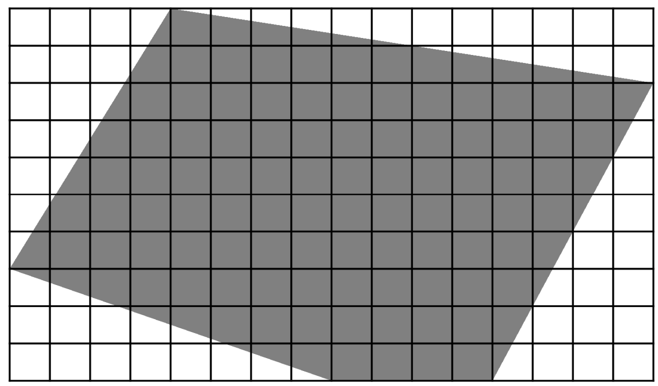
THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-FIRST ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 14, 2018

1. Express $\left(\frac{(2-4)^3}{4^2}\right)^{-5}$ as a rational number in lowest terms.

2. Express $\frac{\sqrt{600}+\sqrt{24}-\sqrt{6}}{\sqrt{600}+\sqrt{24}+\sqrt{6}}$ as a rational number in lowest terms.

3. Each small square in the given 16×10 rectangle has an area of one square unit. Find the area of the shaded region.

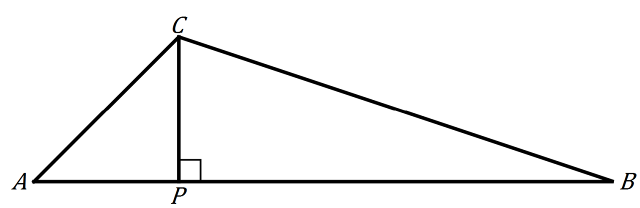


4. A classroom has $\frac{1}{5}$ of its seats occupied. Another 12 students enter the classroom, and it now has $\frac{1}{2}$ of its seats occupied. How many seats does the classroom have?

5. Let $F(x) = \sqrt{15+x} - \sqrt{6-x}$. How many integers are in the domain of $F(x)$?

6. What is the smallest positive integer n for which $1350n$ is a perfect cube?

7. The area of triangle ABC is 50 square units. P is a point on side $\overline{AB}$ with $\overline{CP}$ perpendicular to $\overline{AB}$ as shown in the figure. If $AP = CP = x$ and $PB = 3x$, what is the value of x?



8. Glass A contains 10 ounces of apple juice. Glass B contains 10 ounces of orange juice. Riya pours 1 ounce of juice from glass A into glass B, and mixes the juice thoroughly. Then she pours 1 ounce of juice from glass B into glass A, and mixes the juice thoroughly. What fraction of juice in the glass A is orange juice?

9. For the functions $f(x) = \sqrt{x+1}$ and $g(x) = x^2 + a$, the composite function $f(g(x))$ passes through the point $(1,5)$ in the plane. What is the value of a?

10. Bees make honey by extracting water from nectar. Nectar contains 70% water, while honey contains only 16% water. How many pounds of nectar are needed to make 5 pounds of honey?

11. A triangle has sides of length 24, 56 and n, where n is an integer. Find the difference between the largest possible perimeter of the triangle and the smallest possible perimeter of the triangle.

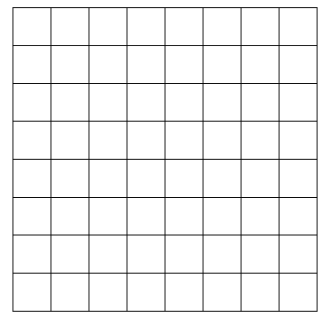
12. Express $\dfrac{2018!}{2018! + 2019!}$ as a rational number in lowest terms.

13. Dawei tosses two nickels and four dimes in the air. What is the probability that the total value of the coins that land heads-up is exactly 20 cents? Express the answer as a rational number in lowest terms.

14. To get to work on time, Ed tries to average 40 mph for the 10 mile drive. Today he had to slow down for the first 5 minutes of his trip, but made it to work on time by going 48 mph the rest of the way. What was Ed's average speed for the first 5 minutes?

15. François's Diner offers a Breakfast Combo consisting of five items selected from {egg, muffin, yogurt, waffle} with no more than two of the same item. How many different Breakfast Combos are possible?

16. A square is divided into an 8 by 8 grid of 64 smaller squares, each of area 1, as shown in the figure. How many total squares with area 1 through 64 are contained in the figure?



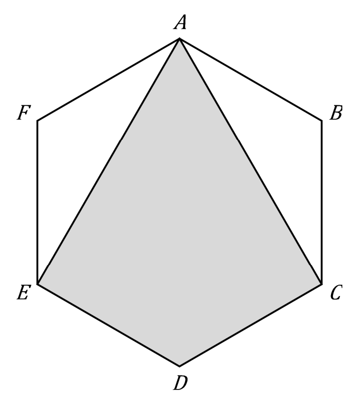
17. What is the shortest distance in the plane from the origin to the line $4x + 3y = 12$?

18. $A(-2,3)$, $B(1,0)$, $C(10,3)$ and $D(a, b)$ are points in the plane. Point D is equidistant from points A, B and C. What is the value of a?

19. An isosceles trapezoid has parallel sides of lengths 10 cm and 14 cm, and diagonals of length 13 cm. What is the area of this trapezoid?

20. How many integers between 1 and 100 inclusive do not contain any even digits (0, 2, 4, 6, 8)?

21. A regular hexagon $ABCDEF$ has area 1 square unit. What is the area of $ACDE$?



22. A box has six rectangular faces. The areas of three of the faces are 6 square units, 15 square units and 49 square units respectively. What is the volume of the box?

23. Find all complex numbers $z = a + bi$ such that $\frac{\overline{z}}{5} + \frac{-1+3i}{z} = 1$, where a and b are real numbers, i is the imaginary unit $\sqrt{-1}$, and $\bar{z} = a - bi$ is the complex conjugate of z.

24. Find $\sin(2\alpha)$ if $\sin\alpha - \cos\alpha = \frac{5}{4}$.

25. Find all ordered pairs (x, y) that satisfy the following system of equations.

$$\begin{aligned} 2x^2 + y^2 - 4x + 2y &= 1 \\ 3x^2 - 2y^2 - 6x - 4y &= 5 \end{aligned}$$

26. If $a = \sqrt{6}^{\sqrt{6}}$ and $\log_a b = \sqrt{6}$, find b.

27. At some time t_1 between 1:00 pm and 1:30 pm, the hour and minute hands of a clock are perpendicular. At a later time t_2 between 1:30 pm and 2:00 pm on the same day, they are perpendicular again. How many minutes pass between t_1 and t_2? Express the answer as a rational number in lowest terms.

28. Find all real numbers x such that the mean and median of the five numbers 6, 3, 16, 11 and x are equal.

29. The binary operation $\otimes$ is defined by $a \otimes b = a^2 - 2ab$.
Find all values of c for which $\big((-2) \otimes 3\big) \otimes c = (-2) \otimes (3 \otimes c)$.

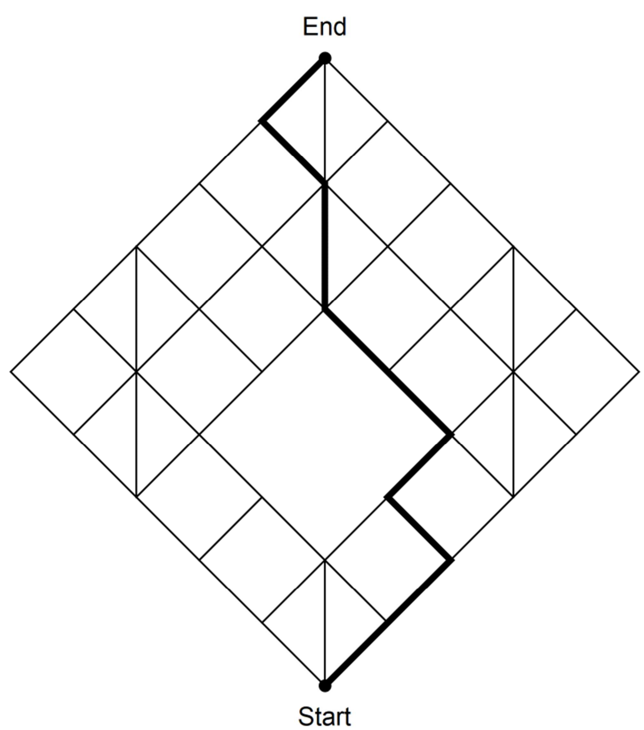
30. Find the value of $\log_8\left(\sqrt[3]{25}\right)\log_{49}(81)\log_3(256)\log_5\left(\frac{1}{7}\right)$. Express the answer as a rational number in lowest terms.

31. Find the largest real value of the function $f(x) = \sqrt{5x-45} - \sqrt{36-3x}$.

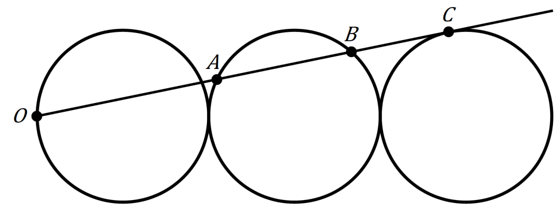
32. An integer whose digits read the same frontwards and backwards is called a palindrome. How many integers between 1 and 5,000,000 inclusive are palindromes? One such example is the integer 32723.

33. Find the minimum value of $|x-1| - |1-2x| + |x+3|$.

34. How many different paths are there from Start to End traveling diagonally upward to the left, diagonally upward to the right, or straight upward along lines in the figure? One such path is shown.



35. Three circles, each of radius 1, are centered at $(1,0)$, $(3,0)$ and $(5,0)$ respectively. Line $\overline{OC}$ goes through the origin O, intersects the middle circle at points A and B, and is tangent to the rightmost circle at point C. Find AB.



36. Kat rolls a fair 6-sided die with sides numbered 1 through 6. Krysta rolls a fair n-sided die with sides numbered 1 through n. If the probability that Krysta's roll is greater than Kat's roll is $\frac{3}{4}$, find n.

37. In triangle ABC, $AB = 5$, $AC = 5$ and $BC = 2\sqrt{5}$. Point M is on $\overline{AC}$ with $AM = 3$. Circles with centers O_1 and O_2 are circumscribed about the triangles ABM and CBM respectively. Find the distance O_1O_2.

38. How many pairs of positive integers (m, n) with $m + n \le 2018$ satisfy $m^3 + \dfrac{1}{n^3} = 8000n^3 + \dfrac{8000}{m^3}$?

39. Let p, q and r be the three different roots of the polynomial $f(x) = x^3 - 4x^2 + 3x + 5$.

Find the value of $\dfrac{1}{p^2} + \dfrac{1}{q^2} + \dfrac{1}{r^2}$.

40. Find the greatest integer less than $\left(\sqrt{5} + \sqrt{2}\right)^4$.

41. In a random arrangement of the letters of the word GALAXY, what is the probability that no position is occupied by the same letter as in the original arrangement? Express the answer as a rational number in lowest terms.