THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTIETH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 15, 2017

1. Express $\dfrac{\frac{1}{2}-\frac{2}{3}}{\frac{1}{5}-\frac{1}{6}}$ as a rational number in lowest terms.

2. The sum of two prime numbers is 99. What is the product of these two prime numbers?

3. The endpoints of segment $\overline{AB}$ have coordinates $A(x, 10)$ and $B(-2, -6)$. Find x if the midpoint of $\overline{AB}$ is $M(8,2)$.

4. Find the remainder when $2x^5 - 36$ is divided by $x - 2$.

5. Find the y-value for the common solution to the system of equations:

$$\frac{2}{x} + \frac{1}{y} = 5$$

$$\frac{-1}{x} + \frac{3}{y} = 8$$

6. Simplify the expression: $\dfrac{1}{\sqrt{4}+\sqrt{7}} + \dfrac{1}{\sqrt{7}+\sqrt{10}} + \dfrac{1}{\sqrt{10}+\sqrt{13}} + \dfrac{1}{\sqrt{13}+\sqrt{16}}$

    Express the answer as a rational number in lowest terms.

7. An obtuse angle of a rhombus is 6 degrees less than 5 times the degree measure of one of its acute angles. What is the degree measure of the acute angle?

8. When a fair coin is tossed five times, what is the probability that exactly four heads are obtained? Express the answer as a rational number in lowest terms.

9. Segment $\overline{AB}$ has length 3. Point M lies on $\overline{AB}$. Find AM if $AM^2 - BM^2 = 6$. Express the answer as a rational number in lowest terms.

10. If a and b are real numbers with $b \neq 0$, the operation $\oplus$ is defined by $a \oplus b = a^2 - \frac{a}{b}$.
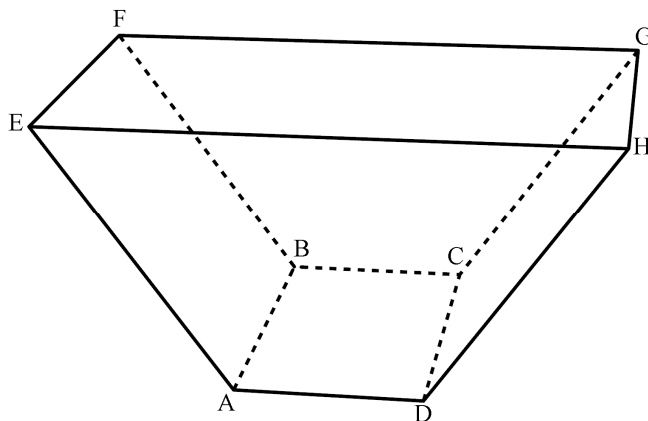   Find $2 \oplus \left(3 \oplus \frac{1}{5}\right)$. Express the answer as a rational number in lowest terms.

11. Express $\dfrac{(20!)^2-(19!)^2}{(21!)^2}$ as a rational number in lowest terms.

12. A store sells black tea for 95 cents per ounce and green tea for $1.43 per ounce. The store's manager plans to create a Special Blend that is a mixture of black tea and green tea, and will sell the Special Blend for $1.10 per ounce. How many ounces of green tea should be contained in one pound of the Special Blend?

13. Find digits A and B (each between 0 and 9 inclusive) such that the number $32A35717B$ is divisible by 72.

14. A $googol$ is 10^{100} and a $googolplex$ is 10^{googol}. Find $\log(\log(\log(googolplex)))$.

15. A vat for storing sand has rectangular base $ABCD$ and rectangular top $EFGH$. Faces $AEFB$ and $CGHD$ are rectangles. Faces $AEHD$ and $BFGC$ are isosceles trapezoids. $AE = DH = 3\sqrt{2}$ feet, $AD = 3$ feet and $\angle AEH = 45°$. If the volume of the vat is 90 cubic feet, what is the perimeter of its rectangular top $EFGH$?


16. Five books are arranged in a row on a bookshelf. The books are removed and returned to the shelf in random order. What is the probability that exactly 3 of the 5 books are returned to their original positions? Express the answer as a rational number in lowest terms.

17. In triangle ABC, $\angle A = 60°$, $\angle B = 70°$ and $AB = 2$. Point D lies on $\overline{AC}$ and $AD = 1$. Find the degree measure of the smallest angle in $\triangle BDC$.

18. A large circle has diameter 20. Inside this large circle are nine small circles of equal radius whose centers lie along two perpendicular diameters of the large circle. The small circles are tangent to one another and to the large circle as shown in the figure. Find the area of the region that lies outside all of the small circles and inside the large circle.
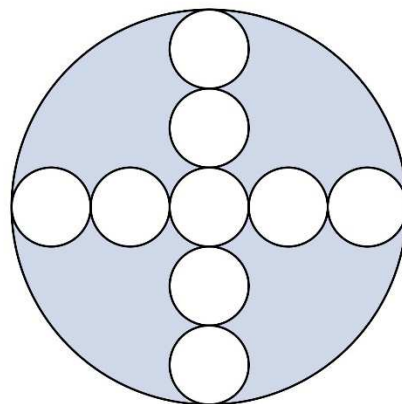

19. Find the sum of all solutions to the equation $(x^2 + 18x + 16)^{x^2 - 5x + 4} = 1$.

20. A rhombus has vertices whose coordinates are $(3,4)$, (a, b), $(9,10)$ and $(c, 6)$ as shown in the figure. Find $a + b + c$.
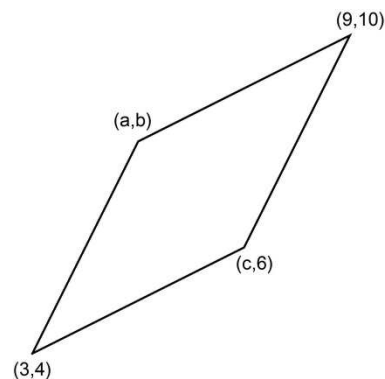

21. One day Jesse leaves school and starts walking toward home at a constant speed. His sister Nora leaves school three minutes later than Jesse and rides her bike at a constant speed, following the same route. Nora passes Jesse two-thirds of the way from school to home and arrives at home two and a half minutes later. How long does it take Jesse to walk home from school?

22. Express $\sqrt[3]{38 + \sqrt{1445}} + \sqrt[3]{38 - \sqrt{1445}}$ in simplest form.

23. Square $ABCD$ has side length 2. Points M, N, O and P are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively. Lines are drawn from A, B, C and D to these midpoints as shown in the figure. Find the area of the shaded region bounded by the lines $\overline{AN}$, $\overline{BO}$, $\overline{CP}$ and $\overline{DM}$. Express the answer as a rational number in lowest terms.
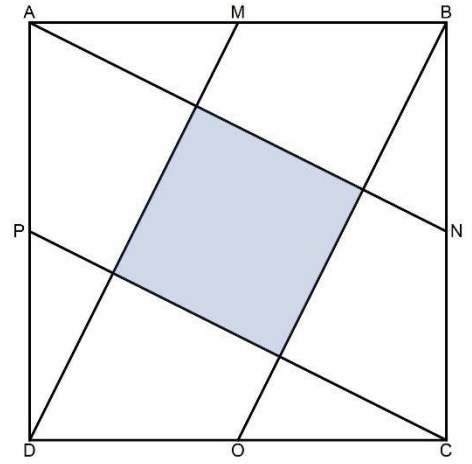
24. ΔABC is an equilateral triangle. A circle of radius 12 cm passes through B and is tangent to $\overline{AC}$. Find the perimeter of ΔABC.
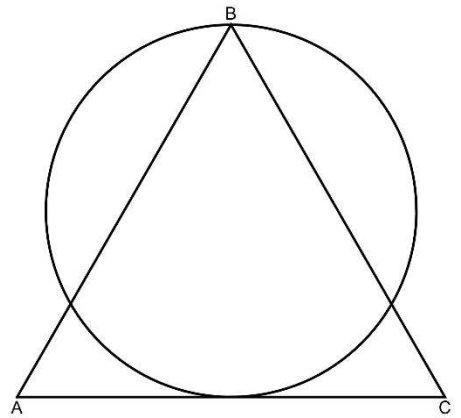
25. If $\sin x = 3 \cos x$, what is the value of $\sin(2x)$? Express the answer as a rational number in lowest terms.

26. Find the value of b that satisfies the following equation.

$$\log_b(10^3) + \log_b(10^4) + \log_b(10^5) + \cdots + \log_b(10^{10}) = 156$$

27. How many even integers between 1000 and 9999 (inclusive) have no repeated digits?

28. Evaluate: $\dfrac{\cos 64° \cos 4° - \cos 86° \cos 26°}{\cos 71° \cos 41° - \cos 49° \cos 19°}$

Express the answer in simplest form.

29. The base of a cube has vertices A, B, C and D as shown in the figure. Vertex E is directly above B. If the surface area of the cube is 54 square feet, find the volume of the tetrahedron whose vertices are A, B, C and E. Express the answer as a rational number in lowest terms.
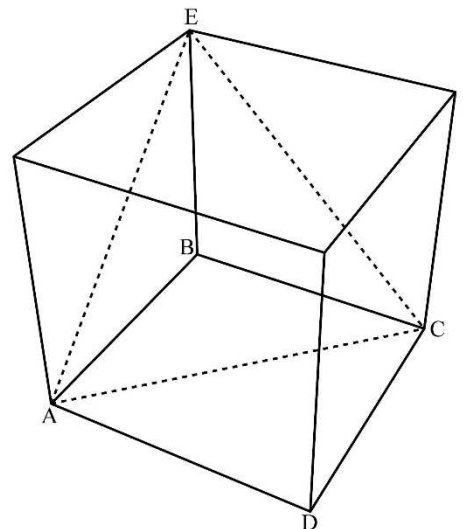
30. Find the units digit of 2017^{7102}.

31. Let P be a point on the circle $x^2 + y^2 + 4x - 8y + 11 = 0$ and Q be a point on the circle $x^2 + y^2 - 20x + 2y + 85 = 0$. Find the smallest possible distance between P and Q.

32. Find the value of $\displaystyle\sum_{n=1}^{2017} n \sin\left(\frac{n\pi}{2}\right)$

33. Triangle ABC has area 180 square units. Five equally spaced lines are drawn parallel to $\overline{AC}$, dividing ΔABC into six regions of equal height as shown in the figure. What is the area of the largest of these six regions?
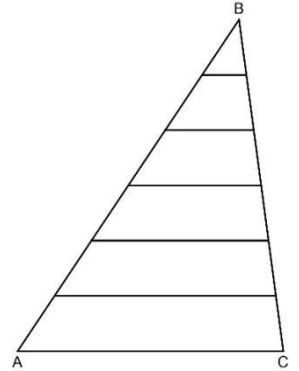
34. If $25^x = 4^y$ and $5^{4x} = 10^6$, find $\frac{1}{x} + \frac{1}{y}$. Express the answer as a rational number in lowest terms.

35. Find the sum of all solutions to $\sin x - \cos x = \frac{1}{10}$ with $0 \le x \le 2\pi$.

36. Point M is inside an equilateral triangle ABC, with $MC = 4$ and $MA = MB = 8$. Find the altitude of ΔABC.

37. At Starlight High, 66% of seniors take calculus, 58% take physics and 50% take music. If 36% of seniors take calculus and physics, 36% take calculus and music and 36% take physics and music, what is the largest possible percentage of seniors who do not take any of the three subjects?

38. Find all real numbers c such that $|x - 3| - |x + 5| - |x - 6| = c$ has exactly three real solutions.

39. Let S_0 be a square of side length 4. A sequence of smaller squares is constructed by joining the midpoints of each previous square. The midpoints of square S_0 are the corners of square S_1; the midpoints of square S_1 are the corners of square S_2; and so forth. For $n \ge 0$, let A_n be the area of the region that lies inside square S_{2n} and outside square S_{2n+1}. The first three such regions are shown (shaded) in the figure.
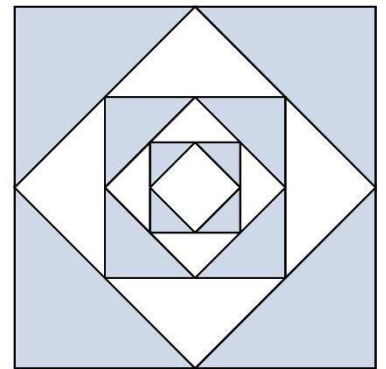
Find $\displaystyle\sum_{n=0}^{\infty} A_n$.

40. In a random arrangement of the letters in **BLUESUEDESHOES** what is the probability that no **E** occurs anywhere to the right of any **U**? Express the answer as a rational number in lowest terms.

41. In trapezoid $ABCD$, sides $\overline{AD}$ and $\overline{BC}$ are parallel, $AD = 34$, $BC = 2$, $AB = 24$ and $\angle A + \angle D = 90°$. Find the radius of a circle that passes through A and B and that is tangent to the line $\overline{CD}$. Express the answer as a rational number in lowest terms.