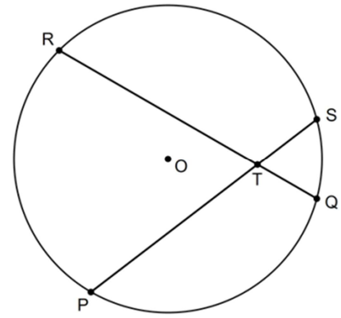
THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FIFTY-NINTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 9, 2016

1. Express $\dfrac{\frac{1}{2}-\frac{1}{4}+\frac{1}{6}}{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}}$ as a rational number in lowest terms.

2. If $\dfrac{a^3b^5}{(ab^2)^4}\cdot\dfrac{(a^2b)^5}{a^3b}$ is expressed in the form $a^r b^s$, find the value of $r+s$.

3. Line L passes through the points $(-2,5)$ and $(4,-7)$. What is the slope of the line perpendicular to L ?

4. P, Q, R and S are points on a circle centered at O. Chords $\overline{PS}$ and $\overline{QR}$ intersect at point T as shown in the figure. Find the degree measure of $\angle RTP$ if arc $\overset{\frown}{RS} = 152°$ and arc $\overset{\frown}{PQ} = 148°$.

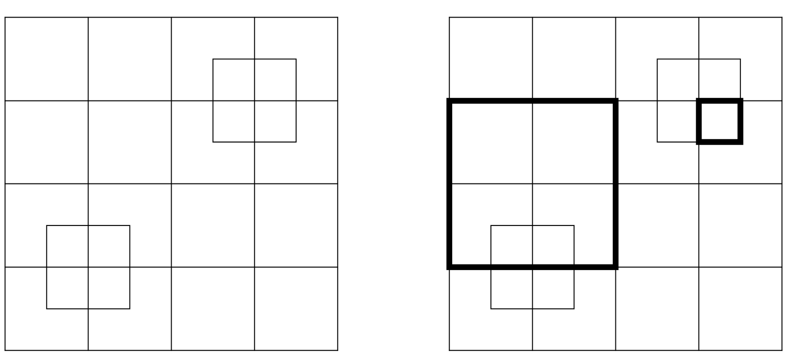


5. Nora rides up a hill on a ski lift that travels at the rate of 7 mph and then skis the same distance down the hill at the rate of 14 mph. What is Nora's average speed for the round trip up and down the hill?

6. Find the value of the product xy if $\dfrac{8^{x+y}}{4^{x+2y}} = 512$ and $\dfrac{25^{2x+y}}{5^{3x}} = 125$.

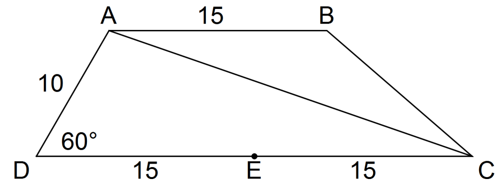
7. Suppose that $z = a + \dfrac{b}{d+\frac{c}{e+f}}$ where each of a, b, c, d, e and f is 1, 2 or 3. Find the maximum possible value of z.

8. How many squares can be formed using edges in the figure on the left below? Two such squares are shown in the figure on the right.

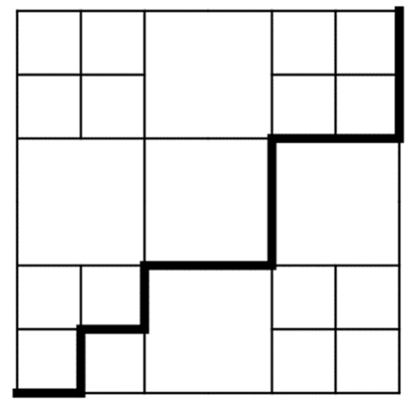


9. A rectangle whose height is four times its width is inscribed in a circle of radius 1. Find the area of the rectangle.

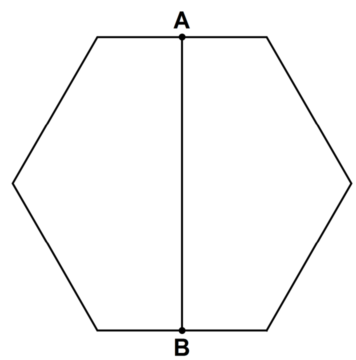
10. In trapezoid $ABCD$, sides $\overline{AB}$ and $\overline{CD}$ are parallel, $\angle ADE = 60°$, $AD = 10$ and $AB = DE = EC = 15$. What is the area of triangle ABC ?



11. Find the number of paths from the lower left corner to the upper right corner of the given grid, if the only allowable moves are along grid lines upward or to the right. One such path is shown.

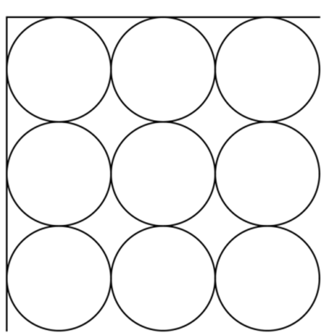


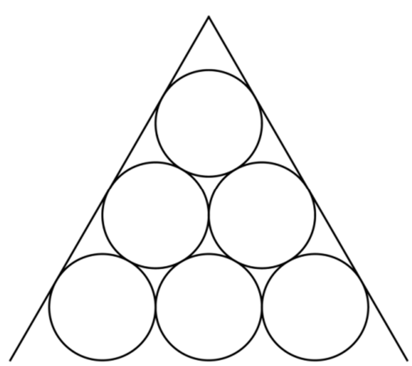
12. The area of a regular hexagon is $24\sqrt{3}$ square units. Line segment $\overline{AB}$ joins the midpoints of a pair of opposite sides of the hexagon. Find the length AB.



13. The two wheels of a bicycle have different sizes; let C be the circumference of the rear wheel. When the bike travels 120 meters, the front wheel makes 6 more turns than the rear wheel. If the circumferences of the front and rear wheels were increased by 25% and 20% respectively, the front wheel would make 4 more turns than the rear wheel over the same distance of 120 meters. Find C.

14. A square and an equilateral triangle are filled with the same number of congruent circles, which are tangent to one another and to the sides of the square and the triangle respectively. A portion of the square and a portion of the triangle are shown below. There are 14 more circles along one side of the triangle than there are along one side of the square. How many circles are there along one side of the square?





15. The height of a trapezoid is 7 and the lengths of its two diagonals are 11 and 9. Determine the area of the trapezoid.

16. Suppose $f(x) = ax^6 + bx^4 + x^3 + x + 13$. If $f(-5) = 11$, what is the value of $f(5)$?

17. Suppose that θ is an angle such that $\cot(\theta) + \csc(\theta) = 3$. What is the value of $\cot(\theta) - \csc(\theta)$?

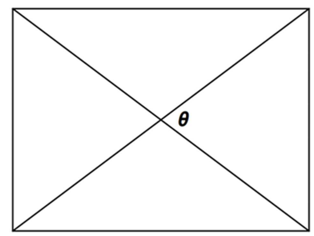
18. The sum of the lengths of the twelve edges of a rectangular box is 148 cm and the distance from one vertex of the box to the furthest vertex is 23 cm. What is the total surface area of the rectangular box?

19. A polynomial $p(t)$ has remainder 2 when divided by $t - 1$ and remainder 17 when divided by $t - 4$. What is the remainder when $p(t)$ is divided by $(t-1)(t-4)$?

20. How many positive integers are divisors of 2016?

21. When the decimal number 15! is expressed as an integer in base 12, the result ends in k zeros. What is the value of k?

22. The diagonals of a rectangle with sides 4 and 3 intersect at an angle θ. Find $\sin(\theta)$.



23. Find the sum of the coefficients in the expansion of the polynomial

$$p(x) = (2x^2 - 3)^{57}(3x^{18} - 9x^{12} + 8x^5 + 1)^4(27x^{103} - 18x^{52} - 4).$$

24. If $x^2 + y^2 = 18$ and $xy = 3$, find the value of $\left(\frac{1}{x} + \frac{1}{y}\right)^4$.

25. Find the value of $\log_3(49)\log_5(\sqrt{2})\log_7(81)\log_8\left(\frac{1}{125}\right)$.

26. Let $a_k = \log_{k^2}(50!)$. Find $\displaystyle\sum_{k=2}^{50} \frac{1}{a_k}$.

27. $\overline{AB}$ is a chord of a circle. If $AB = 14$ cm and the distance from the midpoint of $\overline{AB}$ to the nearest point on the circle is 3 cm, find the diameter of the circle. Express your answer as a rational number in lowest terms.

28. The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 4$ and $a_n = \dfrac{a_{n-1}}{a_{n-2}}$ for $n \geq 3$. Find the sum of the first 2016 terms of the sequence.

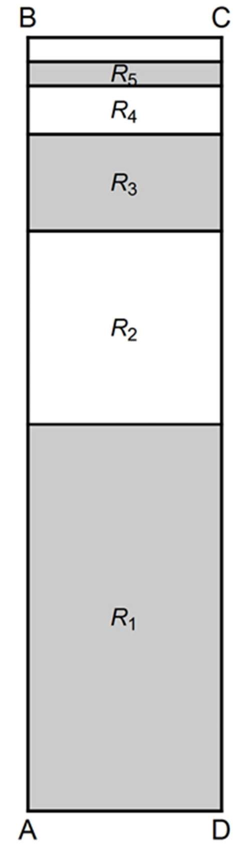
29. Find the largest positive integer n such that $(2016!)^2$ is divisible by 14^n.

30. A palindrome is a sequence of letters that reads the same frontwards and backwards. For example, BCYYYCB is a palindrome. How many different palindromes can be formed by arranging one or more of the letters of GREENMOUNTAINS? For example, M and EGE are two such palindromes that can be formed.

31. Find all values of p for which the two equations $3x^2 - 4x + p - 2 = 0$ and $x^2 - 2px + 5 = 0$ have a common root.

32. Rectangle $ABCD$ has width $AD = 2$ and height $AB = 8$. Lines are drawn parallel to the rectangle's base $\overline{AD}$, forming a sequence of smaller rectangles $\{R_n\}$ within the large rectangle $ABCD$. The height of rectangle R_1 is half the height of the large rectangle $ABCD$, and the height of rectangle R_n is half the height of rectangle R_{n-1} for all $n \geq 2$. The first five of these smaller rectangles are shown in the figure, with the odd numbered rectangles shaded. Let A_n be the area of rectangle R_n.

Find $\displaystyle\sum_{n=1}^{\infty} A_{2n-1}$.



33. Find the value of the product $\sin(20°)\sin(40°)\sin(60°)\sin(80°)$. Express your answer as a rational number in lowest terms.

34. Triangle ABC is inscribed in a circle of radius 2. One of the angles of ΔABC is $\frac{\pi}{3}$ and another of its angles is $\frac{\pi}{4}$. Find the area of ΔABC.

35. Rectangle $ABCD$ has vertices at the points $A(4,4), B(4,8), C(12,8)$ and $D(12,4)$. Find m such that the line $y = mx$ divides rectangle $ABCD$ into two regions of equal area. Express your answer as a rational number in lowest terms.

36. How many ways can 14 identical balls be distributed among 3 distinct boxes with at least 2 balls in each box?

37. Two regular polygons are inscribed in the same circle. One polygon has 2016 sides and the other has 1134 sides. What is the maximum number of vertices that the two polygons can have in common?

38. If $x^2 + y^2 = 6x + 12y + 7$, what is the largest possible value of $3x + 2y$?

39. What is the smallest positive integer n such that $\sqrt{n} - \sqrt{n-1} < 0.005$?

40. When a complex number z is expressed in the form $z = a + bi$ where a and b are real numbers and $i^2 = -1$, the modulus of z, $|z|$, is defined as $|z| = \sqrt{a^2 + b^2}$. Find the maximum value of $|z|$, given that $\left| z + \frac{1}{z} \right| = 1$.

41. Let f be the transformation of the xy-plane defined by $f(x,y) = (x^2 - y^2, 2xy)$. The points (x,y) such that $f\left(f\left(f(x,y)\right)\right) = (-7,24)$ are the vertices of a convex polygon in the plane. Find the area of the polygon.