

# General rogue waves in the focusing and defocusing Ablowitz–Ladik equations

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## Abstract

General rogue waves in the focusing and defocusing Ablowitz–Ladik equations are derived by the bilinear method. In the focusing case, it is shown that rogue waves are always bounded. In addition, fundamental rogue waves reach peak amplitudes which are at least three times that of the constant background, and higher-order rogue waves can exhibit patterns such as triads and circular arrays with different individual peaks. In the defocusing case, it is shown that rogue waves also exist. In addition, these waves can blow up to infinity in finite time.

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(Some figures may appear in colour only in the online journal)

## 1. Introduction

Rogue waves are large and spontaneous nonlinear waves which ‘come from nowhere and disappear with no trace’ [1]. These waves have drawn a lot of attention in the nonlinear wave community recently since they are linked to damaging freak waves in the ocean and transient high-intensity optical waves in fibers [2, 3]. Explicit expressions of rogue waves have been derived for a large number of nonlinear integrable systems. Examples include the nonlinear Schrödinger (NLS) equation, [4–13], the derivative NLS equation [14, 15], the three-wave interaction equation [16], the Davey–Stewartson (DS) equations [17, 18], and many others [19–27]. Experimental observations of rogue waves have also been reported in optical fibers and water tanks [28–30].

Almost all rogue-wave solutions reported so far are for continuous wave equations. Discrete wave equations, on the other hand, are also important since they can model various physical systems such as wave dynamics in optical lattices. Then an interesting question

is rogue-wave behaviors in discrete systems [31]. For the focusing Ablowitz–Ladik (AL) equation and discrete Hirota equation, non-traveling fundamental rogue waves and special second-order rogue waves were derived in [19, 32–34]. It was shown that these rogue waves can reach higher peak amplitudes compared to their continuous counterparts.

In this paper, we derive general arbitrary-order rogue waves in the focusing and defocusing AL equations by the bilinear method. These solutions are presented through determinants, and they contain  $2N + 1$  non-reducible free real parameters, where  $N$  is the order of rogue waves (this number of free parameters is more than that in rogue-wave solutions derived before). In the focusing case, we show that rogue waves are always bounded. In addition, fundamental rogue waves reach peak amplitudes which are at least three times that of the constant background, and higher-order rogue waves can exhibit patterns such as triangular and circular arrays with different individual peaks. In the defocusing case, we show that rogue waves still appear, which is surprising. In this case, we find that rogue waves of every order can blow up to infinity in finite time, even though non-blowup rogue waves also exist.

## 2. General rogue-wave solutions

The AL equation has two types, the focusing and defocusing ones. The focusing AL equation can be written as [35, 36]

$$i \frac{d}{dt} u_n = (1 + |u_n|^2)(u_{n+1} + u_{n-1}), \tag{1}$$

and the defocusing AL equation is

$$i \frac{d}{dt} u_n = (1 - |u_n|^2)(u_{n+1} + u_{n-1}). \tag{2}$$

Regarding rogue waves in these AL equations, we have the following theorems.

**Theorem 1.** *General  $N$ th order rogue waves in the AL equations (1), (2) are given by*

$$u_n(t) = \frac{\rho}{\sqrt{1 - \rho^2}} \frac{g_n}{f_n} e^{i(\theta n - \omega t)}, \tag{3}$$

where  $\rho$  and  $\theta$  are free real constants,  $\omega = 2 \cos \theta / (1 - \rho^2)$ ,

$$f_n = \tau_n(0), \quad g_n = \tau_n(1) / (1 + \rho)^{2N},$$

$$\tau_n(k) = \det_{1 \leq i, j \leq N} (m_{2i-1, 2j-1}^{(n)}(k)) \Big|_{p=q=1+\rho},$$

$$m_{ij}^{(n)}(k) = A_i B_j m^{(n)}(k),$$

$$m^{(n)}(k) = \frac{1}{pq - 1 + \rho^2} (pq)^n \left( \frac{1 - \rho^2 - q}{1 - 1/p} \right)^k e^{i \left( \frac{1}{pq} - \frac{1}{1 - \rho^2} \right) (q e^{i\theta} - p e^{-i\theta}) t},$$

$$A_i = \sum_{\nu=0}^i \frac{a_\nu}{(i - \nu)!} [(p - 1) \partial_p]^{i-\nu},$$

$$B_j = \sum_{\mu=0}^j \frac{\bar{a}_\mu}{(j - \mu)!} [(q - 1) \partial_q]^{j-\mu},$$

$a_v$  are complex constants, overbar  $\bar{\phantom{x}}$  represents complex conjugation, and

$$a_0 = 1, \quad a_2 = a_4 = \dots = a_{\text{even}} = 0. \tag{4}$$

When  $|\rho| < 1$ , these rogue waves satisfy the focusing AL equation (1); and when  $|\rho| > 1$ , they satisfy the defocusing AL equation (2).

The above expression (3) for rogue waves involves differential operators  $A_i$  and  $B_j$ . A more explicit and purely algebraic expression for these rogue waves (without the use of such differential operators) is presented in the following theorem.

**Theorem 2.** General  $N$ th order rogue waves (3) for AL equations can be rewritten as

$$u_n(t) = (-1)^N \frac{\rho}{\sqrt{1-\rho^2}} \frac{\sigma_n(1, 0)}{\sigma_n(0, 0)} e^{i(\theta n - \omega t)}, \tag{5}$$

where  $\rho$ ,  $\theta$  and  $\omega$  are the same as those in theorem 1,

$$\sigma_n(k, l) = \det_{1 \leq i, j \leq N} \left( \tilde{m}_{2i-1, 2j-1}^{(n)}(k, l) \right), \tag{6}$$

$$\tilde{m}_{ij}^{(n)}(k, l) = \sum_{v=0}^{\min(i, j)} \left( \frac{1-\rho}{1+\rho} \right)^v \Phi_{iv}^{(n)}(k, l) \Psi_{jv}^{(n)}(k, l),$$

$$\Phi_{iv}^{(n)}(k, l) = \frac{1}{2^v} \sum_{\alpha=0}^{i-v} a_\alpha S_{i-v-\alpha}(\mathbf{x} + v\mathbf{s}),$$

$$\Psi_{jv}^{(n)}(k, l) = \frac{1}{2^v} \sum_{\beta=0}^{j-v} \bar{a}_\beta S_{j-v-\beta}(\mathbf{y} + v\mathbf{s}),$$

$a_\alpha$  are complex constants,  $S_v(\mathbf{x})$  are elementary Schur polynomials defined by

$$\sum_{v=0}^{\infty} S_v(\mathbf{x}) \lambda^v = \exp \left( \sum_{v=1}^{\infty} x_v \lambda^v \right),$$

$\mathbf{x} = (x_1, x_2, \dots)$ ,  $\mathbf{y} = (y_1, y_2, \dots)$  and  $\mathbf{s} = (s_1, s_2, \dots)$  are defined by

$$x_v = (n+k)r_v(\rho) + lr_v(1/\rho) - r_v(1) + \rho x/v! + (1-\rho^2)(v+1)r_{v+1}(\rho)y - k\delta_{v1},$$

$$y_v = (n+l)r_v(\rho) + kr_v(1/\rho) - r_v(1) + \rho y/v! + (1-\rho^2)(v+1)r_{v+1}(\rho)x - l\delta_{v1},$$

$$\sum_{v=1}^{\infty} r_v(\rho) \lambda^v = \ln \frac{1+\rho e^\lambda}{1+\rho}, \quad \sum_{v=1}^{\infty} s_v \lambda^v = \ln \left( \frac{2}{\lambda} \tanh \frac{\lambda}{2} \right), \tag{7}$$

$\delta_{v1}$  denotes the Kronecker delta and  $x = ite^{-i\theta}/(1-\rho^2)$ ,  $y = -ite^{i\theta}/(1-\rho^2)$ . The determinant  $\sigma_n(k, l)$  can also be expressed as

$$\sigma_n(k, l) = \sum_{v_1=0}^1 \sum_{v_2=v_1+1}^3 \dots \sum_{v_N=v_{N-1}+1}^{2N-1} \left( \frac{1-\rho}{1+\rho} \right)^{v_1+v_2+\dots+v_N}$$

$$\times \det_{1 \leq i, j \leq N} \left( \Phi_{2i-1, v_j}^{(n)}(k, l) \right) \det_{1 \leq i, j \leq N} \left( \Psi_{2i-1, v_j}^{(n)}(k, l) \right), \tag{8}$$

where we have defined  $\Phi_{iv}^{(n)}(k, l) = \Psi_{jv}^{(n)}(k, l) = 0$  for  $i < v$ .

Regarding boundary conditions of these rogue waves at large times, we have the following theorem.

**Theorem 3.** As  $t \rightarrow \pm\infty$ , solutions  $u_n(t)$  in theorems 1 and 2 approach a constant background,

$$u_n(t) \rightarrow (-1)^N \frac{\rho}{\sqrt{1-\rho^2}} e^{i(\theta n - \omega t)} \tag{9}$$

uniformly for all  $n$  as long as  $\cos \theta \neq 0$ .

This theorem confirms that solutions  $u_n(t)$  in theorems 1 and 2 are indeed rogue waves, i.e., they rise from a constant background and then retreat back to this same background.

Regarding regularity (boundedness) of these rogue waves, we have the following theorem.

**Theorem 4.** General rogue-wave solutions to the focusing AL equation (1) (with  $|\rho| < 1$ ) in theorems 1 and 2 are non-singular for all times.

Proofs of these theorems will be presented in section 4.

**Remark 1.** In these rogue-wave solutions,  $\rho$  controls the background amplitude, and  $\theta$  is the phase gradient across the lattice. Obviously, the value of  $\theta$  can be restricted to  $-\pi < \theta \leq \pi$ . Since the AL equations are invariant with respect to a time shift, we can normalize the imaginary part of  $a_1$  to be zero through a time shift. Then non-reducible free parameters in these  $N$ th order rogue waves are  $\rho, \theta, \text{Re}(a_1)$  and  $a_3, a_5, \dots, a_{2N-1}$ , totaling  $2N + 1$  real parameters. The parameter  $\text{Re}(a_1)$  is equivalent to a shift  $n \rightarrow n - n_0$  in the solution, with  $n_0$  being a real parameter. With this  $n$ -shift, we can set  $a_1 = 0$ . In this case,  $n_0$  becomes a free parameter in the solution instead of  $\text{Re}(a_1)$ . Without loss of generality, one may restrict  $-1/2 < n_0 \leq 1/2$  through a shift of the lattice index  $n$ .

**Remark 2.** The number of irreducible free parameters in these rogue waves of the AL equations is three more than the corresponding number  $2N - 2$  in the NLS equation [12]. The reason is that the NLS equation has three additional invariances which are lacking in the AL equations: the spatial-translation invariance, the Galilean-transformation invariance, and the scaling invariance. These three invariances reduce the number of free parameters in rogue waves of the NLS equation by three, thus it is three less than that in the AL equations. More will be said on this issue in the next section.

**Remark 3.** It was pointed out in [13] that the coefficients  $s_\nu$  in equation (7) are related to Bernoulli numbers  $B_\nu$  as

$$s_\nu = -\frac{2^\nu - 2}{\nu! \nu} B_\nu, \quad (\nu \geq 2), \quad s_1 = 0,$$

where the Bernoulli numbers  $B_\nu$  are defined by

$$\sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} \lambda^\nu = \frac{\lambda}{e^\lambda - 1}.$$

### 3. Dynamics of rogue waves

In this section, we examine dynamics of rogue waves in AL equations.

#### 3.1. Fundamental rogue waves

Fundamental rogue waves are obtained by setting  $N = 1$  in equation (3) or (5). After simple algebra, these rogue waves are

$$u_n(t) = -\frac{\rho}{\sqrt{1-\rho^2}} e^{i(\theta n - \omega t)} \left[ 1 + \frac{2i\rho^2\omega t + (1+\rho)(a_1 - \bar{a}_1) - 1}{\rho^2(1+\rho)^2|R|^2 + \frac{1}{4}(1-\rho^2)} \right],$$

where

$$R = \frac{1}{1 + \rho} \left[ n + i \left( \frac{e^{-i\theta}}{1 + \rho} - \frac{e^{i\theta}}{1 - \rho} \right) t \right] + \frac{\bar{a}_1 - 1/2}{\rho}.$$

After shifts of  $t$ ,  $n$ , and utilizing phase and time-shift invariances of the AL equations, the above fundamental rogue waves can be rewritten as

$$u_n(t) = \frac{\rho}{\sqrt{1 - \rho^2}} e^{i(\theta n - \omega t)} \left[ 1 + \frac{2i\rho^2\omega t - 1}{\rho^2(n + \omega t \tan \theta - n_0)^2 + \rho^4\omega^2 t^2 + \frac{1}{4}(1 - \rho^2)} \right], \quad (10)$$

where  $\rho$ ,  $\theta$  and  $n_0$  are free real parameters. In view of remark 1, we restrict

$$-\pi < \theta \leq \pi, \quad -1/2 < n_0 \leq 1/2$$

in this subsection. From the explicit expression (10), we see that  $|u_n|$  depends on  $n$  only through the combination of  $\rho n$ , thus  $\rho$  controls the spatial width of this rogue wave (smaller  $\rho$  yields broader waves). Of course,  $\rho$  also controls the background amplitude of this rogue wave. This background amplitude is

$$r = \frac{|\rho|}{\sqrt{|1 - \rho^2|}}, \quad (11)$$

as is easily seen from equation (10).

Now we compare this fundamental rogue wave (10) with that reported in [19]. There are two main differences between them. One is that our solution contains one more free parameter  $\theta$  (the phase gradient), whose role will be explained in the later text. The other difference is that our solution yields rogue waves for both focusing and defocusing AL equations, while that in [19] only yields rogue waves for the focusing AL equation.

It is noted that the solution (10) approaches a constant background when  $t \rightarrow \pm\infty$  as long as  $\omega \neq 0$ , i.e.,  $\theta \neq \pm\pi/2$ . If  $\theta = \pm\pi/2$ , then this solution becomes

$$u_n(t) = \frac{\rho}{\sqrt{1 - \rho^2}} e^{\pm i n \pi / 2} \left[ 1 - \frac{1}{\rho^2 \left( n \pm \frac{2}{1 - \rho^2} t - n_0 \right)^2 + \frac{1}{4}(1 - \rho^2)} \right],$$

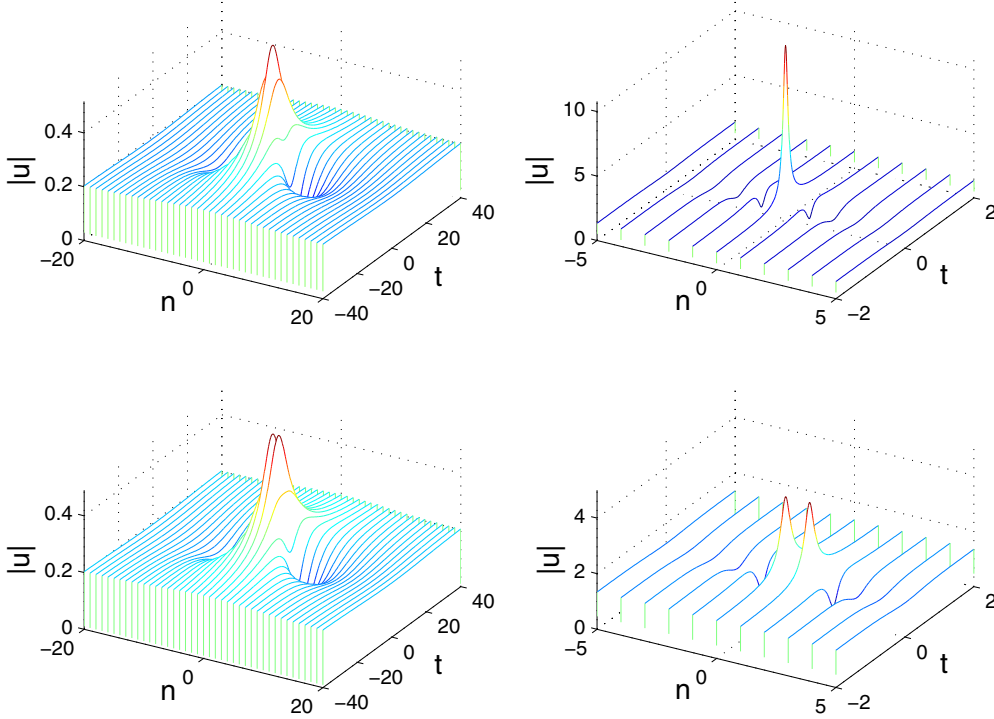
which is a soliton moving on a constant background rather than a rogue wave. For consideration of rogue waves, we will require  $\theta \neq \pm\pi/2$  in the rest of this article.

Dynamics of this rogue wave (10) differs significantly for the focusing and defocusing AL equations (corresponding to  $|\rho| < 1$  and  $|\rho| > 1$  respectively). Thus we will discuss these two cases separately below.

**3.1.1. Focusing case.** In this case,  $|\rho| < 1$ , and equation (10) is the fundamental rogue wave of the focusing AL equation (1). Since  $|\rho| < 1$ , the background amplitude (11) of this rogue wave can be arbitrary, i.e.,  $0 < r < \infty$ . In addition, the denominator in equation (10) is never zero, thus this wave is bounded for all time and lattice sites. It is also seen from equation (10) that  $\theta$  can be viewed as a velocity parameter of this rogue wave, with the velocity being  $-\omega \tan \theta$ , i.e.,  $2 \sin \theta / (\rho^2 - 1)$ . Thus rogue waves with  $\theta = 0, \pi$  can be called non-traveling, and those with other  $\theta$  values called traveling.

Let us first consider non-traveling rogue waves with  $\theta = 0$  (the  $\theta = \pi$  case is very similar). At this  $\theta$  value,  $\omega = \omega_0 \equiv 2/(1 - \rho^2)$ , and the fundamental rogue wave (10) reduces to

$$u_n(t) = \frac{\rho}{\sqrt{1 - \rho^2}} e^{-i\omega_0 t} \left[ 1 + \frac{2i\rho^2\omega_0 t - 1}{\rho^2(n - n_0)^2 + \rho^4\omega_0^2 t^2 + \frac{1}{4}(1 - \rho^2)} \right]. \quad (12)$$



**Figure 1.** Non-traveling fundamental rogue waves (12) in the focusing Ablowitz–Ladik equation. Top row: on-site waves ( $n_0 = 0$ ); bottom row: inter-site waves ( $n_0 = 1/2$ ); left column: broad waves ( $\rho = 0.2$ ); right column: narrow waves ( $\rho = 0.8$ ).

This rogue wave is equivalent to that reported in [19]. The peak amplitude of this rogue wave is reached at  $t = 0$  and the lattice site  $n$  which is closest to the shift parameter  $n_0$ . The highest peak amplitude occurs when  $n_0 = 0$  (or any integer value of  $n_0$ ). In this case, the highest peak amplitude is

$$u_{\max} = \frac{|\rho|}{\sqrt{1 - \rho^2}} \frac{3 + \rho^2}{1 - \rho^2}. \tag{13}$$

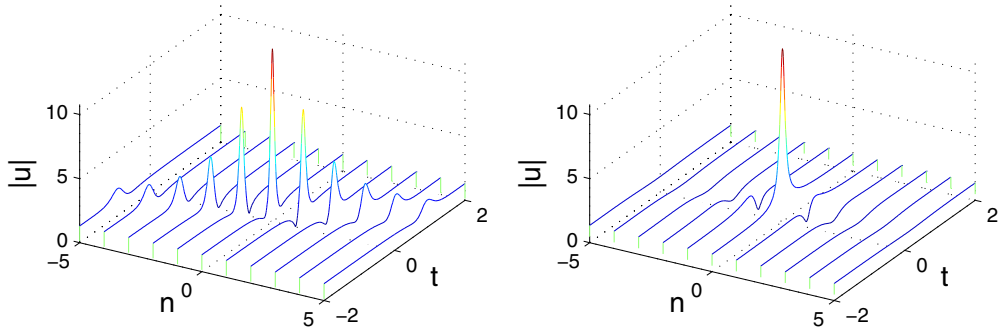
In terms of the background amplitude  $r$  defined in equation (11), this highest peak amplitude is

$$u_{\max} = r(3 + 4r^2). \tag{14}$$

This peak amplitude is at least three times the background amplitude  $r$ , and can be much higher when the background is high. This amplitude is reached at a single lattice site  $n = 0$ , and can be called on-site rogue waves.

The lowest peak amplitude of this rogue wave occurs when  $n_0 = 1/2$ . In this case, the peak amplitude is  $3r$ , which is exactly three times the background. This peak amplitude is reached at two adjacent lattice sites  $n = 0$  and  $n = 1$  simultaneously and can be called inter-site rogue waves.

These non-traveling fundamental rogue waves (12) are illustrated in figure 1. The upper row shows two on-site rogue waves (with  $n_0 = 0$ ), and the lower row shows two inter-site rogue waves (with  $n_0 = 1/2$ ). On the left column,  $\rho = 0.2$ , which is small. On the right column,  $\rho = 0.8$ . We can see from this figure that on-site rogue waves can run much higher than inter-site ones, especially when  $\rho$  is not small (see right column). When  $\rho$  is small, rogue



**Figure 2.** Traveling fundamental rogue waves (10) in the focusing Ablowitz–Ladik equation with  $\rho = 0.8$  and  $n_0 = 0$ . Left:  $\theta = -1.2$ ; right:  $\theta = -0.2$ .

waves are broad (see left column). In this case, the difference between on-site and inter-site waves is less pronounced.

Now we consider traveling rogue waves (10) with  $\theta \neq 0, \pi$ . These rogue waves have not been reported before [19]. Two such solutions, with  $\rho = 0.8, n_0 = 0$  and  $\theta = -1.2, -0.2$  are displayed in figure 2. In the left figure, we see a rogue wave rising from the constant background, traversing across the lattice, and then disappearing into the background again. In the right figure, the traversing motion of the rogue wave is less visible, because this rogue wave rises to its peak amplitude and retreats back to the constant background more quickly.

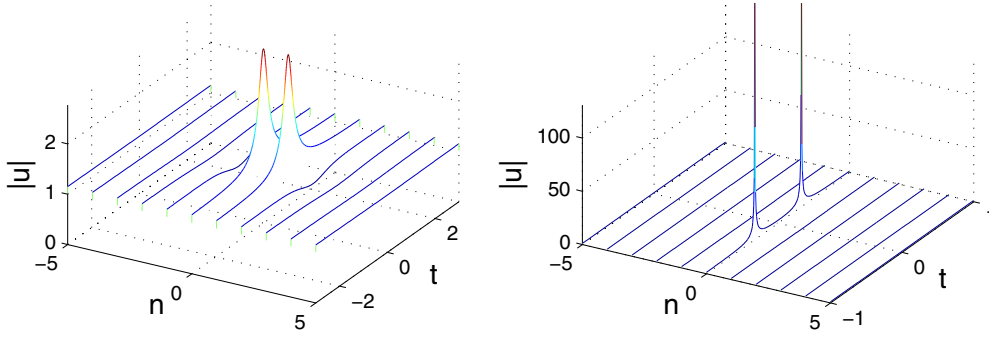
When  $\rho \rightarrow 0$ , fundamental rogue waves (10) become very broad. In this case, the AL equation (1) reduces to the NLS equation, and rogue waves (10) approach the fundamental rogue waves of the NLS equation. In this limit, the parameter  $\theta$  is the counterpart of the moving velocity of NLS rogue waves. The NLS equation admits Galilean invariance, thus any traveling rogue wave can be derived from a non-traveling one through Galilean transformation. However, the AL equation is not Galilean invariant. Because of that,  $\theta$  is a non-reducible parameter in rogue waves of the AL equation.

**3.1.2. Defocusing case.** Now we consider the defocusing case, where  $|\rho| > 1$ , and solution (10) satisfies the defocusing AL equation (2). In this case, solution (10) still approaches the constant background as  $t \rightarrow \pm\infty$ , and rises to higher amplitude in the intermediate times, thus is also a rogue wave. The existence of rogue waves in the defocusing AL equation is surprising. Notice that the background amplitude  $r$  of these rogue waves is always larger than 1 since  $|\rho| > 1$  (see equation (11)). Indeed, we can show that in the defocusing AL equation, rogue waves with background amplitudes less than 1 cannot exist since such backgrounds are modulationally stable (see next subsection).

Rogue waves in the defocusing AL equation exhibit new features that have no counterparts in the focusing AL equation. Since  $|\rho| > 1$ , the denominator in equation (10) may become zero, thus this rogue wave may blow up to infinity in finite time. To illustrate, let us take  $\theta = 0$ . Then from the explicit expression (12), we see that this rogue wave will explode to infinity if

$$|n_0| < \frac{1}{2r}, \tag{15}$$

where  $r$  is the background amplitude in equation (11). When  $n_0 = 1/2$ , this rogue wave will be a regular rogue wave and never blow up since  $r > 1$ . An example is shown in figure 3 (left) with  $\rho = 2$ . This is an inter-site rogue wave, resembling that in figure 1 (lower right panel) of the focusing AL equation. However, if  $n_0 = 0$ , then the rogue wave (12) will always blow up. An example is shown in figure 3 (right) with  $\rho = 2$ . We see that this rogue wave blows up to



**Figure 3.** Fundamental rogue waves (10) in the defocusing Ablowitz–Ladik equation with  $\theta = 0$  and  $\rho = 2$ . Left:  $n_0 = 1/2$ ; right:  $n_0 = 0$ .

infinity at the lattice site  $n = 0$  and times  $t = \pm 3\sqrt{3}/16$ . At other  $n_0$  values of  $0 < |n_0| < 1/2$ , this rogue wave will blow up for background values determined by the condition (15). The  $n_0$ -range for wave blowup shrinks as the background amplitude increases.

One may recall that exploding rogue waves have been reported for the DS II equation before [18]. However, blowup in the DS II equation appears only for second- and higher-order rogue waves, but the blowup here occurs even for fundamental rogue waves (with  $N = 1$ ).

These exploding rogue waves in the defocusing AL equation might also remind the readers of singular rational solutions in the defocusing NLS equation [37, 38]. However, these two types of solutions are fundamentally different for at least two reasons: (a) singular rational solutions of the defocusing NLS equation are singular for all times, but exploding rogue waves of the defocusing AL equation develop singularity only at a certain specific time; (b) exploding rogue waves of the defocusing AL equation have no continuous limit in the defocusing NLS equation, because this continuous limit occurs when  $\rho \rightarrow 0$  (see the end of section 3.1.1), but all rogue waves (3) of the defocusing AL equation have  $|\rho| > 1$ .

Before concluding this subsection, we would like to mention that for the defocusing AL equation truncated to only three lattice sites with fixed boundary conditions, a blowup solution was reported in [39].

**3.1.3. Connection with modulation instability.** Why do rogue waves with background amplitudes higher than 1 exist in the defocusing AL equation? The reason is that such backgrounds are modulationally unstable. This modulation instability is analyzed below.

The defocusing AL equation (2) admits a constant-background solution

$$u_n(t) = r e^{-2i(1-r^2)t}, \tag{16}$$

where  $r$  is the background amplitude. To study the modulation instability of this constant-background solution, we perturb this solution by normal modes

$$u_n(t) = e^{-2i(1-r^2)t} (r + f e^{\lambda t + i\beta n} + \bar{g} e^{\bar{\lambda} t - i\beta n}), \tag{17}$$

where  $\lambda$  and  $\beta$  are the growth rate and wavenumber of the perturbation, and  $f, g \ll 1$ . Substituting this perturbed solution in equation (2) and neglecting terms of higher order in  $f$  and  $g$ , we obtain the following equation for the growth rate  $\lambda$ ,

$$\lambda^2 = 4(r^2 - 1)(1 - \cos \beta)[(r^2 + 1) + (r^2 - 1) \cos \beta]. \tag{18}$$

This formula shows that when the background amplitude  $r > 1$ ,  $\lambda^2$  is positive for all wavenumbers  $\beta$  with  $\cos \beta \neq 1$ , thus this constant background is modulationally unstable.



As a consequence, rogue waves with background amplitudes higher than 1 can exist in the defocusing AL equation (2).

For lower background amplitudes  $0 < r < 1$ , however, the formula (18) shows that  $\lambda^2$  is never positive for any wavenumber  $\beta$ , thus backgrounds lower than 1 are modulationally stable in the defocusing AL equation. Consequently rogue waves with such lower backgrounds cannot exist.

This modulation stability analysis can also be performed for the focusing AL equation (1). In this case, the constant-background solution is

$$u_n(t) = r e^{-2i(1+r^2)t}.$$

Perturbing this solution by normal modes similar to (17) and following similar procedures, we can obtain the following equation for the growth rate  $\lambda$ ,

$$\lambda^2 = 4(r^2 + 1)(1 - \cos \beta)[(r^2 - 1) + (r^2 + 1) \cos \beta].$$

This formula shows that, for any background amplitude  $r$ ,  $\lambda^2$  is positive for wavenumbers  $\beta$  with  $\cos \beta > (1 - r^2)/(1 + r^2)$ ; thus all constant backgrounds in the focusing AL equation are modulationally unstable. This explains why rogue waves with arbitrary constant backgrounds exist in the focusing AL equation (1).

### 3.2. Second-order rogue waves

Now we consider second-order rogue waves in the AL equations. These second-order rogue waves can be obtained from formula (3) or (5) by taking

$$N = 2, \quad a_1 = a_2 = 0,$$

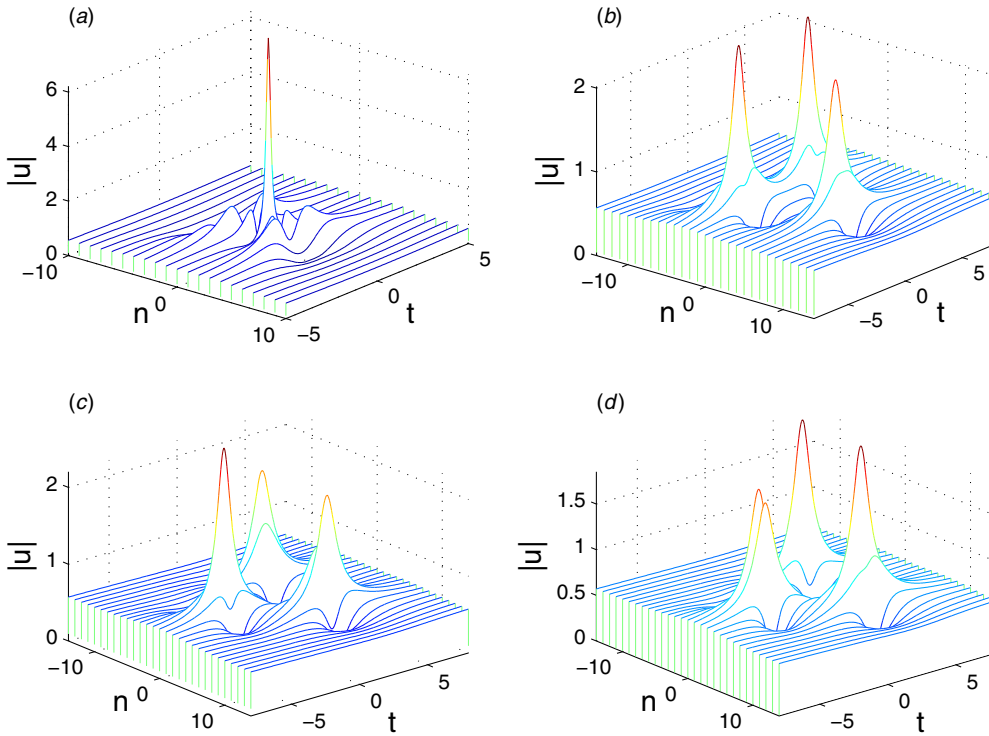
and shifting  $n$  to  $n - n_0$ , with  $n_0$ ,  $\theta$ ,  $\rho$  and  $a_3$  being free parameters. For simplicity, we take  $\theta = 0$  in our discussions below.

**3.2.1. Focusing case.** In this case,  $|\rho| < 1$ . For  $\rho = 1/2$ , four second-order rogue waves are displayed in figure 4 (the  $n_0$  and  $a_3$  parameters are specified in the captions). We see that these second-order rogue waves are all bounded (no blowup). In addition, they can exhibit either a single dominant hump (see panel (a)), or three humps (see panels (b)–(d)), depending on parameters. These behaviors are analogous to second-order rogue waves of the NLS equation [6–13]. However, differences between AL and NLS rogue waves are also apparent. The main difference is that the three humps of the AL rogue waves generally have different heights, while those of the NLS rogue waves generally have the same height. The reason for this difference is that, in the AL rogue waves, some of these three humps are on-site and the others inter-site. On-site humps have higher heights than inter-site ones (see the previous subsection).

Second-order rogue waves in the focusing AL equation have been reported before [19]. Those rogue waves contain only two free real parameters (the counterparts of  $\rho$  and  $n_0$  in this article), thus they are a special class of second-order rogue waves. Due to the lack of the complex free parameter  $a_3$ , those second-order rogue waves in [19] cannot exhibit three-hump structures such as figures 4(b)–(d).

For each given  $|\rho| < 1$ , we have also explored the rogue wave with the highest peak amplitude among all second-order rogue waves with free  $n_0$  and  $a_3$  values. We find that the highest possible peak amplitude is

$$|u|_{\max} = r(5 + 20r^2 + 16r^4), \quad (19)$$



**Figure 4.** Second-order rogue waves in the focusing Ablowitz–Ladik equation for  $\rho = 1/2$  and  $\theta = 0$ . (a)  $n_0 = -3/2, a_3 = -1/54$ ; (b)  $n_0 = -3/2, a_3 = 5/3$ ; (c)  $n_0 = -3/2, a_3 = 5i/3$ ; (d)  $n_0 = 0, a_3 = 5i/3$ .

where  $r = |\rho|/\sqrt{|1 - \rho^2|}$  is the background amplitude (see equation (11)). Interestingly, this highest-amplitude formula is identical to that reported in [19] even though the second-order rogue waves obtained in that work were special.

In this rogue wave with the highest peak amplitude (19), the corresponding  $n_0$  and  $a_3$  values are

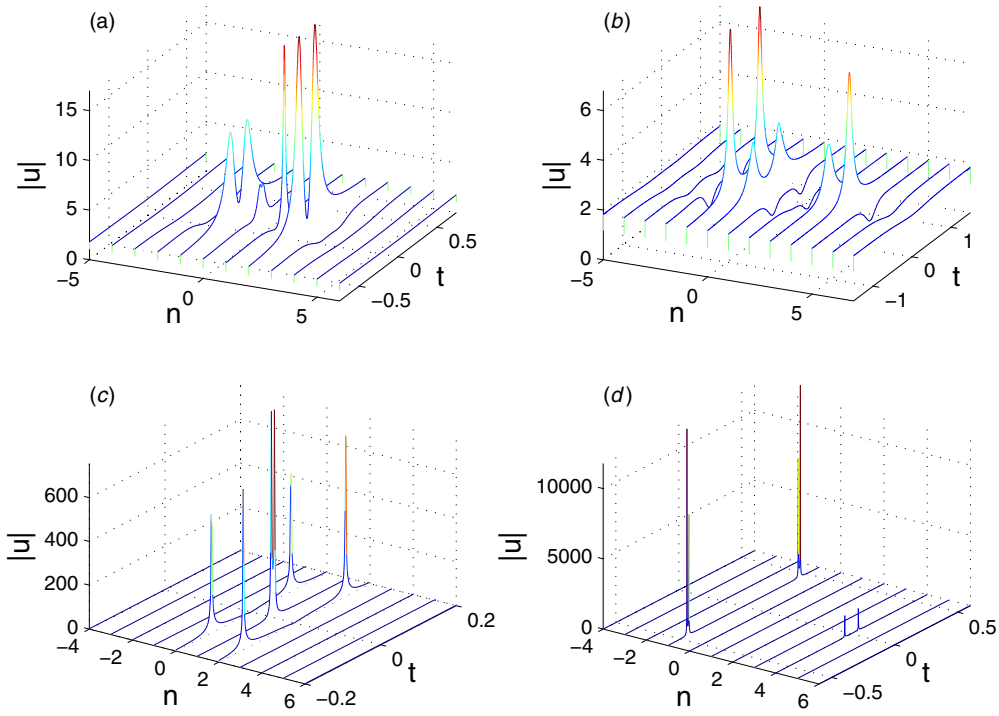
$$n_{0, \max} = -\frac{1 + \rho}{2\rho}, \quad a_{3, \max} = \frac{1}{12} \frac{\rho - 1}{(\rho + 1)^2}, \quad (20)$$

and this peak amplitude occurs at

$$n_{\max} = 0, \quad t_{\max} = 0. \quad (21)$$

For  $\rho = 1/2$ ,  $n_{0, \max} = -3/2$ , and  $a_{3, \max} = -1/54$ . The corresponding rogue wave is as displayed in figure 4(a). This rogue wave reaches peak amplitude  $121/(9\sqrt{3})$ , which is about 13 times higher than the background amplitude  $1/\sqrt{3}$ .

**3.2.2. Defocusing case.** Next we consider second-order rogue waves in the defocusing AL equation, where  $|\rho| > 1$ . For  $\rho = 1.2$ , four of these rogue waves are displayed in figure 5. We see that these second-order rogue waves can be bounded for certain parameter values (see panels (a), (b)). However, for many other parameter values, they blow up in finite time (see panels (c), (d)). This existence of both bounded and exploding second-order rogue waves is similar to that in fundamental rogue waves of the defocusing AL equation (see figure 3).



**Figure 5.** Second-order rogue waves in the defocusing Ablowitz–Ladik equation for  $\rho = 1.2$  and  $\theta = 0$ . Top row: bounded rogue waves; bottom row: exploding rogue waves. The solution parameters are (a)  $a_3 = 0, n_0 = -0.38$ ; (b)  $a_3 = 5/6, n_0 = 0$ ; (c)  $a_3 = 0, n_0 = 0$ ; (d)  $a_3 = 4/3, n_0 = 0$ .

### 3.3. Higher-order rogue waves

Dynamics of third and higher order rogue waves in the AL equations can be studied in a similar way by using the general formula (3) or (5). For instance, we consider third-order rogue waves in the focusing AL equation by taking

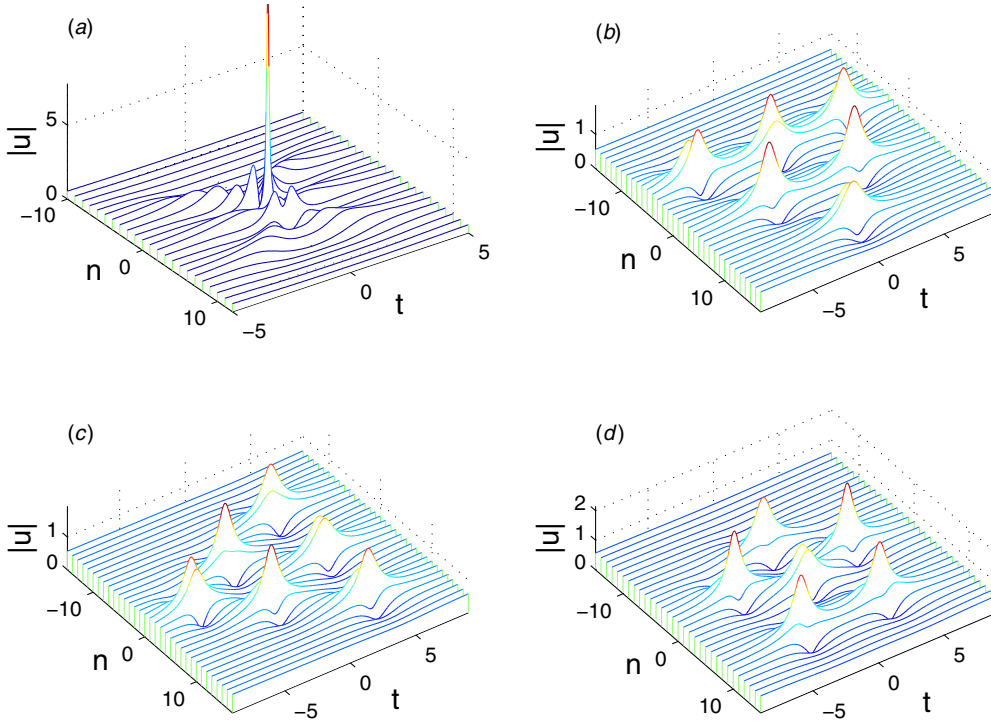
$$N = 3, \theta = 0, \rho = 1/2, a_1 = a_2 = a_4 = n_0 = 0,$$

with  $a_3$  and  $a_5$  as free parameters. For four choices of  $(a_3, a_5)$  values, the corresponding rogue waves are displayed in figure 6. It is seen that this rogue wave can exhibit a single high peak, or six lower peaks, depending on the  $(a_3, a_5)$  values. Notice that the six peaks in figure 6 form triangular or circular patterns, analogous to the NLS equation [10–12]. However, the six peaks here have uneven amplitudes, unlike the NLS equation where the six peaks have almost identical amplitudes.

## 4. Derivation of rogue-wave solutions

In this section, we derive general rogue-wave solutions and prove their boundary and regularity properties in theorems 1–4.

We first establish a few lemmas. In lemma 1 we introduce the so-called Grammian solutions to certain bilinear differential-difference equations, which are relevant to our study. By assuming that the matrix elements obey appropriate dispersion relations, we can show that the determinant (which we call  $\tau$  function) satisfies these bilinear equations. If we choose



**Figure 6.** Third-order rogue waves in the focusing Ablowitz–Ladik equation for  $\rho = 1/2, \theta = 0$  and  $n_0 = 0$ . (a)  $a_3 = a_5 = 0$ ; (b)  $a_3 = 2, a_5 = 0$ ; (c)  $a_3 = 2i, a_5 = 0$ ; (d)  $a_3 = 0, a_5 = 2$ .

suitable matrix elements, this  $\tau$  function gives polynomial solutions, which is explained in lemma 2. The next crucial step is to apply reduction to these polynomial solutions. This reduction is achieved in lemma 3. Then by constraining parameters in the matrix elements, the  $\tau$  function satisfies certain reality and conjugacy conditions, hence its bilinear equations reduce to the AL equations through a variable transformation. Rogue waves in the AL equations then are expressed through this  $\tau$  function.

**Lemma 1.** Let  $m_{ij}^{(n)}, \varphi_i^{(n)}$  and  $\psi_j^{(n)}$  be functions of continuous independent variables  $x, y$  and discrete ones  $k, l$ , satisfying the following differential and difference (dispersion) relations,

$$\begin{aligned} \partial_x m_{ij}^{(n)}(k, l) &= \varphi_i^{(n)}(k, l) \psi_j^{(n-1)}(k, l), \\ \partial_y m_{ij}^{(n)}(k, l) &= \varphi_i^{(n-1)}(k, l) \psi_j^{(n)}(k, l), \\ m_{ij}^{(n+1)}(k, l) &= (1 - \rho^2) m_{ij}^{(n)}(k, l) + \varphi_i^{(n)}(k, l) \psi_j^{(n)}(k, l), \\ m_{ij}^{(n)}(k + 1, l) &= (1 - \rho^2) m_{ij}^{(n)}(k, l) - \varphi_i^{(n-1)}(k + 1, l) \psi_j^{(n)}(k, l), \\ m_{ij}^{(n)}(k, l + 1) &= (1 - \rho^2) m_{ij}^{(n)}(k, l) - \varphi_i^{(n)}(k, l) \psi_j^{(n-1)}(k, l + 1), \\ \partial_x \varphi_i^{(n)}(k, l) &= \varphi_i^{(n+1)}(k, l), \\ \partial_y \varphi_i^{(n)}(k, l) &= -(1 - \rho^2) \varphi_i^{(n-1)}(k, l), \\ \varphi_i^{(n)}(k - 1, l) &= \varphi_i^{(n)}(k, l) - \varphi_i^{(n-1)}(k, l), \\ \varphi_i^{(n)}(k, l + 1) &= (1 - \rho^2) \varphi_i^{(n)}(k, l) - \varphi_i^{(n+1)}(k, l), \end{aligned}$$

$$\begin{aligned}
 (\partial_x + 1)\psi_j^{(n)}(k, l) &= -\psi_j^{(n-1)}(k + 1, l), \\
 (\partial_y - 1)\psi_j^{(n)}(k, l) &= \psi_j^{(n+1)}(k, l - 1), \\
 \psi_j^{(n)}(k + 1, l) &= (1 - \rho^2)\psi_j^{(n)}(k, l) - \psi_j^{(n+1)}(k, l), \\
 \psi_j^{(n)}(k, l - 1) &= \psi_j^{(n)}(k, l) - \psi_j^{(n-1)}(k, l).
 \end{aligned}
 \tag{22}$$

Then the determinant

$$\tau_n(k, l) = \det_{1 \leq i, j \leq N} \left( m_{ij}^{(n)}(k, l) \right)$$

satisfies the bilinear equations

$$\begin{aligned}
 (D_x + 1)\tau_n(k - 1, l) \cdot \tau_n(k, l) &= \tau_{n+1}(k - 1, l)\tau_{n-1}(k, l), \\
 (D_y - 1)\tau_n(k, l + 1) \cdot \tau_n(k, l) &= -\tau_{n-1}(k, l + 1)\tau_{n+1}(k, l), \\
 \tau_{n+1}(k - 1, l)\tau_{n-1}(k, l + 1) - (1 - \rho^2)\tau_n(k - 1, l)\tau_n(k, l + 1) \\
 &= \rho^2\tau_n(k - 1, l + 1)\tau_n(k, l).
 \end{aligned}
 \tag{23}$$

Here  $D$  is Hirota's bilinear differential operator defined by

$$P(D_x, D_y)F(x, y) \cdot G(x, y) = P(\partial_x - \partial_{x'}, \partial_y - \partial_{y'})F(x, y)G(x', y')|_{x'=x, y'=y},$$

and  $P$  is a polynomial of  $D_x$  and  $D_y$ .

**Proof.** By using the dispersion relations (22), we can show that the derivatives and shifts of the  $\tau$  function are expressed by the bordered determinants as follows,

$$\begin{aligned}
 \partial_x \tau_n(k, l) &= \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n)}(k, l) \\ -\psi_j^{(n-1)}(k, l) & 0 \end{vmatrix}, \\
 \partial_y \tau_n(k, l) &= \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) \\ -\psi_j^{(n)}(k, l) & 0 \end{vmatrix}, \\
 \tau_{n+1}(k, l) &= (1 - \rho^2)^{N-1} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n)}(k, l) \\ -\psi_j^{(n)}(k, l) & 1 - \rho^2 \end{vmatrix}, \\
 \tau_{n-1}(k, l) &= \frac{1}{(1 - \rho^2)^N} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) \\ \psi_j^{(n-1)}(k, l) & 1 \end{vmatrix}, \\
 \tau_n(k - 1, l) &= \frac{1}{(1 - \rho^2)^N} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) \\ -\psi_j^{(n)}(k - 1, l) & 1 \end{vmatrix}, \\
 \tau_n(k, l + 1) &= (1 - \rho^2)^{N-1} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n)}(k, l) \\ \psi_j^{(n-1)}(k, l + 1) & 1 - \rho^2 \end{vmatrix}, \\
 (\partial_x + 1)\tau_n(k - 1, l) &= \frac{1}{(1 - \rho^2)^N} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) & \varphi_i^{(n)}(k, l) \\ -\psi_j^{(n)}(k - 1, l) & 1 & 1 \\ -\psi_j^{(n-1)}(k, l) & -1 & 0 \end{vmatrix}, \\
 (\partial_y - 1)\tau_n(k, l + 1) &= (1 - \rho^2)^{N-1} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) \\ \psi_j^{(n-1)}(k, l + 1) & 1 - \rho^2 & 1 \\ -\psi_j^{(n)}(k, l) & 1 - \rho^2 & 0 \end{vmatrix}, \\
 \tau_{n+1}(k - 1, l) &= \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n)}(k, l) \\ -\psi_j^{(n)}(k - 1, l) & 1 \end{vmatrix},
 \end{aligned}$$

$$\tau_{n-1}(k, l + 1) = \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) \\ \psi_j^{(n-1)}(k, l + 1) & 1 \end{vmatrix},$$

$$\tau_n(k - 1, l + 1) = \frac{1}{\rho^2} \begin{vmatrix} m_{ij}^{(n)}(k, l) & \varphi_i^{(n)}(k, l) & \varphi_i^{(n-1)}(k, l) \\ -\psi_j^{(n)}(k - 1, l) & 1 & 1 \\ \psi_j^{(n-1)}(k, l + 1) & 1 - \rho^2 & 1 \end{vmatrix}.$$

By using the Jacobi formula of determinants, we obtain the identities,

$$[(\partial_x + 1)\tau_n(k - 1, l)]\tau_n(k, l) = \tau_n(k - 1, l)\partial_x\tau_n(k, l) + \tau_{n+1}(k - 1, l)\tau_{n-1}(k, l),$$

$$[(\partial_y - 1)\tau_n(k, l + 1)]\tau_n(k, l) = \tau_n(k, l + 1)\partial_y\tau_n(k, l) - \tau_{n-1}(k, l + 1)\tau_{n+1}(k, l),$$

$$\rho^2\tau_n(k - 1, l + 1)\tau_n(k, l) = \tau_{n+1}(k - 1, l)\tau_{n-1}(k, l + 1) - (1 - \rho^2)\tau_n(k - 1, l)\tau_n(k, l + 1),$$

which completes the proof of bilinear equations (23).  $\square$

The above lemma is quite powerful for constructing various types of solutions to bilinear equations (23), since the matrix elements can be any functions satisfying the dispersion relations (22). For example, a class of polynomial solutions can be obtained from it by the choice of matrix elements (see next lemma).

**Lemma 2.** We define matrix elements  $m_{ij}^{(n)}$  by

$$m_{ij}^{(n)}(k, l) = A_i B_j m^{(n)}(k, l),$$

where

$$m^{(n)}(k, l) = \frac{1}{pq - 1 + \rho^2} (pq)^n \left( \frac{1 - \rho^2 - q}{1 - 1/p} \right)^k \left( \frac{1 - \rho^2 - p}{1 - 1/q} \right)^l e^{\xi + \eta},$$

$$\xi = px - \frac{1 - \rho^2}{p}y, \quad \eta = -\frac{1 - \rho^2}{q}x + qy,$$

$A_i$  and  $B_j$  are differential operators with respect to  $p$  and  $q$  respectively, defined as

$$A_i = \sum_{\nu=0}^i \frac{a_\nu}{(i - \nu)!} [(p - 1)\partial_p]^{i-\nu}, \tag{24}$$

$$B_j = \sum_{\mu=0}^j \frac{b_\mu}{(j - \mu)!} [(q - 1)\partial_q]^{j-\mu}, \tag{25}$$

and  $a_\nu, b_\mu$  are constants. Then for any sequences of indices  $I_1, I_2, \dots, I_N$  and  $J_1, J_2, \dots, J_N$ , the determinant,

$$\tau_n(k, l) = \det_{1 \leq i, j \leq N} \left( m_{I_i, J_j}^{(n)}(k, l) \right)$$

satisfies the bilinear equations (23).

**Proof.** It is easy to see that the above  $m^{(n)}(k, l)$  and

$$\varphi^{(n)}(k, l) = p^n (1 - 1/p)^{-k} (1 - \rho^2 - p)^l e^\xi,$$

$$\psi^{(n)}(k, l) = q^n (1 - \rho^2 - q)^k (1 - 1/q)^{-l} e^\eta$$

satisfy the following differential and difference relations,

$$\begin{aligned} \partial_x m^{(n)}(k, l) &= \varphi^{(n)}(k, l)\psi^{(n-1)}(k, l), \\ \partial_y m^{(n)}(k, l) &= \varphi^{(n-1)}(k, l)\psi^{(n)}(k, l), \\ m^{(n+1)}(k, l) &= (1 - \rho^2)m^{(n)}(k, l) + \varphi^{(n)}(k, l)\psi^{(n)}(k, l), \\ m^{(n)}(k + 1, l) &= (1 - \rho^2)m^{(n)}(k, l) - \varphi^{(n-1)}(k + 1, l)\psi^{(n)}(k, l), \\ m^{(n)}(k, l + 1) &= (1 - \rho^2)m^{(n)}(k, l) - \varphi^{(n)}(k, l)\psi^{(n-1)}(k, l + 1), \\ \partial_x \varphi^{(n)}(k, l) &= \varphi^{(n+1)}(k, l), \\ \partial_y \varphi^{(n)}(k, l) &= -(1 - \rho^2)\varphi^{(n-1)}(k, l), \\ \varphi^{(n)}(k - 1, l) &= \varphi^{(n)}(k, l) - \varphi^{(n-1)}(k, l), \\ \varphi^{(n)}(k, l + 1) &= (1 - \rho^2)\varphi^{(n)}(k, l) - \varphi^{(n+1)}(k, l), \\ (\partial_x + 1)\psi^{(n)}(k, l) &= -\psi^{(n-1)}(k + 1, l), \\ (\partial_y - 1)\psi^{(n)}(k, l) &= \psi^{(n+1)}(k, l - 1), \\ \psi^{(n)}(k + 1, l) &= (1 - \rho^2)\psi^{(n)}(k, l) - \psi^{(n+1)}(k, l), \\ \psi^{(n)}(k, l - 1) &= \psi^{(n)}(k, l) - \psi^{(n-1)}(k, l). \end{aligned}$$

Thus

$$m_{ij}^{(n)}(k, l) = A_i B_j m^{(n)}(k, l), \quad \varphi_i^{(n)}(k, l) = A_i \varphi^{(n)}(k, l), \quad \psi_j^{(n)}(k, l) = B_j \psi^{(n)}(k, l)$$

satisfy the dispersion relations (22). Consequently the determinant  $\tau_n(k, l)$  satisfies the bilinear equations (23). This completes the proof.  $\square$

We note that the above  $\tau_n(k, l)$  is not just a polynomial in  $(x, y, k, l, n)$  but a polynomial times the exponential of a linear function, because the elements  $m_{ij}^{(n)}(k, l)$  have the form of  $(i + j)$ -th degree polynomial of  $(x, y, k, l, n)$  times  $(pq)^n [(1 - \rho^2 - q)/(1 - 1/p)]^k [(1 - \rho^2 - p)/(1 - 1/q)]^l e^{\xi + \eta}$ . The bilinear equations (23) are invariant when multiplying an exponential factor of a linear function in  $(x, y, k, l, n)$  to  $\tau_n(k, l)$ . Thus through this gauge invariance, the solutions in lemma 2 are equivalent to polynomial solutions. In this class of polynomials, there is a subclass of solutions which satisfy a certain reduction condition, which is described in the following lemma.

**Lemma 3.** *The determinant*

$$\tau_n(k, l) = \det_{1 \leq i, j \leq N} \left( m_{2i-1, 2j-1}^{(n)}(k, l) \right) \Big|_{p=q=1+\rho}, \tag{26}$$

where  $m_{ij}^{(n)}(k, l)$  is defined in lemma 2, satisfies the reduction condition

$$\tau_n(k + 1, l + 1) = (1 + \rho)^{4N} \tau_n(k, l). \tag{27}$$

**Proof.** We have

$$\begin{aligned} m^{(n)}(k + 1, l + 1) &= \frac{1 - \rho^2 - q}{1 - 1/p} \frac{1 - \rho^2 - p}{1 - 1/q} m^{(n)}(k, l) \\ &= \left( p + \rho^2 + \frac{\rho^2}{p - 1} \right) \left( q + \rho^2 + \frac{\rho^2}{q - 1} \right) m^{(n)}(k, l). \end{aligned}$$

From the general Leibniz rule for high-order derivatives of a product function, we get the operator identity

$$\begin{aligned} [(p - 1)\partial_p]^v \left( p - 1 + 1 + \rho^2 + \frac{\rho^2}{p - 1} \right) \\ = \sum_{\kappa=0}^v \binom{v}{\kappa} \left( p - 1 + \delta_{\kappa 0}(1 + \rho^2) + (-1)^\kappa \frac{\rho^2}{p - 1} \right) [(p - 1)\partial_p]^{v-\kappa}. \end{aligned}$$

Using this identity, we get

$$\begin{aligned}
 A_i \left( p + \rho^2 + \frac{\rho^2}{p-1} \right) &= \sum_{v=0}^i \sum_{\kappa=0}^{i-v} \frac{a_v}{\kappa!(i-v-\kappa)!} \\
 &\quad \times \left( p-1 + \delta_{\kappa 0}(1 + \rho^2) + (-1)^\kappa \frac{\rho^2}{p-1} \right) [(p-1)\partial_p]^{i-v-\kappa} \\
 &= \sum_{\kappa=0}^i \frac{1}{\kappa!} \left( p-1 + \delta_{\kappa 0}(1 + \rho^2) + (-1)^\kappa \frac{\rho^2}{p-1} \right) \\
 &\quad \times \sum_{v=0}^{i-\kappa} \frac{a_v}{(i-v-\kappa)!} [(p-1)\partial_p]^{i-v-\kappa} \\
 &= \sum_{\kappa=0}^i \frac{1}{\kappa!} \left( p-1 + \delta_{\kappa 0}(1 + \rho^2) + (-1)^\kappa \frac{\rho^2}{p-1} \right) A_{i-\kappa},
 \end{aligned}$$

and similarly

$$B_j \left( q + \rho^2 + \frac{\rho^2}{q-1} \right) = \sum_{\lambda=0}^j \frac{1}{\lambda!} \left( q-1 + \delta_{\lambda 0}(1 + \rho^2) + (-1)^\lambda \frac{\rho^2}{q-1} \right) B_{j-\lambda}.$$

Thus the matrix elements satisfy the relation

$$\begin{aligned}
 m_{ij}^{(n)}(k+1, l+1) &= A_i B_j \left( p + \rho^2 + \frac{\rho^2}{p-1} \right) \left( q + \rho^2 + \frac{\rho^2}{q-1} \right) m^{(n)}(k, l) \\
 &= \sum_{\kappa=0}^i \frac{1}{\kappa!} \left( p-1 + \delta_{\kappa 0}(1 + \rho^2) + (-1)^\kappa \frac{\rho^2}{p-1} \right) \\
 &\quad \times \sum_{\lambda=0}^j \frac{1}{\lambda!} \left( q-1 + \delta_{\lambda 0}(1 + \rho^2) + (-1)^\lambda \frac{\rho^2}{q-1} \right) m_{i-\kappa, j-\lambda}^{(n)}(k, l).
 \end{aligned}$$

Substituting  $p = 1 + \rho$  and  $q = 1 + \rho$ , we obtain the contiguity relation

$$\begin{aligned}
 m_{ij}^{(n)}(k+1, l+1) \Big|_{p=q=1+\rho} \\
 = \sum_{\substack{\kappa=0 \\ \kappa:\text{even}}}^i \frac{2\rho + \delta_{\kappa 0}(1 + \rho^2)}{\kappa!} \sum_{\substack{\lambda=0 \\ \lambda:\text{even}}}^j \frac{2\rho + \delta_{\lambda 0}(1 + \rho^2)}{\lambda!} m_{i-\kappa, j-\lambda}^{(n)}(k, l) \Big|_{p=q=1+\rho},
 \end{aligned}$$

from which the following matrix relation is derived:

$$\begin{aligned}
 &\begin{pmatrix} m_{11}^{(n)}(k+1, l+1) & m_{13}^{(n)}(k+1, l+1) & \cdots & m_{1,2N-1}^{(n)}(k+1, l+1) \\ m_{31}^{(n)}(k+1, l+1) & m_{33}^{(n)}(k+1, l+1) & \cdots & m_{3,2N-1}^{(n)}(k+1, l+1) \\ \vdots & \vdots & \ddots & \vdots \\ m_{2N-1,1}^{(n)}(k+1, l+1) & m_{2N-1,3}^{(n)}(k+1, l+1) & \cdots & m_{2N-1,2N-1}^{(n)}(k+1, l+1) \end{pmatrix}_{p=q=1+\rho} \\
 &= \begin{pmatrix} (1+\rho)^2 & 0 & \cdots & 0 \\ \frac{2\rho}{2!} & (1+\rho)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2\rho}{(2N-2)!} & \frac{2\rho}{(2N-4)!} & \cdots & (1+\rho)^2 \end{pmatrix}
 \end{aligned}$$



$$\times \begin{pmatrix} m_{11}^{(n)}(k, l) & m_{13}^{(n)}(k, l) & \cdots & m_{1,2N-1}^{(n)}(k, l) \\ m_{31}^{(n)}(k, l) & m_{33}^{(n)}(k, l) & \cdots & m_{3,2N-1}^{(n)}(k, l) \\ \vdots & \vdots & \ddots & \vdots \\ m_{2N-1,1}^{(n)}(k, l) & m_{2N-1,3}^{(n)}(k, l) & \cdots & m_{2N-1,2N-1}^{(n)}(k, l) \end{pmatrix}_{p=q=1+\rho}$$

$$\times \begin{pmatrix} (1+\rho)^2 & \frac{2\rho}{2!} & \cdots & \frac{2\rho}{(2N-2)!} \\ 0 & (1+\rho)^2 & \cdots & \frac{2\rho}{(2N-4)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1+\rho)^2 \end{pmatrix}.$$

By taking determinant of both sides, the reduction condition (27) is proved. □

**Proof of theorem 1.** Since the  $\tau$  function (26) in lemma 3 satisfies both the bilinear equations (23) and the reduction condition (27), it satisfies all the following bilinear equations,

$$\begin{aligned} (D_x + 1)\tau_n(k, l) \cdot \tau_n(k + 1, l) &= \tau_{n+1}(k, l)\tau_{n-1}(k + 1, l), \\ (D_x + 1)\tau_n(k, l + 1) \cdot \tau_n(k, l) &= \tau_{n+1}(k, l + 1)\tau_{n-1}(k, l), \\ (D_y - 1)\tau_n(k, l + 1) \cdot \tau_n(k, l) &= -\tau_{n-1}(k, l + 1)\tau_{n+1}(k, l), \\ (D_y - 1)\tau_n(k, l) \cdot \tau_n(k + 1, l) &= -\tau_{n-1}(k, l)\tau_{n+1}(k + 1, l), \\ \tau_{n+1}(k, l)\tau_{n-1}(k, l) - (1 - \rho^2)\tau_n(k, l)\tau_n(k, l) &= \frac{\rho^2}{(1 + \rho)^{4N}} \tau_n(k + 1, l)\tau_n(k, l + 1). \end{aligned}$$

We now substitute  $x = ict/(1 - \rho^2)$  and  $y = -idt/(1 - \rho^2)$ , where  $c$  and  $d$  are complex constants. Then the time derivative becomes  $i(1 - \rho^2)\partial_t = -c\partial_x + d\partial_y$ , and we obtain

$$\begin{aligned} [i(1 - \rho^2)D_t + c + d]\tau_n(k + 1, l) \cdot \tau_n(k, l) &= c\tau_{n-1}(k + 1, l)\tau_{n+1}(k, l) + d\tau_{n+1}(k + 1, l)\tau_{n-1}(k, l), \\ [-i(1 - \rho^2)D_t + c + d]\tau_n(k, l + 1) \cdot \tau_n(k, l) &= c\tau_{n+1}(k, l + 1)\tau_{n-1}(k, l) + d\tau_{n-1}(k, l + 1)\tau_{n+1}(k, l), \\ \tau_{n+1}(k, l)\tau_{n-1}(k, l) - (1 - \rho^2)\tau_n(k, l)\tau_n(k, l) &= \frac{\rho^2}{(1 + \rho)^{4N}} \tau_n(k + 1, l)\tau_n(k, l + 1). \end{aligned}$$

The determinant solution (26) is now written as

$$\tau_n(k, l) = \det_{1 \leq i, j \leq N} (A_{2i-1}B_{2j-1}m^{(n)}(k, l)) \Big|_{p=q=1+\rho},$$

where operators  $A_i, B_j$  are defined in equations (24)–(25), and

$$m^{(n)}(k, l) = \frac{1}{pq - 1 + \rho^2} (pq)^n \left( \frac{1 - \rho^2 - q}{1 - 1/p} \right)^k \left( \frac{1 - \rho^2 - p}{1 - 1/q} \right)^l e^{i\left(\frac{1}{pq} - \frac{1}{1-\rho^2}\right)(qd - pc)t}.$$

By taking  $b_\mu = \bar{a}_\mu$  and  $d = \bar{c}$ , the conjugacy condition

$$\tau_n(l, k) = \overline{\tau_n(k, l)}$$

is then satisfied. We now define

$$f_n = \tau_n(0, 0), \quad g_n = \tau_n(1, 0)/(1 + \rho)^{2N}, \tag{28}$$

then  $f_n$  is real,  $\tau_n(0, 1)/(1 + \rho)^{2N} = \bar{g}_n$ , and the above bilinear equations yield

$$\begin{aligned} [i(1 - \rho^2)D_t + c + \bar{c}]g_n \cdot f_n &= cg_{n-1}f_{n+1} + \bar{c}g_{n+1}f_{n-1}, \\ f_{n+1}f_{n-1} - (1 - \rho^2)f_n f_n &= \rho^2 g_n \bar{g}_n. \end{aligned}$$

Finally we set  $c = e^{-i\theta}$ , where  $\theta$  is a real constant. Then through the variable transformation,

$$u_n = \frac{\rho}{\sqrt{1-\rho^2}} \frac{g_n}{f_n} e^{i(\theta n - \omega t)},$$

where  $\omega = (e^{i\theta} + e^{-i\theta})/(1 - \rho^2) = 2 \cos \theta / (1 - \rho^2)$ , the above bilinear equations are transformed to

$$i \frac{d}{dt} u_n = (1 + \sigma |u_n|^2)(u_{n+1} + u_{n-1}),$$

where  $\sigma = \text{sgn}(1 - \rho^2)$ . Thus when  $|\rho| < 1$ , the transformed equation is the focusing AL equation (1), while when  $|\rho| > 1$ , the transformed equation is the defocusing AL equation (2). Then rogue-wave solutions (3) for the focusing and defocusing AL equations are established (it is easy to see that functions  $f_n, g_n$  defined in equation (28) are identical to those given in theorem 1). The selection of parameters (4) can be proved in the same way as that for rogue waves of the NLS equation in [12] and is thus not repeated here.  $\square$

**Proof of theorem 2.** By the reparametrization  $p = 1 + \rho P$  and  $q = 1 + \rho Q$ , the matrix element  $m_{ij}^{(n)}(k, l) = A_i B_j m^{(n)}(k, l)$  in lemma 2 is given by

$$m^{(n)}(k, l) = \frac{(-1)^{k+l} \rho^{k+l-1}}{P + Q + \rho(1 + PQ)} (1 + \rho P)^{n+k} (1 + \rho Q)^{n+l} \left(\frac{1 + Q/\rho}{P}\right)^k \left(\frac{1 + P/\rho}{Q}\right)^l e^{\xi + \eta},$$

$$\xi + \eta = \left(1 + \rho P - \frac{1 - \rho^2}{1 + \rho Q}\right)x + \left(1 + \rho Q - \frac{1 - \rho^2}{1 + \rho P}\right)y,$$

and

$$A_i = \sum_{\nu=0}^i \frac{a_\nu}{(i-\nu)!} (P\partial_P)^{i-\nu}, \quad B_j = \sum_{\mu=0}^j \frac{b_\mu}{(j-\mu)!} (Q\partial_Q)^{j-\mu}.$$

Let us consider the following generator  $\mathcal{G}$  of differential operators  $(P\partial_P)^\alpha (Q\partial_Q)^\beta$ ,

$$\mathcal{G} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{\kappa^\alpha \lambda^\beta}{\alpha! \beta!} (P\partial_P)^\alpha (Q\partial_Q)^\beta = \exp(\kappa P\partial_P + \lambda Q\partial_Q).$$

Recalling the identity

$$\mathcal{G}F(P, Q) = F(e^{\kappa}P, e^{\lambda}Q)$$

for any function  $F$ , applying  $\mathcal{G}$  to the above  $m^{(n)}$  and taking  $P = Q = 1$  (i.e.  $p = q = 1 + \rho$ ), we have

$$\begin{aligned} & \mathcal{G}m^{(n)}(k, l) \Big|_{P=Q=1} \\ &= \frac{(-1)^{k+l} \rho^{k+l-1}}{e^\kappa + e^\lambda + \rho(1 + e^{\kappa+\lambda})} (1 + \rho e^\kappa)^{n+k} (1 + \rho e^\lambda)^{n+l} \left(\frac{1 + e^\lambda/\rho}{e^\kappa}\right)^k \left(\frac{1 + e^\kappa/\rho}{e^\lambda}\right)^l e^{\xi + \tilde{\eta}} \\ &= \frac{(-1)^{k+l} (1 + \rho)^{2(n+k+l)-1} / 2\rho}{1 - \frac{1-\rho}{1+\rho} \frac{1-e^\kappa}{1+e^\kappa} \frac{1-e^\lambda}{1+e^\lambda}} \exp \left[ (n+k) \ln \frac{1 + \rho e^\kappa}{1 + \rho} + (n+l) \ln \frac{1 + \rho e^\lambda}{1 + \rho} \right. \\ & \quad \left. + k \ln \frac{1 + e^\lambda/\rho}{1 + 1/\rho} + l \ln \frac{1 + e^\kappa/\rho}{1 + 1/\rho} - k\kappa - l\lambda - \ln \frac{1 + e^\kappa}{2} - \ln \frac{1 + e^\lambda}{2} + \tilde{\xi} + \tilde{\eta} \right], \end{aligned} \quad (29)$$

where

$$\tilde{\xi} + \tilde{\eta} = \left(1 + \rho e^\kappa - \frac{1 - \rho^2}{1 + \rho e^\lambda}\right)x + \left(1 + \rho e^\lambda - \frac{1 - \rho^2}{1 + \rho e^\kappa}\right)y.$$

Differentiating the first expansion in (7) with respect to  $\lambda$ , we get

$$\frac{\rho e^\lambda}{1 + \rho e^\lambda} = \frac{\rho}{1 + \rho} + \sum_{v=1}^{\infty} (v + 1)r_{v+1}(\rho)\lambda^v,$$

thus  $\tilde{\xi} + \tilde{\eta}$  can be written as

$$\begin{aligned} \tilde{\xi} + \tilde{\eta} &= 2\rho(x + y) + \left(\rho(e^\kappa - 1) + (1 - \rho^2) \sum_{v=1}^{\infty} (v + 1)r_{v+1}(\rho)\lambda^v\right)x \\ &\quad + \left(\rho(e^\lambda - 1) + (1 - \rho^2) \sum_{v=1}^{\infty} (v + 1)r_{v+1}(\rho)\kappa^v\right)y. \end{aligned}$$

Moreover since we have the formal expansion

$$\begin{aligned} &\left[1 - \frac{1 - \rho}{1 + \rho} \frac{1 - e^\kappa}{1 + e^\kappa} \frac{1 - e^\lambda}{1 + e^\lambda}\right]^{-1} \\ &= \sum_{\mu=0}^{\infty} \left[\frac{1 - \rho}{1 + \rho} \frac{\kappa\lambda}{4} \exp\left(\ln\left(\frac{2}{\kappa} \tanh \frac{\kappa}{2}\right) + \ln\left(\frac{2}{\lambda} \tanh \frac{\lambda}{2}\right)\right)\right]^\mu, \end{aligned}$$

equation (29) can be rewritten as

$$\begin{aligned} &\frac{(-1)^{k+l} 2\rho e^{-2\rho(x+y)}}{(1 + \rho)^{2(n+k+l)-1}} \mathcal{G}m^{(n)}(k, l) \Big|_{p=q=1} \\ &= \sum_{\mu=0}^{\infty} \left(\frac{1 - \rho}{1 + \rho} \frac{\kappa\lambda}{4}\right)^\mu \exp\left(\sum_{v=1}^{\infty} (x_v + \mu s_v)\kappa^v + \sum_{v=1}^{\infty} (y_v + \mu s_v)\lambda^v\right) \\ &= \sum_{\mu=0}^{\infty} \left(\frac{1 - \rho}{1 + \rho} \frac{\kappa\lambda}{4}\right)^\mu \sum_{\alpha=0}^{\infty} S_\alpha(\mathbf{x} + \mu\mathbf{s})\kappa^\alpha \sum_{\beta=0}^{\infty} S_\beta(\mathbf{y} + \mu\mathbf{s})\lambda^\beta, \end{aligned}$$

where  $x_v$  and  $y_v$  are as defined in theorem 2. By taking the coefficient of  $\kappa^\alpha \lambda^\beta$ , we obtain

$$\begin{aligned} &\frac{(-1)^{k+l} 2\rho e^{-2\rho(x+y)}}{(1 + \rho)^{2(n+k+l)-1}} \frac{(P\partial_p)^\alpha (Q\partial_Q)^\beta}{\alpha! \beta!} m^{(n)}(k, l) \Big|_{p=q=1} \\ &= \sum_{\mu=0}^{\min(\alpha, \beta)} \frac{1}{4^\mu} \left(\frac{1 - \rho}{1 + \rho}\right)^\mu S_{\alpha-\mu}(\mathbf{x} + \mu\mathbf{s}) S_{\beta-\mu}(\mathbf{y} + \mu\mathbf{s}). \end{aligned}$$

Therefore the matrix element  $m_{ij}^{(n)}(k, l)$  in lemma 2 with  $p = q = 1 + \rho$  is explicitly expressed in the polynomial form,

$$\begin{aligned} &\frac{(-1)^{k+l} 2\rho e^{-2\rho(x+y)}}{(1 + \rho)^{2(n+k+l)-1}} m_{ij}^{(n)}(k, l) \Big|_{p=q=1+\rho} \\ &= \sum_{\alpha=0}^i \sum_{\beta=0}^j \sum_{\mu=0}^{\min(i-\alpha, j-\beta)} \frac{a_\alpha b_\beta}{4^\mu} \left(\frac{1 - \rho}{1 + \rho}\right)^\mu S_{i-\alpha-\mu}(\mathbf{x} + \mu\mathbf{s}) S_{j-\beta-\mu}(\mathbf{y} + \mu\mathbf{s}). \end{aligned}$$

By taking  $x = ite^{-i\theta}/(1 - \rho^2)$ ,  $y = -ite^{i\theta}/(1 - \rho^2)$  and  $b_\beta = \bar{a}_\beta$ , the matrix element  $m_{ij}^{(n)}(k, l)$  from the above equation is equal to  $\tilde{m}_{ij}^{(n)}(k, l)$  in theorem 2, multiplied by a factor which is  $(i, j)$ -independent and is inversely proportional to  $(-1)^k (1 + \rho)^{2k}$ . Recalling the definition of

functions  $f_n, g_n$  in equation (28),  $g_n/f_n$  in theorem 1 is then equal to  $(-1)^N \sigma_n(1, 0)/\sigma_n(0, 0)$ , thus the algebraic expression of rogue waves in equations (5)–(6) of theorem 2 is proved.

The alternative expression (8) for  $\sigma_n(k, l)$  in theorem 2 can be derived directly from the original expression (6) through a similar determinant calculus in [12]. We rewrite the  $N \times N$  determinant  $\sigma_n(k, l)$  in (6) into a  $3N \times 3N$  determinant form, then apply the Laplace expansion, which leads to the expression (8) (see [12] for details). Thus, theorem 2 is proved.  $\square$

**Proof of theorem 3.** In the polynomial solution (8) in  $n$  and  $t$ , the leading term comes from the one with  $v_1 = 0, v_2 = 1, \dots, v_N = N - 1$ , i.e.,

$$[(1 - \rho)/(1 + \rho)]^{N(N-1)/2} \det_{1 \leq i, j \leq N} \left( \Phi_{2i-1, j-1}^{(n)}(k, l) \right) \det_{1 \leq i, j \leq N} \left( \Psi_{2i-1, j-1}^{(n)}(k, l) \right).$$

Notice that the highest-degree terms in  $\Phi_{iv}^{(n)}(k, l)$  and  $\Psi_{jv}^{(n)}(k, l)$  are  $a_0 x_1^{i-v}/(i-v)!2^v$  and  $\bar{a}_0 y_1^{j-v}/(j-v)!2^v$  respectively. In addition, the leading term of  $n, t$  in the product  $x_1 y_1$  is given by  $|r_1(\rho)n + i(e^{-i\theta}\rho/(1-\rho^2) - e^{i\theta}2r_2(\rho))t|^2 = r_1(\rho)^2 |n + i(e^{-i\theta}/(1-\rho) - e^{i\theta}/(1+\rho))t|^2$ . Consequently the leading term of  $n, t$  in the polynomial  $\sigma_n(k, l)$  is proportional to

$$\left| n + i \left( \frac{e^{-i\theta}}{1-\rho} - \frac{e^{i\theta}}{1+\rho} \right) t \right|^{N(N+1)} = \left[ \left( n + \frac{2t \sin \theta}{1-\rho^2} \right)^2 + \left( \frac{2t \rho \cos \theta}{1-\rho^2} \right)^2 \right]^{N(N+1)/2}.$$

If  $\cos \theta \neq 0$ , then this term is dominant as  $n^2 + t^2$  goes to infinity in any direction on the  $(n, t)$ -plane. The coefficient of this dominant term does not vanish and is independent of  $k$  and  $l$  by direct calculation. Thus if  $\cos \theta \neq 0$ ,  $\sigma_n(1, 0)/\sigma_n(0, 0)$  approaches 1 when the space-time point  $(n, t)$  goes to infinity, for example, when  $t$  goes to infinity (for each fixed  $n$ ) or  $n$  goes to infinity (for each fixed  $t$ ). In addition, if  $\cos \theta = 0$ , then  $\sigma_n(1, 0)/\sigma_n(0, 0)$  approaches 1 uniformly in  $n$  as  $|t|$  goes to infinity. Hence the boundary condition (9) in theorem 3 is proved.  $\square$

**Proof of theorem 4.** From theorem 2, we see that  $\Psi_{iv}^{(n)}(k, l)$  is the complex conjugate of  $\Phi_{iv}^{(n)}(l, k)$ . Then from the expression of  $\sigma_n(k, l)$  in equation (8), we have

$$\sigma_n(0, 0) = \sum_{v_1=0}^1 \sum_{v_2=v_1+1}^3 \cdots \sum_{v_N=v_{N-1}+1}^{2N-1} \left( \frac{1-\rho}{1+\rho} \right)^{v_1+v_2+\dots+v_N} \left| \det_{1 \leq i, j \leq N} \left( \Phi_{2i-1, v_j}^{(n)}(0, 0) \right) \right|^2.$$

Clearly  $\sigma_n(0, 0) \geq 0$  if  $-1 < \rho < 1$ . Furthermore the term for  $v_1 = 1, v_2 = 3, \dots, v_N = 2N - 1$  is not zero because  $\Phi_{ii}^{(n)}(k, l) = a_0/2^i$  and  $\Phi_{iv}^{(n)}(k, l) = 0$  ( $i < v$ ). Therefore  $\sigma_n(0, 0)$  is strictly positive for  $-1 < \rho < 1$ . Consequently, rogue waves for the focusing AL equation in theorems 1 and 2 are always non-singular, which completes the proof.  $\square$

Of course, this non-singularity of solutions does not hold for the defocusing AL equation (where  $|\rho| > 1$ ), as has been seen in section 3.

### 5. Summary

In this paper, general  $N$ th order rogue waves in the focusing and defocusing Ablowitz–Ladik (AL) equations were derived by the bilinear method. These solutions were given by determinants, and they contain  $2N + 1$  non-reducible free real parameters (which is more than that in rogue-wave solutions derived before). In the focusing case, we showed that rogue waves are always bounded. In addition, they can reach much higher peak amplitudes than their continuous counterparts in the nonlinear Schrödinger equation. Furthermore, higher-order rogue waves can exhibit triangular and circular patterns with different individual peaks. In the

defocusing case, we showed that rogue waves still appear, which is surprising. In this case, we found that rogue waves of any order can blow up to infinity in finite time, even though non-blowup rogue waves can also exist.

It is noted that solutions in the AL equations are closely related to those in the discrete Hirota equation [19]

$$i \frac{d}{dt} v_n = (1 \pm |v_n|^2)[(a + ib)v_{n+1} + (a - ib)v_{n-1}], \quad (30)$$

where  $a$  and  $b$  are any real constants. When  $a = 0$ , this equation becomes the discrete modified KdV equation; and when  $b = 0$ , it becomes the AL equations. Since we have constructed rogue-wave solutions of the AL equations for arbitrary carrier wave frequency  $\theta$ , it is straightforward to derive rogue-wave solutions in the discrete Hirota equation as well. By writing  $a + ib = Re^{i\Theta}$  with real constants  $R$  and  $\Theta$ , we see that

$$v_n(t) = e^{-in\Theta} u_n(Rt)$$

is a solution of equation (30) when  $u_n(t)$  is a solution of the AL equation (1) or (2). Thus, rogue-wave solutions for the discrete Hirota equation can be obtained directly from theorem 1 or 2. For example, from theorem 2 we get the following theorem. Here for simplifying the expression, the sign  $(-1)^N$  in theorem 2 is dropped and notations  $\phi = \theta - \Theta$ ,  $\Omega = R\omega$  are used.

**Theorem 5.** *General  $N$ th order rogue waves for the discrete Hirota equation (30) are given by*

$$v_n(t) = \frac{\rho}{\sqrt{1 - \rho^2}} \frac{\sigma_n(1, 0)}{\sigma_n(0, 0)} e^{i(\phi n - \Omega t)},$$

where  $\rho$ ,  $\phi$  are free real constants,  $\Omega = [(a + ib)e^{i\phi} + (a - ib)e^{-i\phi}]/(1 - \rho^2)$ , and  $\sigma_n(k, l)$  is defined in theorem 2 with  $x$  and  $y$  replaced by  $it(a - ib)e^{-i\phi}/(1 - \rho^2)$  and  $-it(a + ib)e^{i\phi}/(1 - \rho^2)$  respectively.

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