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MULTI-COMPONENT VORTEX SOLUTIONS IN SYMMETRIC COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. A Hamiltonian system of incoherently coupled nonlinear Schrödinger (NLS) equations is considered in the context of physical experiments in photorefractive crystals and Bose-Einstein condensates. Due to the incoherent coupling, the Hamiltonian system has a group of various symmetries that include symmetries with respect to gauge transformations and polarization rotations. We show that the group of rotational symmetries generates a large family of vortex solutions that generalize scalar vortices, vortex pairs with either double or hidden charge, and coupled states between solitons and vortices. Novel families of vortices with different frequencies and vortices with different charges at the same component are constructed and their linearized stability problem is block-diagonalized for numerical analysis of unstable eigenvalues.

CONTENTS

1. Introduction

Optical vortices [27] have attracted ever growing attention of researchers due to the wide variety of applications of the so-called singular optical beams [34], ranging from optical tweezers to quantum information [36]. It is anticipated thar the localized *vortex solitons* in nonlinear media [9, 17] will be used as information carriers in futuristic all-optical photonic devices. Similarly, vortices in matter waves and stirred Bose-Einstein condensates (BEC) [13] provide a close link between self-focusing of light in nonlinear optics and the nonlinear dynamics of matter waves.

Vortex solitons suffer from the symmetry-breaking modulational instability in the attractive (focusing) nonlinear systems [14]. Several routes to stabilize self-localized vortices have been proposed; for example, stable vortex solitons have been predicted to exist in media with competing nonlinearities [9] as well as in nonlocal nonlinear media [5, 39]. A different approach to vortex stabilization is proposed for partially coherent optical vortices [28] since the threshold of modulational instability can be tuned with the degree of partial coherence of the light field [2, 22]. This stabilization of vortex beams has been demonstrated in experiments with photorefractive media [16].

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For a deeper understanding of the stabilizing effect of partial coherence in nonlinear media, the modal expansion was proposed [6], similarly to the statistical description of partially correlated fields. In this approach, an optical field is composed of infinitely many components ψ_k, which propagate at slightly different angles and form the coherence spectrum. The composite field evolves in the nonlinear potential which is induced by all interacting components, e.g., in photorefractive saturable media. The corresponding nonlinear correction to the refractive index, $\delta n(E)$, is a function of a total intensity, $E = \sum_k |\psi_k|^2$. Thus, it is symmetric with respect to all components and it can be regarded as a generalization of the Manakov system [18]. Similarly, the nonlinear interaction potential for multi-component Bose-Einstein condensates is only slightly asymmetric, e.g., for the two-component BEC made of distinct hyperfine states of ^{87}Rb, the contributions from self- and cross-action of each component differ by several percent [29].

The distinctive feature of a vortex excited in a partially coherent field is the characteristic ring dislocation in its cross-correlation function [28]. As was demonstrated recently by Motzek et.al. [24] for the partially coherent vortex with imposed topological charge $m = 1$, the model involving only three interacting components with vortex charges $(m-1, m, m+1)$ is sufficient to adequately describe both the presence of the ring dislocation in the cross-correlation function and its robustness in nonlinear media.

A physically relevant problem of the evolution of partially coherent vortices in nonlinear media can be effectively formulated as the stability problem for *multi-component composite vortex solitons*. Composite vortex solitons were predicted for a piecewise linear potential [25, 26]; we distinguish these states by using the topological charges in the components of the composite field, i.e., by the integer tuples $(\dots, m_{k-1}, m_k, m_{k+1}, \dots)$. Several investigations has shown [9] that, in isotropic media with saturable nonlinear interaction potential, the coupled state between a soliton and a charge-one vortex, $(0, 1)$, is modulationally unstable. The vortex component breaks into a dipole mode [15], which was shown to be linearly stable [40]; the stability predictions extend to rotating dipole modes [30]. Similar states, e.g. the multipole modes [10, 11], were found to be metastable in nonlinear photorefractive media. It remains unexplored if the stabilization close to the bifurcation point can be achieved for these higher-order azimuthal states, similar to the local bifurcation analysis in [40]. Generalization to the higher number of components include the three-component coupled state between solitons and dipoles and between solitons and vortices [8]. Similar solutions exist even when the potential is spatially anisotropic [23]. The stability analysis of these three-component solutions was recently reported in numerical work [38].

In this paper we study existence and stability of coupled states between solitons and vortices in the system of coupled nonlinear Schrödinger equation (NLS) with additional rotational symmetries. The multi-component vortices allow the generalization to the so-called "necklace-ring" vector solitons [7], composed of the azimuthally modulated ring-shaped beams, resembling "optical necklaces" [31–33]. The stability properties of the solutions with larger angular momentum differ significantly from their counterparts with smaller ("hidden") momentum. The stabilization of counter-rotating vortices against azimuthal symmetry-breaking instability was confirmed for the model with saturable nonlinearity [4, 41], with the cubic-quintic nonlinear function [12, 21], and experimentally in photorefractive defocusing media [19]. A novel physical phenomenon, the so-called "charge-flipping," was reported for the model with the cubic-quintic nonlinear function [12]. It is associated with the specific instability modes of the vortex with a hidden angular momentum, $(+m, -m)$. Namely, the weak instability mode with an azimuthal index $s = 2m$ initiates the rotation through the soliton family with the two components exchanging the angular momentum and periodically reversing their charges [12]. This effect is a particular consequence of the rotational symmetries of the system of coupled NLS equations.

We report here the systematic study of the Hamiltonian system of nonlinearly coupled NLS equation and address the following questions. First we characterize the group of rotations which preserves the symplectic structure of the Hamiltonian system. The group includes symmetries with respect to gauge transformations as well as polarization rotations. Then we address the classifications of all stationary localized solutions and find a large family of vortex solutions generated by the group of rotational symmetries that generalizes the scalar vortices and the vortex pairs with either double or hidden charge. In particular, novel families of vortices with different frequencies and vortices with different charges at the same component are constructed for the Manakov system of cubic NLS equations. Finally, we characterize the general properties of the linearized stability problem which are related to the group of symplectic rotations. We block-diagonalize the linearized stability problem for the vortex solutions and develop the analysis of unstable eigenvalues for the coupled states between solitons and the vortex pairs with either double or hidden charges in the system of saturable NLS equations.

This paper is organized as follows. Section 2 describes the model of incoherently coupled NLS equations. Section 3 discusses symplectic symmetries of the Hamiltonian system due to gauge transformations and polarization rotations. Section 4 gives a classification of stationary localized solutions of the Hamiltonian system. Section 5 describes soliton solutions in the space of one dimension. Section 6 gives a classification of soliton and vortex solutions in the space of two dimensions and illustrates a numerical example of the vortex pairs with different charges in the system of two cubic NLS equations (the Manakov system). Section 7 discusses linearized stability of vortex solutions and the block-diagonalization of the linearized problem due to rotational symmetries. Section 8 describes numerical results on existence and stability of coupled states between a soliton and a vortex pair in the system of three saturable NLS equations. Section 9 concludes the paper.

2. Formalism

We consider the system of coupled nonlinear Schrödinger (NLS) equations in the form

$$i\dot{\psi}_k + \Delta\psi_k = W'(E)\psi_k, \qquad k = 1, 2, \ldots, n, \tag{2.1}$$

where $\psi_k(x,t) : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{C}$, Δ is the Laplacian in space $x \in \mathbb{R}^d$, the dot denotes the partial derivative in time $t \in \mathbb{R}$, and $W(E) : \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear C^2 function of the scalar variable $E = \sum_{k=1}^n |\psi_k|^2$, such that $W(0) = W'(0) = 0$.

System (2.1) can be cast as a Hamiltonian system with the standard symplectic structure. Let $\psi_k = u_k + iv_k$, where $(u_k, v_k) \in \mathbb{R}^2$ and denote $\mathbf{u} = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ and $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$. Let I and O denote the identity and zero matrices in $\mathbb{R}^n$. Then, the system (2.1) is equivalent to the Hamiltonian system:

$$\begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \begin{pmatrix} \nabla_{\mathbf{u}} \\ \nabla_{\mathbf{v}} \end{pmatrix} H(\mathbf{u}, \mathbf{v}), \tag{2.2}$$

where H is the Hamiltonian in the form

$$H = \int_{\mathbb{R}^d} \left[\sum_{j=1}^d \left(\partial_{x_j}\mathbf{u} \cdot \partial_{x_j}\mathbf{u} + \partial_{x_j}\mathbf{v} \cdot \partial_{x_j}\mathbf{v} \right) + W\left(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\right) \right] dx, \tag{2.3}$$

and the standard dot product in $\mathbb{R}^n$ is used. The values of H are constant in time t due to the t-translation invariance of the NLS system (2.1). The NLS system (2.1) has also other conserved quantities (see Chapter 2 in [35]). Due to the x-translation invariance, it has the conserved vector momentum $\mathbf{P} = (P_1, \ldots, P_d)^T$ with the components

$$P_j = \int_{\mathbb{R}^d} \left(\mathbf{v} \cdot \partial_{x_j}\mathbf{u} - \mathbf{u} \cdot \partial_{x_j}\mathbf{v} \right) dx, \qquad j = 1, \ldots, d. \tag{2.4}$$

Due to the invariance of Δ with respect to x-rotations, the NLS system (2.1) has the conserved angular momentum, which in the case $d = 2$ takes the form

$$M = \int_{\mathbb{R}^2} \left(\mathbf{u} \cdot \partial_\theta\mathbf{v} - \mathbf{v} \cdot \partial_\theta\mathbf{u} \right) dx, \tag{2.5}$$

where $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$. Due to the phase shift (gauge) invariance, it has n conserved charges,

$$Q_k = \int_{\mathbb{R}^d} \left(u_k^2 + v_k^2\right) dx, \qquad k = 1, \dots, n. \tag{2.6}$$

We shall proceed with analysis of a complete group of rotational symmetries of the NLS system (2.1).

3. Rotational Symmetries

The nonlinear function $W(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v})$ and the Hamiltonian function $H(\mathbf{u}, \mathbf{v})$ are invariant with respect to N-parameter group of rotations in the space $\mathbb{R}^{2n}$, where

$$N = \begin{pmatrix} 2n \\ 2 \end{pmatrix} = \frac{(2n)!}{2!(2n-2)!} = n(2n-1).$$

In order to characterize the group of rotations which preserves the symplectic structure of the Hamiltonian system (2.2), we need the following result.

Lemma 1. *Let $y = Gx$ be an invertible linear transformation from $x \in \mathbb{R}^m$ to $y \in \mathbb{R}^m$. Let $H : \mathbb{R}^m \mapsto \mathbb{R}$ be a G-invariant Hamiltonian function for the system $\dot{x} = J\nabla_x H(x)$, such that $H(Gx) = H(x)$. The transformation $y = Gx$ is symplectic, such that $\dot{y} = J\nabla_y H(y)$, if and only if*

$$J = GJG^T. \tag{3.1}$$

Proof. Although the result is standard in Hamiltonian mechanics (see Sec. A in [20]), we deduce an elementary proof from the identities

$$\dot{y} = G\dot{x}, \qquad \nabla_x H(x) = G^T \nabla_y H(y)$$

and the fact that G is invertible. □

Corollary 1. If the group of infinitesimal symmetries is generated by the near-identity matrix $G = I + g$, where g is infinitesimal, then the symplectic structure is preserved if and only if

$$gJ + Jg^T = O. \tag{3.2}$$

Lemma 2. *Let $(\mathbf{u}, \mathbf{v})^T \in \mathbb{R}^{2n}$ be the vector of variables of the Hamiltonian system (2.2) and*

$$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}.$$

Then, the group of symplectic rotations of the system (2.2) is generated by the matrix

$$g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}, \tag{3.3}$$

where $A^T = -A$ and $B^T = B$ are matrices in $\mathbb{R}^n$.

Proof. The general $n(2n-1)$-parameter group of rotations in $\mathbb{R}^{2n}$ is generated by the matrix $g : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ of infinitesimal symmetries in the form

$$g = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix},$$

where A and C are skew-symmetric matrices and B is a general matrix in $\mathbb{R}^{n\times n}$. Using the constraint (3.2), we find immediately the constraints $A = C$ and $B^T = B$, which restrict the general group of $n(2n-1)$ rotations to the n^2-parameter group of symplectic rotations. □

Corollary 2. Let $\boldsymbol{\psi} \in \mathbb{C}^n$ be the vector of variables in the coupled NLS system (2.1), such that $\boldsymbol{\psi} = \mathbf{u} + i\mathbf{v}$. Then, the group of symplectic rotations of the coupled NLS system (2.1) is generated by the matrix $g_c : \mathbb{C}^n \mapsto \mathbb{C}^n$, where $g_c = A - iB$ with $A^T = -A$ and $B^T = B$.

Related to the group of n^2 symplectic rotations, there exist n^2 quantities, which are constant in time t. By the standard technique (see [20]), the conserved quantities are related to the dot products $i\bar{\boldsymbol{\psi}} \cdot A\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}} \cdot B\boldsymbol{\psi}$. Since the Hamiltonian system (2.2) is defined on $x \in \mathbb{R}^d$, the conserved quantities are equivalent to the set of quadratic functionals

$$Q_{k,m} = \int_{\mathbb{R}^d} \psi_k \bar{\psi}_m dx, \qquad k = 1, \ldots, n, \;\; m = k, \ldots, n. \tag{3.4}$$

There exists n real-valued charges $Q_{k,k} \equiv Q_k$ which coincide with the charges (2.6). The additional $\frac{1}{2}n(n-1)$ conserved quantities $Q_{k,m}$ with $k \neq m$ are complex-valued. The total number of real-valued conserved quantities $Q_{k,m}$ is $n + n(n-1) = n^2$, and this number coincides with the number of symplectic rotational symmetries of the system (2.2). It is easy to verify conservation of $Q_{k,m}$ directly from the coupled NLS system (2.1) for $\psi_k(\cdot, t) \in L^2(\mathbb{R}^d)$.

Example 1 ($n = 2$). In the complex-valued form $\boldsymbol{\psi} \in \mathbb{C}^2$, there exist $n^2 = 4$ transformation matrices, which leave solutions of the NLS system (2.1) with $n = 2$ invariant:

$$G_1 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}, \quad G_3 = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix}, \quad G_4 = \begin{pmatrix} \cos\theta_4 & i\sin\theta_4 \\ i\sin\theta_4 & \cos\theta_4 \end{pmatrix},$$

where $\theta_{1,2,3,4}$ are arbitrary parameters. The parameters θ_1 and θ_2 are related to the phase shift (gauge) invariance, while the parameters θ_3 and θ_4 are related to the rotational symmetry of the Hamiltonian system (2.2). If there exists a solution of the coupled NLS system (2.1) in the form $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$, then there exists a 4-parameter continuation of the solution to the form

$$\tilde{\psi}_1 = \alpha_1 e^{i\theta_1}\psi_1 + \alpha_2 e^{i\theta_2}\psi_2, \tilde{\psi}_2 = -\bar{\alpha}_2 e^{i\theta_1}\psi_1 + \bar{\alpha}_1 e^{i\theta_2}\psi_2, \tag{3.5}$$

where

$$\alpha_1 = \cos\theta_3\cos\theta_4 + i\sin\theta_3\sin\theta_4, \quad \alpha_2 = \sin\theta_3\cos\theta_4 + i\cos\theta_3\sin\theta_4.$$

In the general case $n \geq 2$, there exists n-parameter continuations of solutions due to phase shift (gauge) invariance and $n(n-1)$-parameter continuation of solutions due to rotational symmetries of the Hamiltonian system (2.2).

4. Stationary Localized Solutions of the Coupled System

The class of stationary solutions of the coupled NLS system (2.1) is defined in the form

$$\psi_k = \varphi_k(x)e^{i\omega_k t}, \qquad k = 1, \ldots, n, \tag{4.1}$$

where ω_k are real-valued parameters. The functions $\varphi_k(x) : \mathbb{R}^d \mapsto \mathbb{C}$ solve the PDE boundary-value problem:

$$\varphi_k \in L^2(\mathbb{R}^d) : \quad \Delta\varphi_k - \omega_k\varphi_k = W'(E)\varphi_k, \qquad k = 1, 2, \ldots, n, \tag{4.2}$$

where $E = \sum_{k=1}^n |\varphi_k(x)|^2$. The boundary-value problem (4.2) coincides with the variational problem for critical points of the Lyapunov functional,

$$\Lambda = H + \sum_{k=1}^{n} \omega_k Q_k. \tag{4.3}$$

In order to classify all stationary localized solutions of the boundary-value problem (4.2), we specify two distinct cases in the following result.

Lemma 3. *For any $k \neq m$, either $\omega_k = \omega_m$ or*

$$\omega_k \neq \omega_m, \quad \int_{\mathbb{R}^d} \varphi_k(x)\bar{\varphi}_m(x)dx = \int_{\mathbb{R}^d} \varphi_k(x)\varphi_m(x)dx = 0, \tag{4.4}$$

where $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a stationary solution of the boundary-value problem (4.2).

Proof. The statement is derived by integrating the PDE problem (4.2) and using the Green Theorem. The first equality in (4.4) ensures that $\dot{Q}_{k,m} = 0$ for all $k \neq m$, where $Q_{k,m}$ is given by (3.4) and $\psi_k(\cdot, t) \in L^2(\mathbb{R}^d)$ is given by (4.1). □

We shall consider separately the two particular types of stationary solutions in the boundary-value problem (4.2) and the general (mixed) case.

(I) Case $\omega_k \neq \omega_m$ for all $k \neq m$. Non-trivial two-component solutions with $\omega_1 \neq \omega_2$ are computed in [26] for a piecewise linear potential function $W(E)$ and in [15] for a rational function $W(E)$. In particular, a coupled state between a fundamental soliton and a vortex of charge m was considered in [15, 26]. The constraints (4.4) are satisfied for these solutions, since the vortex charges are different in different components of the solution and integration over the angle in polar coordinates gives zero integrals. Stability of the stationary two-component solutions with $\omega_1 \neq \omega_2$ was addressed in [40]. It is obvious that the group of rotational symmetries (3.5) generates a nonstationary solution $(\tilde{\psi}_1, \tilde{\psi}_2)$ with two frequencies in one component. The existence of such nonstationary solutions with several frequencies explains the existence of exact internal modes (eigenvectors that correspond to purely imaginary eigenvalues) in the linearization of stationary solutions discovered in [40].

(II) Case $\omega_k = \omega_m$ for all $k \neq m$. Let $\omega_1 = \cdots = \omega_n \equiv \omega$. The functional Λ in (4.3) is invariant with respect to the n^2-parameter group of symplectic rotations and the boundary-value problem (4.2) can be rewritten in the elliptic form:

$$\varphi_k \in L^2(\mathbb{R}^d): \quad \Delta\varphi_k = W'_\omega(E)\varphi_k, \qquad k = 1, 2, \ldots, n, \tag{4.5}$$

where $W_\omega = W(E) + \omega E$ is a function of scalar variable $E = \sum_{k=1}^n |\varphi_k(x)|^2$. The divergence form of the Green theorem follows from the PDE problem (4.5):

$$\begin{cases} \operatorname{div}(\bar{\varphi}_k \nabla\varphi_m - \varphi_m \nabla\bar{\varphi}_k) = 0 \\ \operatorname{div}(\varphi_k \nabla\varphi_m - \varphi_m \nabla\varphi_k) = 0 \end{cases} \qquad 1 \leq k, m \leq n. \tag{4.6}$$

It follows from the divergence form (4.6) that the vector $\mathbf{p} = (p_1, \ldots, p_d)^T$ with components

$$p_j = i\left(\bar{\boldsymbol{\varphi}} \cdot \partial_{x_j}\boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \partial_{x_j}\bar{\boldsymbol{\varphi}}\right), \qquad j = 1, \ldots, d$$

satisfies $\nabla \cdot \mathbf{p} = 0$. Therefore, there exists a vector field $\mathbf{A}$, such that $\mathbf{p} = \nabla \times \mathbf{A}$ and the vector momentum $\mathbf{P} = \int_{\mathbb{R}^d} \mathbf{p} dx$ defined in (2.4) is identically zero. Stationary solutions of the PDE problem (4.5) in the space of two dimensions were considered in [4, 11, 41], where, in particular, vortex pairs with either double or hidden charge were addressed. The group of rotational symmetries can be applied to vortex pairs to generate a more general stationary solution which includes dipole and vortex modes of the coupled system [1, 7]. Stability of the stationary two-component solutions with $\omega_1 = \omega_2$ was considered in the saturable coupled NLS equations [4, 41]. It was shown that vortex pairs of both types are unstable but the vortex pair with hidden charge is less unstable compared to the vortex pair with double charge. In contrast, full stabilization of the vortices with hidden momentum can be achieved in the coupled NLS equation with the competing cubic-quintic nonlinearity [12, 21].

(III) General case. In general, localized solutions of the PDE problem (4.2) can be represented by p components with distinct parameters ω_k and $q = n - p$ components with equal parameters ω_k. For instance, the three-component coupled state between the fundamental soliton and a vortex pair with either double or hidden charge was considered in [10, 23, 38]. The local bifurcation of such three-component states is analyzed in Sec. 8.

5. Solitons in One Dimension

We shall consider a construction of one-dimensional solutions in the case II (all $\omega_k = \omega_m$), such that $d = 1$ and $\omega_1 = \cdots = \omega_n \equiv \omega$. It follows from the divergence equations (4.6) for solutions in $L^2(\mathbb{R})$ that

$$\begin{cases} \bar{\varphi}_k(x)\varphi_m'(x) - \varphi_m(x)\bar{\varphi}_k'(x) = 0 \\ \varphi_k(x)\varphi_m'(x) - \varphi_m(x)\varphi_k'(x) = 0 \end{cases} \qquad \forall x \in \mathbb{R}$$

As a result, there exist constants $c_{m,k}, d_{m,k} \in \mathbb{C}$ such that $\varphi_m(x) = c_{m,k}\bar{\varphi}_k(x)$ and $\varphi_m(x) = d_{m,k}\varphi_k(x)$. Therefore, the most general family of stationary solutions of the boundary-value problem (4.5) with $d = 1$ takes the form

$$\varphi_k = a_k \varphi(x), \qquad k = 1, \ldots, n, \tag{5.1}$$

where $\varphi(x) : \mathbb{R} \mapsto \mathbb{R}$ and $a_k \in \mathbb{C}$. Using the normalization $\sum_{k=1}^n |a_k|^2 = 1$, we define the function $\varphi(x)$ from a solution of the normalized boundary-value ODE problem:

$$\varphi \in L^2(\mathbb{R}) : \quad \varphi'' = W_\omega'(\varphi^2)\varphi. \tag{5.2}$$

In order to remove the parameter of x-translation and to define a unique solution $\varphi(x)$, the ODE problem (5.2) can be considered in the space of even functions on $x \in \mathbb{R}$. The family (5.1) can be generated from the particular scalar solution,

$$\varphi_1 = \varphi(x), \qquad \varphi_k = 0, \quad k = 2, \ldots, n, \tag{5.3}$$

by means of the n^2-parameter group of symplectic rotations. As a result, n complex parameters a_k of the family (5.1) under the normalization constraint $\sum_{k=1}^n |a_k|^2 = 1$ are expressed in terms of n^2 real-valued parameters of symplectic rotations. This count shows that $n^2 - 2n + 1 = (n-1)^2$ parameters of the group of symplectic rotations are redundant.

6. Vortices in Two Dimensions

We shall consider a construction of one-dimensional solutions in the case II (all $\omega_k = \omega_m$), such that $d = 2$ and $\omega_1 = \cdots = \omega_n \equiv \omega$. The same reduction (5.1) exists and allows us to generate a family of vector solutions from the scalar solution (5.3) by using the group of symplectic rotations. However, the reduction (5.1) does not generate the most general family of solutions of the boundary-value problem (4.5) with $d = 2$. We shall classify all possible vortex solutions in the PDE problem (4.5) after separation of variables.

We use the polar coordinates (r, θ) for vortex solutions and place the center of the vortex to the origin $r = 0$. Thus, we eliminate two parameters of space translations which are related to the space translation invariance of the NLS system (2.1). Let us separate the polar coordinates as follows:

$$\varphi_k = \phi_k(\theta) R_k(r), \qquad k = 1, \ldots, n, \tag{6.1}$$

where $R_k(r) : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\phi_k(\theta) : [0, 2\pi] \mapsto \mathbb{C}$. The separation of variables in the PDE problem (4.5) leads to two ODE problems:

$$\phi_k(\theta + 2\pi) = \phi_k(\theta) : \quad \phi_k'' + m_k^2 \phi_k = 0, \qquad k = 1, \ldots, n, \tag{6.2}$$

$$R_k \in L^2(\mathbb{R}_+) : \quad R_k'' + \frac{1}{r} R_k' - \frac{m_k^2}{r^2} R_k = W_\omega'(E_0) R_k, \qquad k = 1, \ldots, n, \tag{6.3}$$

subject to the constraint

$$E = \sum_{k=1}^n R_k^2(r) |\phi_k(\theta)|^2 = E_0(r). \tag{6.4}$$

The periodicity conditions for $\phi_k(\theta)$ imply that parameters m_k are integers. The value of m_k is referred to as *the vortex charge* in the kth component. When $m_k = 0$, the kth component is *a soliton*. When $m_k \neq 0$, the kth component is *a vortex*. The following lemma shows that two distinct cases occur in solutions of the coupled system of nonlinear ODEs (6.3).

Lemma 4. *For any $k \neq m$, either $m_k^2 = m_m^2$ and $R_k = c_{m,k} R_m$ for some $c_{m,k} \in \mathbb{R}$ or $m_k^2 \neq m_m^2$ and the functions $R_k(r), R_m(r)$ satisfy the constraints:*

$$\int_0^\infty \frac{R_k(r) R_m(r) dr}{r} = 0, \qquad m_k^2 \neq m_m^2. \tag{6.5}$$

In the latter case, there is no constant $c_{m,k} \in \mathbb{R}$ such that $R_k = c_{m,k} R_m$ and

$$\phi_k = a_k e^{\pm i m_k \theta}, \qquad \phi_m = a_m e^{\pm i m_m \theta}, \tag{6.6}$$

where $a_k, a_m \in \mathbb{C}$.

Proof. It follows from (6.3) that the functions R_k and R_m for all $k \neq m$ satisfy the divergence equation:

$$\frac{d}{dr}\left[r \left(R_m R_k' - R_m' R_k \right) \right] + \frac{m_m^2 - m_k^2}{r} R_m R_k = 0, \qquad r \geq 0. \tag{6.7}$$

When $m_k^2 = m_m^2$ and $R_m, R_k \in L^2(\mathbb{R}_+)$, the divergence equation (6.7) implies that $R_m R_k' - R_m' R_k = 0$ such that $R_k = c_{m,k} R_m$ for some $c_{m,k} \in \mathbb{R}$. When $m_k^2 \neq m_m^2$, Eq. (6.7) can be integrated on $r \in (0, \infty)$ under the conditions that $R_k(r)$ and $R_m(r)$ and their derivatives are bounded at $r \to 0$ and belong to $L^2(\mathbb{R}_+)$. This procedure results in the constraint (6.5), while the reduction $R_k = c_{m,k} R_m$ fails to satisfy the system (6.3) for any $c_{m,k} \in \mathbb{R}$. The particular representation (6.6) is required by the fact that $E_0(r)$ in (6.4) does not depend on θ. □

We shall consider separately the two particular types of stationary solutions in the coupled ODE system (6.3) and the general (mixed) case.

(i) Case $m_k^2 \neq m_m^2$ for all $k \neq m$. A nontrivial feature of the coupled system (6.3) is the existence of invariant reductions $R_k = 0$ for a particular value of k. Therefore, the family of vortices of different charges can be classified by the number of nonempty (nonzero) components $R_k(r)$. The single-component (scalar) vortex has the form (5.3) rewritten in polar coordinates as

$$\varphi_1 = R_1(r) e^{i m_1 \theta}, \qquad \varphi_k = 0, \quad k = 2, \ldots, n. \tag{6.8}$$

By symplectic rotations, the scalar solution (6.8) transforms to the vector solution

$$\varphi_k = a_k R(r) e^{i m \theta}, \qquad k = 1, \ldots, n, \tag{6.9}$$

where $R \equiv R_1$, $m \equiv m_1$, and parameters $a_k \in \mathbb{C}$ satisfy the normalization condition $\sum_{k=1}^n |a_k|^2 = 1$. In the case $n = 2$, the vector solution (6.9) is referred to as *the vortex pair with double charge*. These vector solutions are reviewed in [9].

Similarly, we can define the two-component vortex in the form

$$\varphi_1 = \alpha_1 R_1(r) e^{i m_1 \theta}, \qquad \varphi_2 = \alpha_2 R_2(r) e^{i m_2 \theta}, \qquad \varphi_k = 0, \quad k = 3, \ldots, n, \tag{6.10}$$

where $m_1^2 \neq m_2^2$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. Continuing in the same way, the N-component vortex can be defined for distinct values of m_k^2, $k = 1, \ldots, N$, where $2 \leq N \leq n$. The N-component vortices can be rotated with the group of n^2-parameter symplectic rotations, which result in stationary solutions with *different charges* in the *same components*.

Example 2. Consider the system of cubic NLS equations (2.1) with $n = 2$ and $W'(E) = -E$. Let $\omega = 1$ without loss of generality since the scaling transformation can be used to normalize the parameter ω. Using the transformation (3.5), we continue the solution (6.10) with $\alpha_1 = \alpha_2 = 1$ into a more general family of vortex solutions:

$$\begin{pmatrix} \varphi_1(r,\theta) \\ \varphi_2(r,\theta) \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{i\theta_1} & \alpha_2 e^{i\theta_2} \\ -\bar{\alpha}_2 e^{i\theta_1} & \bar{\alpha}_1 e^{i\theta_2} \end{pmatrix} \begin{pmatrix} R_1(r) e^{i m_1 \theta} \\ R_2(r) e^{i m_2 \theta} \end{pmatrix}. \tag{6.11}$$

The ODE system (6.3) is now rewritten as

$$L_{m_1} R_1 = 0, \qquad L_{m_2} R_2 = 0,$$

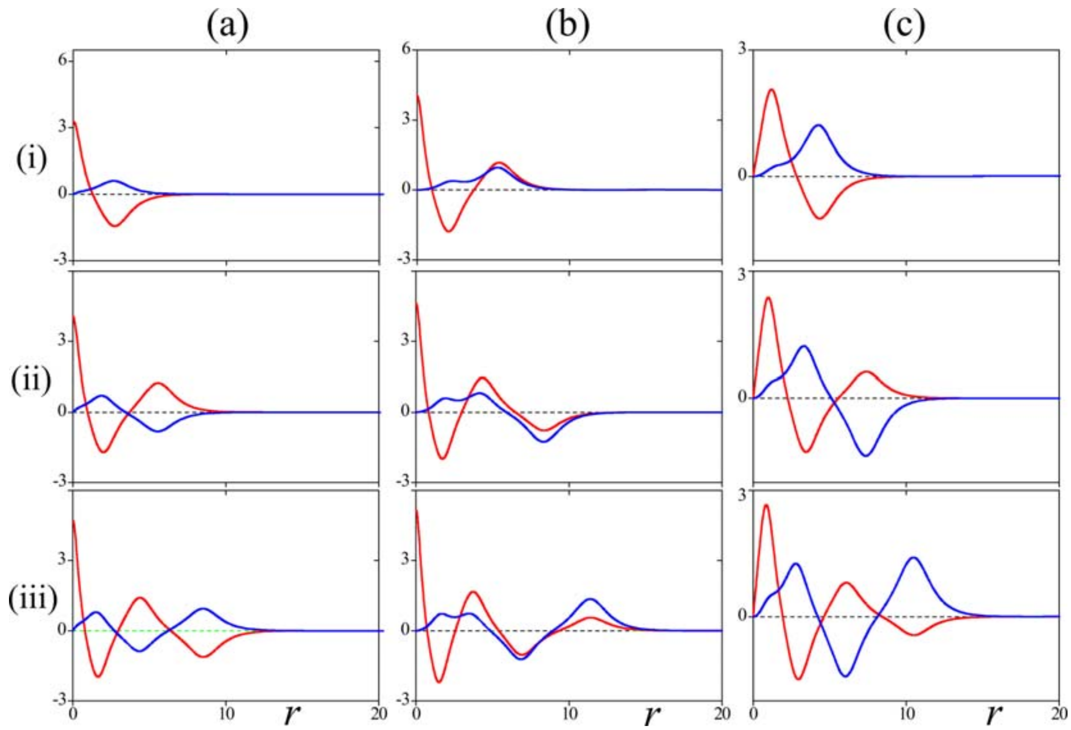

Fig. 1. Envelopes $R_1(r)$ and $R_2(r)$ of the two-component vortex with different charges in the same components for (m_1, m_2): (a) $(0, 1)$, (b) $(0, 2)$, and (c) $(1, 2)$. Rows (i), (ii), and (iii) show the solutions with $n_2 = 0$, $n_2 = 1$, and $n_2 = 2$, respectively.

where

$$L_m \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - 1 + R_1^2 + R_2^2. \tag{6.12}$$

We are looking for solutions $R_1(r)$ and $R_2(r)$ such that $R_k \sim r^{|m_k|}$ as $r \to 0$ and $R_k(r)$ is exponentially decaying as $r \to \infty$. Figure 1 shows numerically obtained solutions for the envelopes $R_1(r)$ and $R_2(r)$ for $(m_1, m_2) = (0, 1); (0, 2); (1, 2)$. If n_k denotes the number of nodes for the function $R_k(r)$ on $r > 0$, then Fig. 1 suggests preservation of integers:

$$|m_1| + n_1 = |m_2| + n_2.$$

We have observed in the numerical integration of the cubic NLS system (2.1) that all vortex solutions in Fig. 1 are linearly unstable. The radial instability of the coupled state between a soliton and a charge-m vortex is responsible for the catastrophic collapse [3] of the central peak which develops faster than any azimuthal instability. This situation is observed for the solutions with $(0, 1)$ and (0,2) types, which correspond to Fig. 1 (a,b). The collapse instability is significantly slowed down when all the components carry vortices, such as in the case shown in Fig. 1(c). In this case, we observe an azimuthal instability splitting the rings just before the splinters collapse [37].

Numerical integration of the cubic NLS system (2.1) for the vortex solution with $m_1 = 1$, $n_1 = 1$, $m_2 = \pm 2$, and $n_2 = 0$ (see Fig. 1 (c,top)) is shown in Figs. 2 and 3. The initial conditions are shown in Figs. 2(a) and 3(a,c), while the results of the temporal evolution are shown in Fig. 2(b,c) for $\alpha_1 = 1$ and $\alpha_2 = 0$ and in Fig. 3(b,d) for $\alpha_1 = \alpha_2 = 1/\sqrt{2}$.

(ii) Case $m_k^2 = m_m^2$ for all $k \neq m$. Let $m_1^2 = \cdots = m_n^2 \equiv m^2$ and $R(r)$ be defined from the boundary-value ODE problem:

$$R \in L^2(\mathbb{R}_+): \quad R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = W'_\omega(R^2)R, \tag{6.13}$$

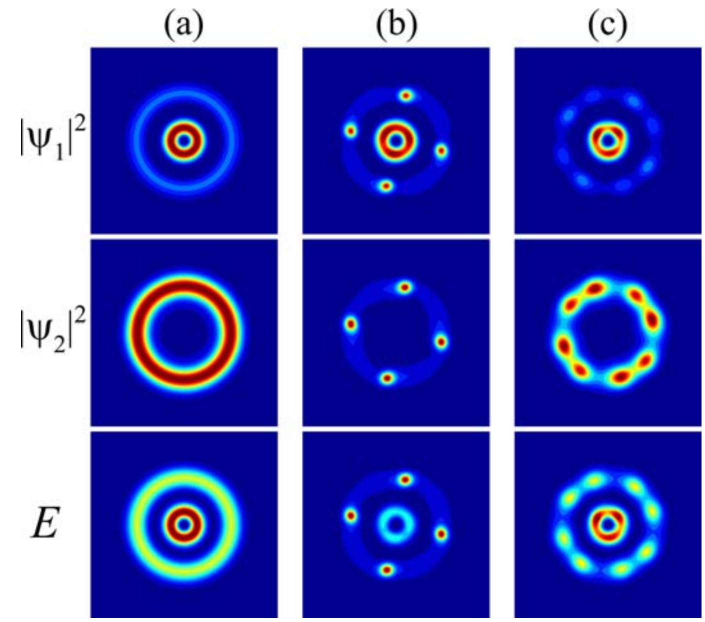

Fig. 2. Intensities $|\psi_1|^2$, $|\psi_2|^2$ and $E = |\psi_1|^2 + |\psi_2|^2$ for the two-component solution (6.11) with $\alpha_1 = 1$ and $\alpha_2 = 0$ at $t = 0$ (a), $t = 7.5$ and $(m_1, m_2) = (+1, +2)$ (b), $t = 8$ and $(m_1, m_2) = (+1, -2)$ (c).

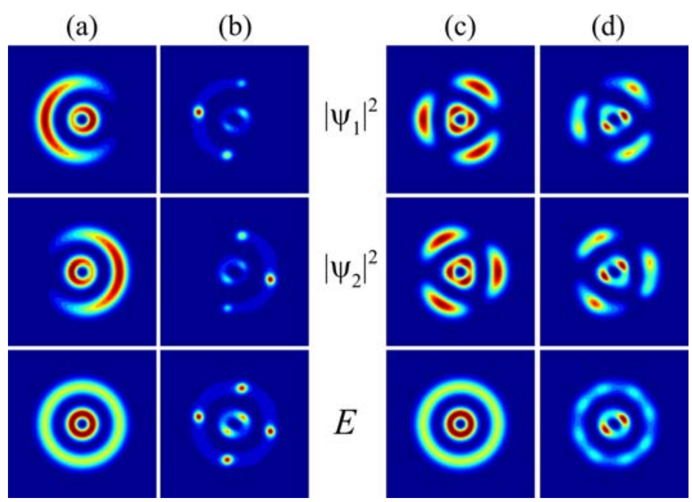

Fig. 3. The same as in Fig. 2 for the two-component solution (6.11) with $\alpha_1 = \alpha_2 = 1/\sqrt{2}$ at $t = 0$ (a), $t = 7.5$ (b) for $(m_1, m_2) = (+1, +2)$ and at $t = 0$ (c), $t = 8$ (d) for $(m_1, m_2) = (+1, -2)$.

under the constraint and normalization $\sum_{k=1}^{n} |\phi_k|^2 = 1$. The ODE system (6.2) admits the general solution

$$\phi_k = a_k e^{im\theta} + b_k e^{-im\theta}, \qquad k = 1, \ldots, n. \tag{6.14}$$

By using the constraint (6.4) on separation of variables, we obtain two constraints on $(2n)$ complex parameters a_k and b_k:

$$\sum_{k=1}^{n} |a_k|^2 + |b_k|^2 = 1, \qquad \sum_{k=1}^{n} a_k \bar{b}_k = 0. \tag{6.15}$$

Let $\mathbf{a} = (a_1, \ldots, a_n)^T \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n)^T \in \mathbb{C}^2$, and $(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^n a_k \bar{b}_k$ be the standard inner product in $\mathbb{C}^n$, such that $||\mathbf{a}||^2 = (\mathbf{a}, \mathbf{a})$. The two constraints (6.15) can be rewritten in the form

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = 1, \qquad (\mathbf{a}, \mathbf{b}) = 0. \tag{6.16}$$

When $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} = \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, the family of stationary solutions (6.14) is generated from the single-component vortex (6.8) by using the group of n^2 symplectic rotations. Since this family has n complex-valued parameters with a single constraint (6.15), it is clear that $n^2 - 2n + 1 = (n-1)^2$ parameters of the symplectic rotations are redundant. When $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, the family of stationary solutions (6.14) is generated from the following two-component vortex solution by using the group of n^2 symplectic rotations:

$$\varphi_1 = \alpha_1 R(r) e^{im\theta}, \qquad \varphi_2 = \alpha_2 R(r) e^{-im\theta}, \qquad \varphi_k = 0, \quad k = 3, \ldots, n, \tag{6.17}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $|\alpha_1|^2 + |\alpha_2|^2 = 1$. In the case $n = 2$, the vortex solution (6.17) is referred to as *the vortex pair with hidden charge*. The most general family of vortex solutions obtained from the two-component vortex solution (6.17) has *equal charges* in the *different components*. This family has $2n$ complex parameters under one real-valued and one complex-valued constraints (6.15). Therefore, it is clear that $1 + n^2 - 4n + 3 = (n-2)^2$ parameters of the symplectic rotations are redundant. The vortex pairs with hidden charge are reviewed in [9].

(iii) General case. In general, p-component vortices with different charges can form coupled states with q two-component vortex pairs with hidden charge, where $p + 2q \leq n$ and $n \geq 3$.

7. Linearization of Stationary Solutions

Stability of the stationary solutions (4.1) in the time evolution of the coupled NLS system (2.1) is studied with the method of linearization:

$$\psi_k = e^{i\omega_k t}\left(\varphi_k(x) + u_k(x) e^{\lambda t} + \bar{v}_k(x) e^{\bar{\lambda} t}\right), \qquad k = 1, \ldots, n, \tag{7.1}$$

where $(u_k, v_k) : \mathbb{R}^d \mapsto \mathbb{C}^2$ are components of the eigenvector in $L^2(\mathbb{R}^d, \mathbb{C}^{2n})$ that correspond to the eigenvalue $\lambda \in \mathbb{C}$. Let $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)^T \in \mathbb{C}^n$, $\mathbf{u} = (u_1, \ldots, u_n)^T \in \mathbb{C}^n$, $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$, $E(x) = \sum_{k=1}^n |\varphi_k(x)|^2$, and $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$. The linearized problem defines spectral stability of stationary solutions with respect to time evolution of the NLS system (2.1). The linearized problem is written in the Hamiltonian form:

$$\mathcal{H} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = i\lambda \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \tag{7.2}$$

where the self-adjoint operator $\mathcal{H}$ on $L^2(\mathbb{R}^d, \mathbb{C}^{2n})$ takes the form

$$\mathcal{H} = \begin{pmatrix} \Omega & O \\ O & \Omega \end{pmatrix} + \left(-\Delta + W'(E)\right) \begin{pmatrix} I & O \\ O & I \end{pmatrix} + W''(E) \begin{pmatrix} \boldsymbol{\varphi} \\ \bar{\boldsymbol{\varphi}} \end{pmatrix} \cdot \left(\bar{\boldsymbol{\varphi}}^T \ \ \boldsymbol{\varphi}^T\right). \tag{7.3}$$

The last term in (7.3) consists of the outer product of two vectors, which is a rank-one matrix for any $x \in \mathbb{R}^d$. We shall characterize general properties of the linearized problem (7.2)–(7.3) which are related to the group of symplectic rotations.

(I) Case $\omega_k \neq \omega_m$ for all $k \neq m$. The rotational symmetries of the NLS system (2.1) lead to existence of nonstationary solutions which are obtained by symplectic rotations of stationary solutions with *distinct* parameters ω_k, $k = 1, \ldots, n$. Derivatives of these solutions with respect to parameters of rotations result in eigenvectors of the linearized problem that correspond to nonzero eigenvalues λ. We will show that each distinct pair of frequencies $\omega_k \neq \omega_m$ for $k \neq m$ corresponds to a pair of purely imaginary eigenvalues $\lambda = \pm i(\omega_k - \omega_m)$. Each pair can be either isolated or embedded in the continuous spectrum of the linearized problem (7.2).

Lemma 5. *Let $\omega_k \neq \omega_m$ for any $k \neq m$. The linearized problem has the exact pair of eigenvectors and eigenvalues:*

$$u_k = \varphi_m(x), \quad v_m = -\bar{\varphi}_k(x), \quad \lambda = i(\omega_m - \omega_k), \tag{7.4}$$

$$u_m = \varphi_k(x), \quad v_k = -\bar{\varphi}_m(x), \quad \lambda = -i(\omega_m - \omega_k), \tag{7.5}$$

where all other components are identically zero.

Proof. Direct proof of the statement is developed by showing that

$$\left(\bar{\boldsymbol{\varphi}}^T \ \ \boldsymbol{\varphi}^T\right) \cdot \left(\begin{array}{c} \mathbf{u} \\ \mathbf{v} \end{array} \right) = \bar{\varphi}_k u_k + \varphi_m u_m + \varphi_k v_k + \varphi_m v_m = 0,$$

for the eigenvectors (7.4) and (7.5). The linearized problem (7.2) reduces then to the PDE problem (4.2) for components $\varphi_k(x)$ and $\varphi_m(x)$. □

A simple count shows that there exists $n(n-1)/2$ pairs of purely imaginary eigenvalues of Lemma 5 in the case where all parameters $(\omega_1, \ldots, \omega_n)$ are distinct. In addition, there exist n zero eigenvalues related to n gauge invariances, where the eigenvectors are given by (7.4) with $m = k$. As a result, the group of n^2 symplectic rotations corresponds exactly to the purely imaginary and zero eigenvalues with the eigenvectors (7.4) and (7.5) of the linearized problem (7.2). These pairs can have negative Krein signatures and they trap potentially unstable eigenvalues of the stability problem (see discussion and references in [40]).

(II) Case $\omega_k = \omega_m$ for all $k \neq m$. Let $\omega_1 = \cdots = \omega_n \equiv \omega$ such that $\Omega = \omega I$. All $n(n-1)$ eigenvalues in Lemma 5 are zero, so that all n^2 symplectic rotations produce a multi-dimensional kernel of $\mathcal{H}$. This feature is related to the fact that the n^2 symplectic rotations map stationary solutions (4.1) with equal values of parameters $(\omega_1, \ldots, \omega_n)$ to the same class of stationary solutions. The linearized problem (7.2)–(7.3) can be simplified if the stationary solutions are obtained from a N-component vortex solution with $N < n$ by symplectic rotations. In particular, we consider symplectic rotations of the scalar vortex (6.8) and the two-component vortex pair with hidden charge (6.17).

(i) Symplectic rotations of the scalar vortex (6.8)**.** The stationary solutions of the vector form (6.9) are generated from the scalar vortex (6.8) by using symplectic rotations. We show that the linearization problem (7.2)–(7.3) for the vector solution (6.9) can be block-diagonalized into a coupled 2-by-2 (non-self-adjoint) linearized operator and $(n-1)$ pairs of uncoupled (self-adjoint) linear Schrödinger operators.

Let $\varphi_k(x) = a_k \varphi(x)$, where $a_k \in \mathbb{C}$ under the normalization condition $\sum_{k=1}^n |a_k|^2 = 1$. Then, the operator $\mathcal{H}$ is rewritten in the form

$$\mathcal{H} = \left(\omega - \Delta + W'(|\varphi|^2)\right) \left(\begin{array}{cc} I & O \\ O & I \end{array} \right) + W''(|\varphi|^2) \left(\begin{array}{cc} |\varphi|^2(\mathbf{a},\mathbf{a}) & \varphi^2(\mathbf{a},\bar{\mathbf{a}}) \\ \bar{\varphi}^2(\bar{\mathbf{a}},\mathbf{a}) & |\varphi|^2(\mathbf{a},\mathbf{a}) \end{array} \right), \tag{7.6}$$

where $(\mathbf{f}, \mathbf{g}) = \sum_{k=1}^n f_k \bar{g}_k$ is used in vector space $\mathbb{C}^n$. Due to the normalization conditions, we have $(\mathbf{a}, \mathbf{a}) = \|\mathbf{a}\|^2 = 1$. By Gram–Schmidt orthogonalization, there exists an orthonormal basis in the vector space $\mathbb{C}^n$:

$$S_1 = \{\mathbf{a}, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n-1}\}, \tag{7.7}$$

where the set $\{\mathbf{c}_j\}_{j=1}^{n-1}$ spans an orthogonal compliment of the vector $\mathbf{a}$ in $\mathbb{C}^n$. The components of the eigenvectors $\mathbf{u}(x)$ and $\mathbf{v}(s)$ can be decomposed over the orthonormal basis S_1:

$$\mathbf{u}(x) = \alpha^+(x)\mathbf{a} + \sum_{j=1}^{n-1} \gamma_j^+(x)\mathbf{c}_j, \quad \mathbf{v}(x) = \alpha^-(x)\bar{\mathbf{a}} + \sum_{j=1}^{n-1} \gamma_j^-(x)\bar{\mathbf{c}}_j.$$

The linear eigenvalue problem (7.2) is projected to the basis S_1, which results in the following system of equations for (α^+, α^-) and (γ_j^+, γ_j^-):

$$\begin{aligned}\left(\omega - \Delta + W'(|\varphi|^2)\right)\alpha^+ + W''(|\varphi|^2)(|\varphi|^2\alpha^+ + \varphi^2\alpha^-) &= i\lambda\alpha^+, \\ \left(\omega - \Delta + W'(|\varphi|^2)\right)\alpha^- + W''(|\varphi|^2)(\overline{\varphi}^2\alpha^+ + |\varphi|^2\alpha^-) &= -i\lambda\alpha^-\end{aligned} \tag{7.8}$$

and

$$\begin{aligned}\left(\omega - \Delta + W'(|\varphi|^2)\right)\gamma_j^+ &= i\lambda\gamma_j^+, \\ \left(\omega - \Delta + W'(|\varphi|^2)\right)\gamma_j^- &= -i\lambda\gamma_j^-.\end{aligned} \tag{7.9}$$

The 2-by-2 non-self-adjoint eigenvalue problem (7.8) is uncoupled from the set of self-adjoint eigenvalue problems (7.9). The stability of vector vortices (6.9) is defined entirely by the stability of scalar vortices (6.8) in the 2-by-2 non-self-adjoint eigenvalue problem (7.8).

(ii) Symplectic rotations of the two-component vortex pair with hidden charge (6.17)**.** The stationary solution of the vector form (6.1) with a general superposition (6.14) can be obtained from the two-component vortex pair with hidden charge (6.17) by using symplectic rotations. We show that the linearized problem (7.2)–(7.3) for the vector solutions (6.1) with a general superposition (6.14) can be block-diagonalized into a coupled 4-by-4 (non-self-adjoint) linearized operator and $(n-2)$ pairs of uncoupled (self-adjoint) linear Schrödinger operators.

Let $\varphi(x)$ be given by (6.1) with the general superposition (6.14) under the normalization condition (6.15). Then, the operator $\mathcal{H}$ is rewritten in the form

$$\mathcal{H} = \left(\omega - \Delta + W'(R^2)\right)\begin{pmatrix} I & O \\ O & I \end{pmatrix} + R^2 W''(R^2)\mathcal{H}_1, \tag{7.10}$$

where

$$\mathcal{H}_1 = \begin{pmatrix} \mathbf{a} \\ \bar{\mathbf{b}} \end{pmatrix} \cdot \left(\bar{\mathbf{a}}^T \ \ \mathbf{b}^T\right) + \begin{pmatrix} \mathbf{b} \\ \bar{\mathbf{a}} \end{pmatrix} \cdot \left(\bar{\mathbf{b}}^T \ \ \mathbf{a}^T\right) + \begin{pmatrix} \mathbf{a} \\ \bar{\mathbf{b}} \end{pmatrix} \cdot \left(\bar{\mathbf{b}}^T \ \ \mathbf{a}^T\right) e^{2im\theta} + \begin{pmatrix} \mathbf{b} \\ \bar{\mathbf{a}} \end{pmatrix} \cdot \left(\bar{\mathbf{a}}^T \ \ \mathbf{b}^T\right) e^{-2im\theta}.$$

The polar coordinates (r, θ) can be separated by using the Fourier series for periodic functions on $\theta \in [0, 2\pi]$:

$$\begin{pmatrix} \mathbf{u}(r,\theta) \\ \mathbf{v}(r,\theta) \end{pmatrix} = \sum_{s \in \mathbb{Z}} \begin{pmatrix} \mathbf{u}_s(r) \\ \mathbf{v}_s(r) \end{pmatrix} e^{is\theta}, \tag{7.11}$$

such that the system (7.2) reduces to the form

$$\begin{aligned}&\left(\omega - \Delta_s + W'(R^2)\right)\begin{pmatrix} \mathbf{u}_s \\ \mathbf{v}_s \end{pmatrix} + R^2 W''(R^2)\left[\begin{pmatrix} \mathbf{a} \\ \bar{\mathbf{b}} \end{pmatrix} \cdot \left(\bar{\mathbf{a}}^T \ \ \mathbf{b}^T\right) + \begin{pmatrix} \mathbf{b} \\ \bar{\mathbf{a}} \end{pmatrix} \cdot \left(\bar{\mathbf{b}}^T \ \ \mathbf{a}^T\right)\right]\begin{pmatrix} \mathbf{u}_s \\ \mathbf{v}_s \end{pmatrix} \\ &\quad + R^2 W''(R^2)\left[\begin{pmatrix} \mathbf{a} \\ \bar{\mathbf{b}} \end{pmatrix} \cdot \left(\bar{\mathbf{b}}^T \ \ \mathbf{a}^T\right)\begin{pmatrix} \mathbf{u}_{s-2m} \\ \mathbf{v}_{s-2m} \end{pmatrix} + \begin{pmatrix} \mathbf{b} \\ \bar{\mathbf{a}} \end{pmatrix} \cdot \left(\bar{\mathbf{a}}^T \ \ \mathbf{b}^T\right)\begin{pmatrix} \mathbf{u}_{s+2m} \\ \mathbf{v}_{s+2m} \end{pmatrix}\right] = i\lambda\begin{pmatrix} \mathbf{u}_s \\ -\mathbf{v}_s \end{pmatrix},\end{aligned}$$

where

$$\Delta_s = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{s^2}{r^2}. \tag{7.12}$$

Let $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$. There exists an orthogonal basis in the vector space $\mathbb{C}^n$:

$$S_2 = \{\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n-2}\}, \tag{7.13}$$

where the set $\{\mathbf{c}_j\}_{j=1}^{n-1}$ spans an orthogonal compliment of the vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{C}^n$. It follows from the normalization condition (6.15) that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal in $\mathbb{C}^n$, while the norms of $\mathbf{a}$ and $\mathbf{b}$ are defined by the parameter $\mu \in (-1, 1)$:

$$|\mathbf{a}|^2 = \frac{1+\mu}{2}, \qquad |\mathbf{b}|^2 = \frac{1-\mu}{2}.$$

such that $|\mathbf{a}|^2 + |\mathbf{b}|^2 = 1$ and $|\mathbf{a}|^2 - |\mathbf{b}|^2 = \mu$. The components of the eigenvectors $\mathbf{u}_s(r)$ and $\mathbf{v}_s(r)$ can be decomposed over the orthonormal basis S_2:

$$\mathbf{u}_s(r) = \alpha_s^+(r)\mathbf{a} + \beta_s^+(r)\mathbf{b} + \sum_{j=1}^{n-2} \gamma_{s,j}^+(r)\mathbf{c}_j, \quad \mathbf{v}_s(r) = \alpha_s^-(r)\bar{\mathbf{b}} + \beta_s^-(r)\bar{\mathbf{a}} + \sum_{j=1}^{n-2} \gamma_{s,j}^-(r)\bar{\mathbf{c}}_j.$$

The linear eigenvalue problem (7.2) is projected to the basis S_2, which results in the following system of equations for $(\alpha_s^+, \alpha_s^-, \beta_{s-2m}^+, \beta_{s-2m}^-)$ and $(\gamma_{s,j}^+, \gamma_{s,j}^-)$:

$$\begin{aligned}
\left(\omega - \Delta_s + W'(R^2)\right) \alpha_s^+ + R^2 W''(R^2)\Upsilon &= i\lambda\alpha_s^+, \\
\left(\omega - \Delta_s + W'(R^2)\right) \alpha_s^- + R^2 W''(R^2)\Upsilon &= -i\lambda\alpha_s^-, \\
\left(\omega - \Delta_{s-2m} + W'(R^2)\right) \beta_{s-2m}^+ + R^2 W''(R^2)\Upsilon &= i\lambda\beta_{s-2m}^+, \\
\left(\omega - \Delta_{s-2m} + W'(R^2)\right) \beta_{s-2m}^- + R^2 W''(R^2)\Upsilon &= -i\lambda\beta_{s-2m}^-,
\end{aligned} \tag{7.14}$$

and

$$\left(\omega - \Delta_s + W'(R^2)\right) \gamma_{s,j}^+ = i\lambda\gamma_{s,j}^+, \quad \left(\omega - \Delta_s + W'(R^2)\right) \gamma_{s,j}^- = -i\lambda\gamma_{s,j}^-, \tag{7.15}$$

where

$$\Upsilon = \frac{1+\mu}{2}(\alpha_s^+ + \beta_{s-2m}^-) + \frac{1-\mu}{2}(\alpha_s^- + \beta_{s-2m}^+).$$

The 4-by-4 non-self-adjoint eigenvalue problem (7.14) is uncoupled from the set of self-adjoint eigenvalue problems (7.15). The stability of vector vortices (6.1) with a general superposition (6.14) is defined entirely by the stability of the two-component vortex pair with hidden charge (6.17) in the 4-by-4 non-self-adjoint eigenvalue problem (7.14). When $\mu = \pm 1$, the non-self-adjoint problem reduces to the 2-by-2 non-self-adjoint problem and 2 uncoupled self-adjoint problems, since the vortex pair (6.17) with $\mu = \pm 1$ degenerates into the scalar vortex (6.8). When $\mu \in (-1, 1)$, no further reductions of the non-self-adjoint problem exist. A symmetric case $\mu = 0$ deserves a special consideration since this case corresponds to equal amplitudes $\alpha_1 = \alpha_2$ in the vortex pair (6.17). Numerical analysis of the 4-by-4 non-self-adjoint eigenvalue problem in the symmetric case $\mu = 0$ is reported in [41] for the saturable NLS equations and in [12] for the cubic–quintic NLS equations.

(iii) General case. In general, the family of stationary vortex solutions is generated from a p-component vortex with different charges and q two-component vortices with hidden charge, where $p + 2q \leq n$. Therefore, the linearized problem (7.2) for these vortex solutions can be block-diagonalized into corresponding blocks. If one is interested in the stability of a composite vortex obtained by symplectic rotations of a simple vortex (called the seed vortex), the following algorithm can be used:

(1) Start with the linearized problem for the seed vortex, e.g., the scalar vortex (6.8) or the vortex pair with hidden charge (6.17).
(2) Obtain a complete set of eigenvalues and eigenvectors of the uncoupled linearized problems.
(3) Rotate the seed vortex and the eigenvectors of the linearized problem with the same group of symplectic rotations used to obtain a composite vortex.

For instance, stability of the vortex pair with double charge (6.9) can be studied from that of the scalar vortex (6.8), and stability of vortices with a general superposition (6.14) can be studied from that of the vortex pair with hidden charge (6.17).

8. Soliton-Vortex Coupled States in Saturable Coupled NLS Equations

We shall consider a local bifurcation of a coupled state between a fundamental soliton in one component and a vortex pair with either double or hidden charge in the other two components. An example of such three-component coupled states occurs in the saturable coupled NLS equations (2.1) with $n = 3$ and

$$W'(E) = -\frac{E}{1 + \sigma E}, \qquad 0 \leq \sigma \leq 1. \tag{8.1}$$

Existence and stability of such structures were addressed recently in [38] by direct numerical simulations. We develop here a local bifurcation analysis which extends our earlier work [40] and correct some inconsistencies of the numerical results in [38].

Let us use the normalization $\omega_1 = 1$. The fundamental soliton of the first component $\varphi_1 = U(x)$ satisfies the scalar boundary-value problem

$$U \in L^2(\mathbb{R}^2): \qquad \Delta U - U = W'(U^2)U, \tag{8.2}$$

where $U(x)$ is a real-valued nonnegative ground state. The local bifurcation of the vortex pair in the second and third components of the solution occurs when the linear operator $\mathcal{L} = -\Delta + W'(U^2)$ admits a nontrivial bound state for $\varphi_2(x)$ and $\varphi_3(x)$:

$$\mathcal{L}\varphi_j = -\omega_j \varphi_j, \qquad j = 2, 3, \tag{8.3}$$

where $\omega_j > 0$. By Lemma 3, if $\omega_{2,3} \neq 1$, then

$$\int_{\mathbb{R}^2} \varphi_{2,3}(x)U(x)dx = \int_{\mathbb{R}^2} \bar{\varphi}_{2,3}(x)U(x)dx = 0.$$

If $U(x)$ is a radially symmetric fundamental soliton and $\varphi_{2,3}(x)$ represent a vortex pair with either double or hidden charge, then the constraints are trivially satisfied due to integration in the angular variable θ. By Lemma 3, if $\omega_2 \neq \omega_3$, then $\int_{\mathbb{R}^2} \varphi_2(x)\varphi_3(x)dx = \int_{\mathbb{R}^2} \bar{\varphi}_2(x)\varphi_3(x)dx = 0$. The constraints are not satisfied if $\varphi_{2,3}(x)$ represent a vortex pair with either double or hidden charge, such that these solutions exist only when $\omega_2 = \omega_3 \equiv \omega$. By Lemma 4, the functions $\varphi_{2,3}(x)$ for the vortex pairs are written in polar coordinates by the separation of variables (6.1) with $R_2 = R_3 \equiv R(r)$ and $m_2 = \pm m_3 \equiv m$.

We shall consider both vortex pairs in the diagonal form:

$$\varphi_1 = U(r), \qquad \varphi_2 = \alpha_2 R(r) e^{im\theta}, \qquad \varphi_3 = \alpha_3 R(r) e^{\pm im\theta}, \tag{8.4}$$

where $m \in \mathbb{N}$,

$$\alpha_2 = \sqrt{\frac{1+\mu}{2}}, \qquad \alpha_3 = \sqrt{\frac{1-\mu}{2}}, \qquad -1 \leq \mu \leq 1,$$

and the functions $U(r)$ and $R(r)$ satisfy the coupled system of ODEs:

$$U''(r) + \frac{1}{r}U'(r) - U(r) = W'(U^2 + R^2)U, \tag{8.5}$$

$$R''(r) + \frac{1}{r}R'(r) - \frac{m^2}{r^2}R(r) - \omega R(r) = W'(U^2 + R^2)R. \tag{8.6}$$

The functions $U(r)$ and $R(r)$ are real-valued nonnegative ground states for $r > 0$ with $U(0) > 0$ and $R(0) = 0$. By the group of symplectic rotations, the vortex pair with double charge (m, m) is equivalent to the scalar vortex with $\alpha_2 = 1$ and $\alpha_3 = 0$, while the vortex pair with hidden charge $(m, -m)$ can be rotated into a more general solution family with a general superposition (6.14). The coupled problem (8.5)–(8.6) is the same as in the two-component NLS system. Local bifurcation of the vortex solution (U, R) with $m = 1$ is considered analytically and numerically for the saturable nonlinearity in [40]. The perturbation series for the vortex solution near the local bifurcation threshold takes the form

$$U = u_0(r) + \epsilon^2 u_2(r) + O(\epsilon^4), \quad R = \epsilon\phi_1(r) + \epsilon^3\phi_3(r) + O(\epsilon^4), \quad \omega = \omega_0 + \epsilon^2\omega_2 + O(\epsilon^4), \tag{8.7}$$

where the function $u_0(r)$ solves the nonlinear ODE

$$u_0'' + \frac{1}{r}u_0' - u_0 = W'(u_0^2)u_0,$$

the function $\phi_1(r)$ is a eigenvector of the operator $\mathcal{L}$ with $U = u_0$ and charge m for the eigenvalue $\omega = \omega_0$,

$$\phi_1'' + \frac{1}{r}\phi_1' - \frac{m^2}{r^2}\phi_1 - \omega_0\phi_1 = W'(u_0^2)\phi_1,$$

the function $u_2(r)$ solves the inhomogeneous linear ODE:

$$u_2'' + \frac{1}{r}u_2' - u_2 - W'(u_0^2)u_2 - 2W''(u_0^2)u_0^2 u_2 = W''(u_0^2)u_0\phi_1^2,$$

and the correction term ω_2 is found from the inhomogeneous linear ODE for $\phi_3(r)$:

$$\omega_2 \int_0^\infty \phi_1^2(r) r dr = \int_0^\infty W''(u_0^2)(2u_0 u_2 + \phi_1^2)\phi_1^2 r dr. \tag{8.8}$$

Numerical results on the local bifurcation in the saturable NLS system (2.1) with (8.1) suggest that $\omega_2 > 0$ (at least for $m = 1$ and $\sigma = 0.5$) [40]. The linearized stability problem for the coupled state takes the form (7.2)–(7.3). The vortex pair (m, m) is equivalent to the scalar vortex. By the block-diagonalization of Sec. 7, the stability problem in the three-component system is equivalent to the stability problem in the two-component system.

Specific details on unstable eigenvalues for the vortex pair with double charge are explained in [40] with the local bifurcation analysis. It is shown there that the vortex pair with $m = 1$ is spectrally stable near the local bifurcation threshold but it has two pairs of potentially unstable eigenvalues: the purely imaginary eigenvalues of negative energy for $s = 1$ and the double zero eigenvalue for $s = 2$, where s occurs in the Fourier series decomposition (7.11). By Lemma 5, the pair of purely imaginary eigenvalues for $s = 1$ is forced to stay on the imaginary axis. By the perturbation analysis and numerical computations in [40], the double zero eigenvalue for $s = 2$ bifurcates into a pair of purely imaginary eigenvalues of negative energy and leads to instability far from the local bifurcation threshold. (This instability for $s = 2$ is re-confirmed in the numerical computations of [38] (Fig. 3), where a full three-component stability was considered without block-diagonalization of the linearized system. Results of [38] also report a weak instability for $s = 1$ which does not agree with our analysis and numerical results in [40]. This weak instability is an artificial result caused by the numerical inaccuracies in [38].)

Next, we examine the linear stability for the vortex pair $(m, -m)$ coupled to the fundamental soliton by the asymptotic analysis. When the decomposition (7.1) is supplemented by the separation of variables

$$u_1 = u_+(r)e^{-is\theta}, \quad u_2 = v_+(r)e^{i(m-s)\theta}, \quad u_3 = w_+(r)e^{-i(m+s)\theta},$$
$$v_1 = u_-(r)e^{-is\theta}, \quad v_2 = v_-(r)e^{-i(m+s)\theta}, \quad v_3 = w_-(r)e^{i(m-s)\theta},$$

the linearized problem is rewritten in the explicit form,

$$\begin{aligned}
i\lambda u_+ &= -\Delta_s u_+ + u_+ + V(r)u_+ + V_{11}(r)(u_+ + u_-) + V_{12}(\bar\alpha_2 v_+ + \alpha_2 v_- + \bar\alpha_3 w_+ + \alpha_3 w_-),\\
-i\lambda u_- &= -\Delta_s u_- + u_- + V(r)u_- + V_{11}(r)(u_+ + u_-) + V_{12}(\bar\alpha_2 v_+ + \alpha_2 v_- + \bar\alpha_3 w_+ + \alpha_3 w_-),\\
i\lambda v_+ &= -\Delta_{s-m} v_+ + \omega v_+ + V(r)v_+ + \alpha_2 V_{12}(r)(u_+ + u_-) + \alpha_2 V_{22}(\bar\alpha_2 v_+ + \alpha_2 v_- + \bar\alpha_3 w_+ + \alpha_3 w_-),\\
-i\lambda v_- &= -\Delta_{s+m} v_- + \omega v_- + V(r)v_- + \bar\alpha_2 V_{12}(r)(u_+ + u_-) + \bar\alpha_2 V_{22}(\bar\alpha_2 v_+ + \alpha_2 v_- + \bar\alpha_3 w_+ + \alpha_3 w_-),\\
i\lambda w_+ &= -\Delta_{s+m} w_+ + \omega w_+ + V(r)w_+ + \alpha_3 V_{12}(r)(u_+ + u_-) + \alpha_3 V_{22}(\bar\alpha_2 v_+ + \alpha_2 v_- + \bar\alpha_3 w_+ + \alpha_3 w_-),\\
-i\lambda w_- &= -\Delta_{s-m} w_- + \omega w_- + V(r)w_- + \bar\alpha_3 V_{12}(r)(u_+ + u_-) + \bar\alpha_3 V_{22}(\bar\alpha_2 v_+ + \alpha_2 v_- + \bar\alpha_3 w_+ + \alpha_3 w_-),
\end{aligned}$$

where Δ_s is defined by (7.12) and the potential terms are

$$V = W'(U^2 + R^2), \quad V_{11} = W''(U^2 + R^2)U^2, \quad V_{12} = W''(U^2 + R^2)UR, \quad V_{22} = W''(U^2 + R^2)R^2.$$

At the local bifurcation threshold, when $\epsilon = 0$ in the perturbation expansion (8.7), the linearized problem has two pairs of purely imaginary eigenvalues for $\lambda = \pm i(1 - \omega_0)$ and $s = \pm m$, two double zero eigenvalues for $s = \pm 2m$ and a multiple zero eigenvalue for $s = 0$. The multiple zero eigenvalue for $s = 0$ is controlled by the symmetries of the coupled NLS equations (2.1) and it remains zero beyond the local bifurcation threshold. The two pairs of purely imaginary eigenvalues for $s = \pm m$ are controlled by Lemma 5 and they remain neutrally stable beyond the local bifurcation threshold. The two double zero eigenvalues for

$s = \pm 2m$ could bifurcate into unstable eigenvalues as happens for the coupled two-component soliton-vortex state [40]. We shall hence consider the splitting of these eigenvalues and the consequences of the splitting for stability of the coupled three-component soliton-vortex state.

We construct a perturbation series expansion for the solution of the linearized problem for $s = 2m$ near the local bifurcation threshold:

$$\begin{aligned} u_\pm &= \epsilon u_1^\pm(r) + O(\epsilon^3), & v_+ &= c_2\phi_1(r) + \epsilon^2 v_2^+(r) + O(\epsilon^4), & v_- &= \epsilon^2 v_2^-(r) + O(\epsilon^4), \\ w_+ &= \epsilon^2 w_2^+(r) + O(\epsilon^4), & w_- &= c_3\phi_1(r) + \epsilon^2 w_2^-(r) + O(\epsilon^4), & \lambda &= \epsilon^2\lambda_2 + O(\epsilon^4), \end{aligned} \tag{8.9}$$

where $(c_2, c_3) \in \mathbb{C}^2$ are projection coordinates for the two-dimensional kernel of the linearized problem for $s = 2m$. The linear inhomogeneous system for $u_1^\pm(r)$ at $O(\epsilon)$ admits the implicit solution $u_1^\pm(r) = (\bar{\alpha}_2 c_2 + \alpha_3 c_3)\tilde{u}_2(r)$, where $\tilde{u}_2(r)$ is found from the linear inhomogeneous ODE

$$\Delta_{2m}\tilde{u}_2 - \tilde{u}_2 - W'(u_0^2)\tilde{u}_2 - 2W''(u_0^2)u_0^2\tilde{u}_2 = W''(u_0^2)u_0\phi_1^2.$$

The linear inhomogeneous system for $v_2^+(r)$ and $w_2^-(r)$ at $O(\epsilon^2)$ is solvable subject to the projection equations:

$$i\lambda_2 x_1 = -\mu C_0 x_1, \qquad i\lambda_2 x_2 = -C_0 x_1, \tag{8.10}$$

where

$$x_1 = \bar{\alpha}_2 c_2 + \alpha_3 c_3, \qquad x_2 = \bar{\alpha}_2 c_2 - \alpha_3 c_3,$$

the constant C_0 is defined by

$$C_0 \int_0^\infty \phi_1^2(r) r dr = \int_0^\infty W''(u_0^2)(2u_0\tilde{u}_2 + \phi_1^2)\phi_1^2 r dr,$$

and the equation (8.8) has been used. Two eigenvalues exist in the reduced eigenvalue problem (8.10): $\lambda_2 = i\mu C_0$ and $\lambda_2 = 0$. The zero eigenvalue is preserved at $\lambda = 0$ due to the translation of the vortex pair with hidden charge (8.4) along the arbitrary parameter μ. When $\mu \neq 0$, the nonzero eigenvalue is purely imaginary near the local bifurcation threshold. (Another purely imaginary eigenvalue $\lambda_2 = -i\mu C_0$ exists for $s = -2m$.) The pair of purely imaginary eigenvalues has negative energy, similar to the case of the scalar vortex in [40], which is recovered here for $\mu = \pm 1$. Therefore, this pair will lead to instability far from the local bifurcation threshold when the pair collides either with eigenvalues with positive energy or with the continuous spectrum of the linearized problem.

When $\mu = 0$, the eigenvalue $\lambda_2 = i\mu C_0$ is zero. This additional zero is preserved as the generalized kernel of the linearized problem for $s = 2m$ becomes two-dimensional when $\alpha_2 = \alpha_3 = 1/\sqrt{2}$. Indeed, the eigenvector of the linearized problem for $\lambda = 0$

$$u_+ = u_- = v_- = w_+ = 0, \qquad v_+ = -w_- = R(r),$$

satisfies the Fredholm alternative theorem for the generalized kernel. Therefore, the zero eigenvalues for $s = \pm 2m$ remain zero for the symmetric vortex pair with hidden charge (when $\mu = 0$) beyond the local bifurcation threshold.

In the end, we give specific details of unstable eigenvalues for the coupled state between fundamental soliton and the vortex pair with hidden charge. Since no other potentially unstable eigenvalues exist at the local bifurcation threshold [40], the analysis above guarantees stability of the coupled state near the local bifurcation threshold. Moreover, potentially unstable eigenvalues with $s = \pm 1$ and $s = \pm 2$ are proved to lead not to instability of the symmetric vortex pair with $\mu = 0$ far from the local bifurcation threshold. This perturbation analysis explains the main conclusion of [8, 41] that the vortex pair with hidden charge is more stable than the vortex pair with double charge when it is coupled with the stable fundamental soliton. This result is also in agreement with the numerical results of [38] reported on Fig. 3, where no instability was discovered for $s = 2$ ($m = 1$). (The weak instability for $s = 1$ in [38] is again an artificial result caused by the inaccuracies of the numerical scheme.)

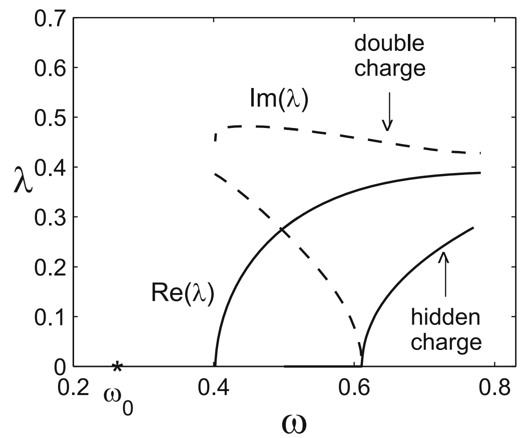

Fig. 4. Unstable eigenvalues of the coupled state between fundamental soliton and the vortex pair with double and hidden charges for $\sigma = 0.5$.

The results above do not exclude other mechanisms of instabilities which may occur far from the local bifurcation threshold. Figure 4 shows unstable eigenvalues of the coupled state between the fundamental soliton and the vortex pairs with double and hidden charges for $\sigma = 0.5$. The unstable eigenvalue for the vortex pair with double charge $m = 1$ corresponds to $s = 2$, in agreement with the previous discussion (this part of the graph is published in [40]). The unstable eigenvalue for the symmetric vortex pair with hidden charge $m = 1$ corresponds to $s = 3$ (this part of the graph is in agreement with Fig. 3 in [38]).

The unstable eigenvalues for the vortex pair with double charge $m = 1$ are complex. They originate from the continuous spectrum at $\omega \approx 0.41$ and exist for $\omega > 0.41$. The unstable eigenvalue for the symmetric vortex pair with hidden charge $m = 1$ is real. The pair of real eigenvalues coalesces at the origin for $\omega \approx 0.61$ and becomes a pair of purely imaginary eigenvalues for $\omega < 0.61$. Near the value $\omega \approx 0.39$, the pair of purely imaginary eigenvalues merges with the continuous spectrum and disappears.

It is clear from Fig. 8 that the vortices with hidden charge are more stable than vortices with double charge due to two reasons. First, the stability region of hidden-charge vortices $\omega_0 < \omega < 0.61$ is wider than the stability region of double-charge vortices $\omega_0 < \omega < 0.41$, where $\omega_0 = 0.26$ is the local bifurcation threshold for $\sigma = 0.5$. Second, when both vortices are unstable, growth rates of the hidden-charge vortices are smaller than those of the double-charge vortices.

Concerning evolution of unstable vortices far from the local bifurcation threshold, it has been studied in the previous literature. The vortex pair with double charge either breaks into a rotating dipole solution or splits into two fundamental solitons that move apart (see Figs. 7 and 8 in [40]). The coupled state between fundamental soliton and a vortex pair with hidden charge breaks up into three fundamental solitons that move apart (see Fig. 8 in [38]). The reason for this difference is quite simple: hidden-charge vortices have zero net angular momentum. Due to momentum conservation of the original system of coupled NLS equations, the filaments do not rotate. On the contrary, double-charge vortices have nonzero net angular momentum and the filaments must rotate. The net angular momentum of the hidden-charge vortices is the physical reason why unstable eigenvalues of the corresponding linearized problem are real rather than complex.

9. Conclusion

Although the model with a purely incoherent coupling between nonlinear modes is only an approximation to real physical models, this model is important in the perturbation theory. If partial coherence is accounted, which breaks the rotational symmetries of the coupled NLS system, zero eigenvalues corresponding to the additional group of symplectic rotations bifurcate to the complex domain. The family of vortex solutions may undertake various bifurcations, which can be studied by using the Lyapunov-Schmidt theory. In the context of these problems, the results obtained in our paper represent a necessary step in the construction and analysis of the perturbation series expansions. An example of the perturbation series expansion is reported in [29].

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REFERENCES

1. M. Ahles, K. Motzek, A. Stepken, F. Kaiser, C. Weilnau, and C. Denz, "Stabilization and breakup of coupled dipole-mode beams in an anisotropic nonlinear medium," *J. Opt. Soc. Amer. B*, **19**, 557–562 (2002).
2. C. Anastassiou, M. Soljacic, M. Segev, E. D. Eugenieva, D. N. Christodoulides, D. Kip, Z. H. Musslimani, and J. P. Torres, "Eliminating the transverse instabilities of Kerr solitons," *Phys. Rev. Lett.*, **85**, 4888–4891 (2000).
3. L. Berge, "Wave collapse in physics: principles and applications to light and plasma waves," *Phys. Rep.*, **303**, 260–370 (1998).
4. M. S. Bigelow, Q. H. Park, and R. W. Boyd, "Stabilization of the propagation of spatial solitons," *Phys. Rev. E*, **66**, 046631 (2002).
5. D. Briedis, D. E. Petersen, D. Edmundson, W. Krolikowski, and O. Bang, "Ring vortex solitons in nonlocal nonlinear media," *Optics Express*, **13**, 435 (2005).
6. D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, "Theory of incoherent self-focusing in biased photorefractive media," *Phys. Rev. Lett.*, **78**, 646–649 (1997).
7. A. S. Desyatnikov and Yu. S. Kivshar, "Necklace-ring vector solitons," *Phys. Rev. Lett.*, **87**, 033901 (2001).
8. A. S. Desyatnikov, Yu. S. Kivshar, K. Motzek, F. Kaiser, C. Weilnay, and C. Denz, "Multicomponent dipole-mode spatial solitons," *Optics Lett.*, **27**, 634–636 (2002).
9. A. S. Desyatnikov, Yu. S. Kivshar, and L. Torner, "Optical vortices and vortex solitons," *Prog. Optics*, **47**, 219–319.
10. A. S. Desyatnikov, D. Neshev, E. A. Ostrovskaya, Yu. S. Kivshar, W. Krolikowski, B. Luther-Davies, J. J. Garcia-Ripoll, and V. M. Perez-Garcia, "Multipole spatial vector solitons," *Optics Lett.*, **26**, 435–437 (2001).
11. A. S. Desyatnikov, D. Neshev, E. A. Ostrovskaya, Yu. S. Kivshar, G. McCarthy, W. Krolikovski, and B. Luther-Davies, "Multipole composite spatial solitons: theory and experiment," *J. Opt. Soc. Amer. B*, **19**, 586 (2002).
12. A. S. Desyatnikov, D. Mihalache, D. Mazilu, B. A. Malomed, C. Denz, and F. Lederer, "Two-dimensional solitons with hidden and explicit vorticity in bimodal cubic-quintic media," *Phys. Rev. E*, **71**, 026615 (2005).
13. A. L. Fetter and A. A. Svidzinsky, "Vortices in a trapped dilute Bose–Einstein condensate," *J. Phys. Condensed Matter*, **13**, R135–R194 (2001).
14. W. J. Firth and D. V. Skryabin, "Optical solitons carrying orbital angular momentum," *Phys. Rev. Lett.*, **79**, 2450 (1997).
15. J. J. Garcia-Ripoll, V. M. Perez-Garcia, E. A. Ostrovskaya, and Yu. S. Kivshar, "Dipole-mode vector solitons," *Phys. Rev. Lett.*, **85**, 82–85 (2000).

16. C. C. Jeng, M. F. Shih, K. Motzek, and Yu. Kivshar, “Partially incoherent optical vortices in self-focusing nonlinear media,” *Phys. Rev. Lett.*, **92**, 043904–4 (2004).
17. Yu. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, San Diego (2003).
18. S. V. Manakov, “On the theory of two-dimensional stationary self focussing of electromagnetic waves,” *Zh. Eksp. Teor. Fiz.*, **38**, 248–253 (1974).
19. A. V. Mamaev, M. Saffman, and A. A. Zozulya, “Propagation of a mutually incoherent optical vortex pair in anisotropic nonlinear media,” *J. Optics B*, **6**, S318 (2004).
20. K. R. Meyer and G. R. Hall, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Springer-Verlag, New York (1992).
21. D. Mihalache, D. Mazilu, I. Towers, B. A. Malomed, and F. Lederer, “Stable two-dimensional spinning solitons in a bimodal cubic-quintic model with four-wave mixing,” *J. Optics A*, **4**, 615 (2002).
22. K. Motzek, F. Kaiser, W. H. Chu, M. F. Shih, and Yu. S. Kivshar, “Soliton transverse instabilities in anisotropic nonlocal self-focusing media,” *Optics Lett.*, **29**, 280–282 (2004).
23. K. Motzek, F. Kaiser, C. Weilnau, C. Denz, G. McCarthy, W. Krolikowski, A. Desyatnikov, and Yu. S. Kivshar, “Multi-component vector solitons in photorefractive crystals,” *Optics Commun.*, **209**, 501–506 (2002).
24. K. Motzek, Yu. S. Kivshar, M. F. Shih, and G. A. Swartzlander, “Spatial coherence singularities and incoherent vortex solitons,” *J. Opt. Soc. Amer. B*, **22**, 1437–1442 (2005).
25. Z. H. Musslimani, M. Segev, and D. N. Christodoulides, “Multicomponent two-dimensional solitons carrying topological charges,” *Optics Lett.*, **25**, 61 (2000).
26. Z. H. Musslimani, M. Segev, D. N. Christodoulides, and M. Soljacic, “Composite multihump vector solitons carrying topological charge,” *Phys. Rev. Lett.*, **84**, 1164–1167 (2000).
27. J. F. Nye and M. V. Berry, “Dislocations in wave trains,” *Proc. Roy. Soc. London A*, **336**, 165–190 (1974).
28. D. M. Palacios, I. D. Maleev, A. S. Marathay, and G. A. Swartzlander, “Spatial correlation singularity of a vortex field,” *Phys. Rev. Lett.*, **92**, 143905–4 (2004).
29. Q. H. Park and J. H. Eberly, “Nontopological vortex in a two-component Bose–Einstein condensate,” *Phys. Rev. A*, **70**, 021602(R) (2004).
30. D. V. Skryabin, J. M. McSloy, and W. J. Firth, “Stability of spiralling solitary waves in Hamiltonian systems,” *Phys. Rev. E*, **66**, 055602-4 (2002).
31. M. Soljacic, S. Sears, and M. Segev, “Self-trapping of “necklace” beams in self-focusing Kerr media,” *Phys. Rev. Lett.*, **81**, 4851–4854 (1998).
32. M. Soljacic and M. Segev, “Self-trapping of “necklace-ring” beams in self-focusing Kerr media,” *Phys. Rev. E*, **62**, 2810–2820 (2000).
33. M. Soljacic and M. Segev, “Integer and fractional angular momentum borne on self-trapped necklace-ring beams,” *Phys. Rev. Lett.*, **86**, 420–423 (2001).
34. M. S. Soskin and M. V. Vasnetsov, “Singular pptics,” *Prog. Optics*, **42**, 219–276 (2001).
35. C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, Springer-Verlag, New York (1999).
36. M. Vasnetsov and K. Staliunas, Eds, *Optical Vortices*, Horizons in World Physics, **228**, Nova Science, Huntington, New York (1999).
37. L. T. Vuong, T. D. Grow, A. Ishaaya, A. L. Gaeta, G. W. Hooft, E. R. Eliel, and G. Fibich, “Collapse of optical vortices,” *Phys. Rev. Lett.* (in press).
38. J. Wang, F. Ye, L. Dong, T. Zai, Y. P. Li, “Composite vector solitons with topological charges and their stability analysis,” *Proc. SPIE*, **5646**, 6–16 (2005).
39. A. I. Yakimenko, Yu. A. Zaliznyak, and Yu. S. Kivshar, “Stable vortex solitons in nonlocal self-focusing nonlinear media,” *Phys. Rev. E*, **71**, 065603 (2005).
40. J. Yang and D. E. Pelinovsky, “Stable vortex and dipole vector solitons in a saturable nonlinear medium,” *Phys. Rev. E*, **67**, 016608 (2003).

41. F. Ye, J. Wang, L. Dong, Y.-P. Li, “Suppresion of modulational instability of ring vector solitons,” *Optics Commun.*, **230**, 219–223 (2004).

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