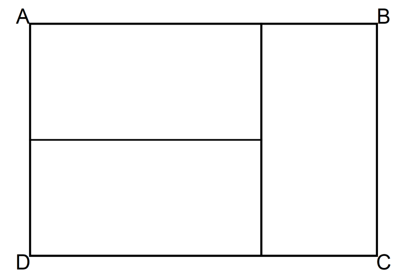
THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-SECOND ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 13, 2019

1. Find the value of $\dfrac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}{\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}}$. Express the answer as a rational number in lowest terms.

2. What is 10% of 20% of 40% of 80% of 5000?

3. Rectangle $ABCD$ is composed of three congruent rectangles as shown in the figure. If the area of $ABCD$ is 96 square inches, what is the perimeter of rectangle $ABCD$?



4. There are three boxes on a scale. The heaviest box is twice the weight of the lightest box and the lightest box weighs 2 pounds less than the middle-weight box. The average of the three weights is 26 pounds. How much does the lightest box weigh?

5. Find x if $\dfrac{(b^x c^{2y})^4}{b^{2y} c^{12x}} = \dfrac{b}{c}$. Express the answer as a rational number in lowest terms.

6. Ted can paint a wall in $1\frac{1}{3}$ hours alone and his speedy friend Tim takes 20 minutes to paint the same wall alone. How many minutes will it take to paint the wall if they work together?

7. Find $3852_9 - 2363_9$ and express the result in base 9.

8. Ye Olde Pet Beauty Shoppe has 15 dogs and 12 cats scheduled for baths one day. 12 of the dogs and 11 of the cats hate getting a bath. Given that an animal loved their bath at the Shoppe that day, what is the probability that the bath lover was a dog? Express the answer as a rational number in lowest terms.

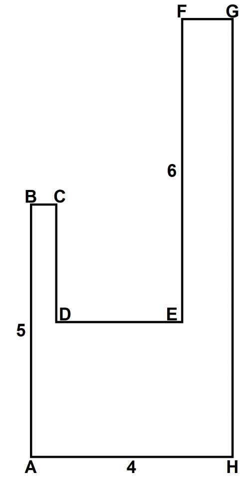
9. What is the smallest positive integer n such that $\sqrt{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot n}$ is an integer?

10. Some even integers can be written as the sum of two odd primes in more than one way. For example, $10 = 5 + 5$ and $10 = 3 + 7$. What is the smallest even integer that can be written as the sum of two odd primes in exactly three distinct ways? Rearrangements of the same primes are not distinct representations (for example, $3 + 7$ and $7 + 3$ are equivalent).

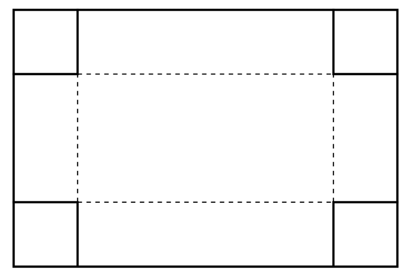
11. Simplify the expression $\dfrac{(5\sqrt{3}+\sqrt{50})(5-\sqrt{24})}{\sqrt{15}-\sqrt{10}}$.

12. Evaluate the expression $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\cdots\left(1-\dfrac{1}{2019^2}\right)$. Express the answer as a rational number in lowest terms.

13. In polygon $ABCDEFGH$ (shown in the figure), all edges that meet at a corner are perpendicular to one another. $AB = 5$, $AH = 4$, and $EF = 6$. Find the perimeter of polygon $ABCDEFGH$.



14. The length of a rectangular sheet of cardboard is 3 feet longer than its width. A lidless box is constructed by removing a 6 inch by 6 inch square from each of the four corners of the sheet and then folding along the dotted lines (see figure). If the volume of the box is 9 cubic feet, what was the area of the original rectangular sheet of cardboard before the squares were removed?



15. Two large hoses and one small hose can fill a water tank in four hours. The same water tank can also be filled in four hours by four small hoses and one large hose. How long will it take for four large hoses and four small hoses to fill the water tank?

16. The average score on an exam is 75. The average of students who scored below 70 is 60. The average of students who scored 70 or more is 80. If the total number of students in the class is 20, how many students scored below 70?

17. The difference, sum, and product of two numbers are in the ratio 1:7:24 respectively. What is the product of the two numbers?

18. A bowl contains n marbles. If two marbles are drawn randomly from the bowl without replacement, the probability that both are purple is $\frac{5}{14}$. What is the smallest possible value of n?

19. In a park, trees have been planted in a circle. Nicole walks along this circle and counts all the trees. Valerie does the same, but starts at a different tree. Nicole's tree number 6 is Valerie's tree number 95, and Nicole's tree number 23 is Valerie's tree number 10. How many trees are in the circle?

20. Caitlin has a collection of marbles, all of which are blue or green. She is creating pairs of 1 blue marble and 1 green marble. After a while she notices that $\frac{2}{7}$ of all the blue marbles are paired up with $\frac{3}{5}$ of all the green marbles. What fraction of Caitlin's marble collection has been paired up?

21. A bowl contains 5 red candies, 8 blue candies, and 13 brown candies. If Tony selects 3 candies from the bowl without replacement, what is the probability that he gets one of each color? Express the answer as a rational number in lowest terms.

22. Points A, B, C and D lie in a straight line in that order. $AB = BC = 1$ and $CD = 2$. Point M is taken inside segment $\overline{BC}$ so that $BM{:}MC = AM{:}MD$. Find AM.

23. In triangle ABC, $\angle A = 30°$ and bisectors of angles B and C intersect at point O. Find $\angle BOC$.

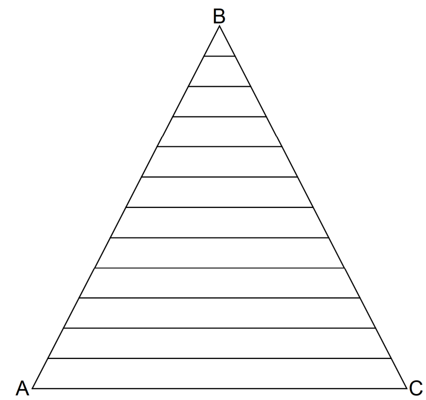
24. In a random arrangement of the letters in **REARRANGEMENT**, what is the probability that all duplicate letters will be grouped together (all **R**'s next to each other, all **E**'s next to each other, etc.)? Express the answer as a rational number in lowest terms.

25. A person walking alongside the road notices that a bus passes her from behind every 7 minutes, and a bus approaches her from the opposite direction every 5 minutes. Assume that buses follow one another at equal time intervals, and that this interval is the same in both directions. Find this time interval (relative to a standing observer).

26. The polynomial $f(x) = ax^{24} + bx^{16} + 30x + c$ has coefficients a, b and c chosen so that $f(12) = 2019$. Find the value of $f(-12)$.

27. If θ satisfies $\tan\theta + \cot\theta = 16$, find the value of $\sin(2\theta)$.

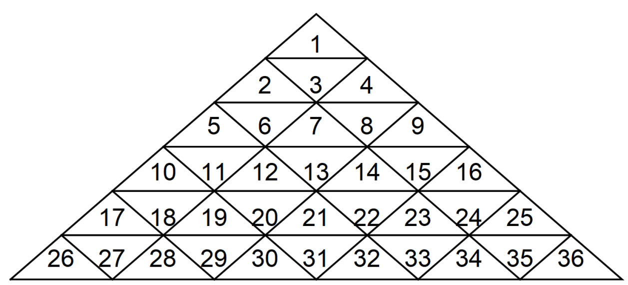
28. Triangle ABC is isosceles, with $AC = 4$ and $AB = BC$. A sequence of line segments parallel to $\overline{AC}$ divide ΔABC into 12 regions of equal height as shown in the figure. Find the sum of the lengths of the 12 parallel line segments (including $\overline{AC}$).



29. A point (x, y) is a lattice point if both x and y are integers. How many lattice points lie inside or on the circle $x^2 + y^2 = 64$?

30. Let $f(x) = \dfrac{25}{3x - 2}$ and $g(x) = x^2$. Find the largest value of x such that $f\big(g(x)\big) = g\big(f(x)\big)$.

31. A triangular game board consists of 36 different triangular spaces as shown in the figure. How many ways can a red checker and a blue checker be placed on different spaces on the board so that the two occupied spaces do not share a common edge?



32. Dave rolls a fair 6-sided die, obtaining the number N. He then flips N fair coins. If Dave obtains exactly 3 heads, what is the probability that he rolled a 3? Express the answer as a rational number in lowest terms.

33. Each face of a right square pyramid has area 6 square cm. What is the volume of the pyramid?

34. If x is a solution to $\sin^6(x) + \cos^6(x) = \frac{3}{7}$, what is the value of $\cos(4x)$?

35. Find the largest positive integer n such that 2019! is divisible by 15^n.

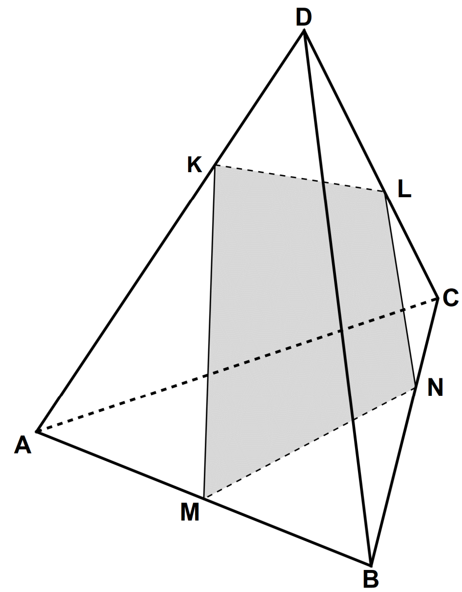
36. Rectangle $ABCD$ has vertices at the points $A(3,0)$, $B(3,44)$, $C(11,44)$ and $D(11,0)$. Line L goes through the point $(-11,6)$ and cuts through the rectangle, dividing the rectangle into two regions of equal area. Find the slope of line L.

37. Find all $x > 0$ such that $\log_x(9) - 2\log_3(x) = 3$.

38. Find all real x that satisfy $|x - 6| - |x + 15| + |x - 10| = 34$.

39. $p(x)$ and $q(x)$ are each polynomials of degree 125. If $p(n) = q(n)$ for all integers n with $1 \leq n \leq 125$ and $p(126) = q(126) + 4$, find the value of $p(-1) - q(-1)$.

40. In pyramid $ABCD$, the base ABC is an equilateral triangle with side 12, and $DA = DB = DC = 9$. Points K, L and M are on edges $\overline{DA}$, $\overline{DC}$ and $\overline{AB}$ respectively, so that $AK = 6$, $CL = 4$, and $AM = 5$. Plane KLM intersects edge $\overline{BC}$ at point N. Find CN.



41. What is the remainder when 2018^{2019} is divided by 360?