion

In this article, we presented the first comprehensive dynamical theory for embedded solitons in the context of the perturbed KdV5 Equation (3). We analytically established that embedded solitons can exist as an isolated solution, or as a continuous family of solutions, inside the continuous spectrum of the linear system. Under small wave speed limit, our results on symmetric nonlocal solitary waves qualitatively reproduced all major results by exponential asymptotics techniques on the fifth-order KdV equation, including the exponential smallness of the tail amplitudes and the amplitude-phase relation. We further studied the dynamics of embedded solitons in detail by classifying all possible scenarios of KdV5 soliton evolution under general perturbations. In particular, we proved that, under Hamiltonian perturbations, embedded solitons are semistable. Stable embedded solitons were found for non-Hamiltonian perturbations.

It is noted that, the embedded solitons studied in this article are all single-humped. In fact, we have shown in Section 2.2 that these embedded solitons all have sech^2 shape. Such solitons are called fundamental in the literature. Multihumped embedded solitons have also been found in a number of Hamiltonian systems by the author and others [14, 17, 18, 21]. In particular, they have been identified in extended Hamiltonian KdV Equations [14, 21]. It is possible that such embedded solitons also exist in the perturbed KdV5 Equation (3). However, evidence shows that multihumped embedded solitons in Hamiltonian systems are linearly unstable [18]. Whether they can be stable in non-Hamiltonian systems such as (3) remains an open question.

Last, we discuss the physical implications of our results to gravity waves on shallow water. The extended KdV Equation (1) has been derived by a number of authors for long gravity waves as an attempt to include higher-order effects into the celebrated KdV equation [1–6]. As we mentioned in

Table 2
Classification of Soliton Evolution Under Non-Hamiltonian Perturbations ($b \neq 2a$)

Parameter Conditions	$\frac{dC}{dT}$ Graph	Features
$(a, b, c) \propto (1, -1, 3)$	0	A family of neutrally stable embedded solitons
$(a, b, c) \propto (2, 7, 3)$	0	A family of asymmetric nonlocal waves

Parameter Conditions		$\frac{dC}{dT}$ Graph	C_{ES}	C_{AW}
$c \in (4a + b, 6a + 3b)$	$(2a - b)(6a + 3b - c) > 0$	Figure 3a	Unstable	Stable
$c \in (12a - 3b, 26a - 7b)$	$(2a - b)(6a + 3b - c) < 0$	Figure 3b	Stable	Unstable
$c \in (4a + b, 6a + 3b)$	$(2a - b)(6a + 3b - c) > 0$	Figure 3c	Unstable	
$c \notin (12a - 3b, 26a - 7b)$	$(2a - b)(6a + 3b - c) < 0$	Figure 3d	Stable	
$c \notin (4a + b, 6a + 3b)$	$(2a - b)(26a - 7b - c) > 0$	Figure 3e		Stable
$c \in (12a - 3b, 26a - 7b)$	$(2a - b)(26a - 7b - c) < 0$	Figure 3f		Unstable
$c \notin (4a + b, 6a + 3b)$	$(6a + 3b - c)(26a - 7b - c) > 0$	Figure 3g		
$c \notin (12a - 3b, 26a - 7b)$	$(6a + 3b - c)(26a - 7b - c) < 0$	Figure 3h		

the introduction, both Hamiltonian and non-Hamiltonian systems have been derived, and the (c_1, c_2, c_3) coefficients in those works are not in agreement. Those coefficients do not fall into the parameter regime of the perturbed KdV5 Equation (3) either. However, the systems derived in [1, 2, 6] and Equation (3) have things in common. It has been shown by [21] that, for the Hamiltonian coefficients obtained in [6], a single-hump embedded soliton exists at a unique wave speed. For the non-Hamiltonian coefficients derived in [1, 2], we have found numerically that an isolated single-hump embedded soliton exists as well (unpublished results). On the basis of our results for the perturbed KdV5 Equation (3), we expect that the single-hump embedded soliton of [21] is semistable. A solitary wave initially moving faster than the embedded soliton will slow down and approach this embedded soliton; whereas, such a wave initially moving slower than the embedded soliton will decay into radiation. For non-Hamiltonian perturbations in [1, 2], we expect the single-hump embedded soliton to be stable or unstable. In the former case, the embedded soliton would be a robust object that may be observable in experiments. Of course, the real water waves are always subject to viscosity-induced damping, which was not accounted for in the perturbed KdV5 Equation (3) and all the extended KdV equations derived in [1, 2, 4–6]. Thus, we need to be cautious when applying our theory in this article to the real system. Nonetheless, we have offered a different picture for gravity wave evolution on shallow water that is in dramatic contrast with the traditional one based on the fifth-order KdV equation. In the traditional picture, an initially localized water wave always decays because of emission of tail radiation. In our new picture, the water wave can approach an embedded soliton state, which moves at a constant, generally unique, speed. The implications of our theory to other embedded-soliton-bearing physical systems, such as the three-wave interaction system [20] and the coupled KdV equations [17], are similar, and are not detailed here.

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