THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-THIRD ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 10, 2020

1. What is $\frac{3}{4}$ of 80% of 520?

2. Express as a rational number in lowest terms:

$$\left(1+\frac{1}{2}\right)^{-1}\left(1+\frac{1}{3}\right)^{-1}\left(1+\frac{1}{4}\right)^{-1}\left(1+\frac{1}{5}\right)^{-1}$$

3. A rectangular box has dimensions 3 feet $\times$ 4 feet $\times$ 12 feet. What is the greatest distance between two corners of the box?

4. A bag contains only yellow and blue marbles. Initially there are exactly 15 marbles in the bag. After 5 additional blue marbles are added to the bag, the probability of randomly drawing a blue marble from the bag is 45%. How many blue marbles were in the bag initially?

5. If a, b and c are positive integers such that $\dfrac{a+b}{a}=4$ and $\dfrac{b+c}{b}=5$, what is the value of $\dfrac{a+b+c}{a}$?

6. What is the value of the expression $\dfrac{3^{-1}-0.3333}{3^{-2}-0.111}$? Express your answer as a rational number in lowest terms.

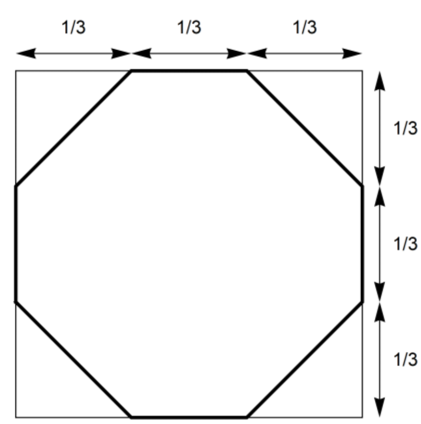
7. Vincenzo's Pizza sells large circular pizzas 18" in diameter and medium circular pizzas 12" in diameter. How many medium pizzas are needed so that the total area of the medium pizzas is equal to the area of 4 large pizzas?

8. The sum of the first four terms of an arithmetic sequence is 50 less than the sum of the next four terms. What is the difference between two consecutive terms in this sequence? Express your answer as a fraction in lowest terms.

9. Express 123_4+567_8 as a number in base 12.

10. A day can be either cloudy or sunny; 40% of the days are cloudy. A day can be either windy or not windy; 25% of the cloudy days are windy, and 15% of the sunny days are windy. What percentage of all days is it windy?

11. A non-equilateral octagon is to be constructed by taking a square sheet of metal and cutting off the four corners as shown in the figure. The cuts are made 1/3 of the way from each corner. If the sheet measures 60 cm across the diagonal, what is the perimeter of the octagon?



12. Bill scores 70%, 86%, and 82% on three tests, which are each worth 20% of his grade. The final exam counts as 40% of his grade, but if his final exam score is higher than his lowest test score, his lowest test score will be replaced with the final exam score. What is the minimum score he needs to get on the final exam to earn an A (90%) in the class?

13. Lily and her sister Betty get on a lift together at the bottom of a ski hill. The lift travels along the same line as the ski trail. Lily gets off the lift four-fifths of the way up the hill, and skis down at a speed that is twice as fast as the speed of the lift. Betty gets off the lift at the top of the hill, and skis down at 1.5 times the speed that Lily skis. At the moment that the first person gets to the bottom of the hill, how far up the hill (as a fraction of the trail length) will the other person still be?

14. The positive integers are written consecutively to make a number beginning with 1234567891011121314 ...
What digit is in the 2020th position?

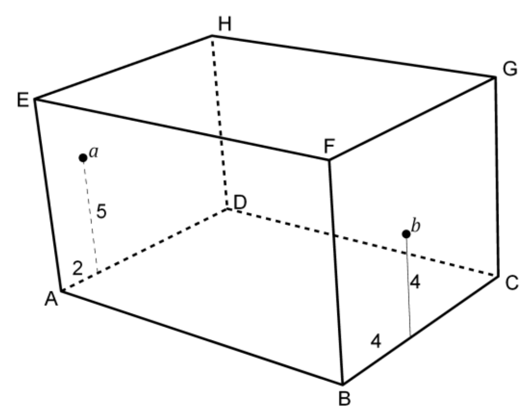
15. Suppose the numbers 1, 2, 4, 8, 16, 32, 64, 128 and 256 are arranged in a three-by-three grid so that each number appears exactly once and the product in all rows and columns is the same. What is the value of the product of each row and column?

16. Four adults can paint a small barn in 12 hours. Six teenagers take 20 hours to do the same job. All adults paint at the same rate, and all teenagers paint at the same rate (different than the rate the adults paint). How long would it take 2 adults and 3 teenagers, working together, to paint the barn?

17. The staircase in Kayla's house has 10 steps. If Kayla can climb the stairs taking either one step or two steps at a time, how many different ways can she climb the stairs?

18. How many minutes after 3:00 will the hour hand and the minute hand on a clock form a 180 degree angle? Express your answer as a rational number in lowest terms.

19. A right rectangular prism has base $\overline{ABCD}$ and top $\overline{EFGH}$ as shown in the figure, with $AB = 14$ cm, $BC = 10$ cm and $AE = 8$ cm. Points a and b are on faces $\overline{ADHE}$ and $\overline{BCGF}$ respectively. Point a is 2 cm in from vertex A and 5 cm up. Point b is 4 cm in from vertex B and 4 cm up. What is the length of the shortest path from a to b along the faces?



20. Find all positive two-digit integers n such that the quotient when n is divided by the sum of its digits is 2 and the remainder is 7.

21. What positive integer n satistifies $\log_{10}(225!) - \log_{10}(223!) = 2 + \log_{10}\left(\frac{n!}{6!}\right)$?

22. A fox runs at the rate of 9 miles per hour across a bridge with train tracks on it. When the fox is one-third of the way across the bridge, she hears a train coming from behind. If the fox runs back the way she came, she will exit the bridge just as the train enters the bridge. If the fox runs toward the far end of the bridge, she will exit the bridge just as the train reaches her. How fast is the train traveling in miles per hour?

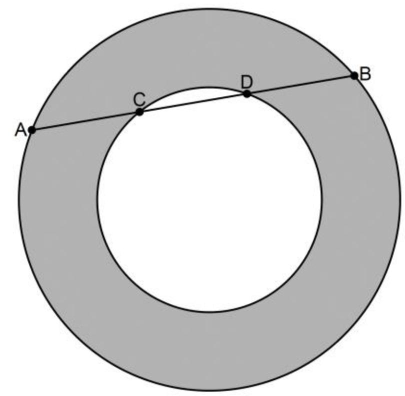
23. Three circles have centers at the points A, B, and C respectively. Each of the three circles is tangent to the other two circles. The points of tangency are K, M, and P, with P lying on segment $\overline{AC}$. Find $\angle KPM$ if $\angle ABC = 50°$.

24. Solve the inequality $\sqrt{6+2x} > -1-x$. Express your answer in interval notation.

25. Simplify the expression $\dfrac{5}{1+2i}+\dfrac{13}{2+3i}+\dfrac{25}{3+4i}+\dfrac{41}{4+5i}$ where $i=\sqrt{-1}$.

Express your answer in the form $a+bi$, where a and b are real numbers.

26. Points A and B lie on the larger of two concentric circles. $\overline{AB}$ intersects the smaller circle at points C and D as shown in the figure, and $AC = CD = DB = 10$. Find the area of the region that is inside the larger circle and outside the smaller circle.



27. The function $f(x)$ satisfies the equation $3\,f(x) - 6\,f\left(\dfrac{1}{x}\right) = \dfrac{8}{x}$ for all $x \neq 0$. Find the value of $f(4)$.

28. Find the value of $\log_3(2)\log_4(3)\log_5(4)\log_6(5)\log_7(6)\log_8(7)$.

29. Find the (x,y) coordinates of the center of the circle that passes through the points $(3,-1)$, $(5,-5)$ and $(9,-3)$.

30. How many ways can the letters in **POMPOM** be arranged so that no pair of consecutive letters are the same?

31. How many integer solutions are there for the inequality $|x|+|y|<100$?

32. Find all triples (x,y,z) that satisfy the following system of equations other than $(1,1,1)$.

$$x+y+z=3$$
$$x+2y-z=2$$
$$x+yz+zx=3$$

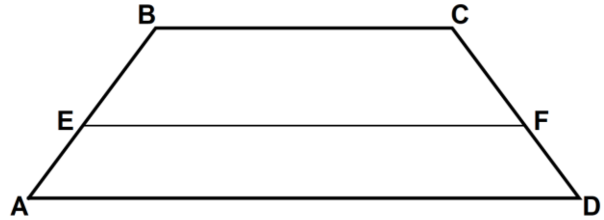
33. Express in simplest form:

$$\frac{2-\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$$

34. Two integers are relatively prime if they have no common factors other than 1. How many integers between 1 and 2020 inclusive are relatively prime to 2020?

35. If $x^2 + y^2 = 40$ and $\frac{1}{x} + \frac{1}{y} = \frac{2}{3}$, find all possible values of xy.

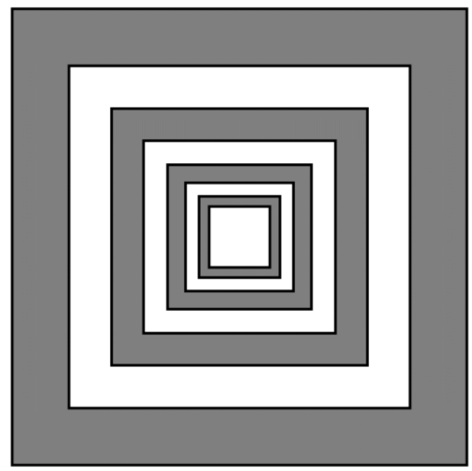
36. $ABCD$ is an isosceles trapezoid with $\overline{AD}$ parallel to $\overline{BC}$, and lengths $AD = 13$, $BC = 7$ and $AB = CD = 5$. Points E and F are on $\overline{AB}$ and $\overline{CD}$ respectively, so that $\overline{EF}$ is parallel to $\overline{AD}$ and divides trapezoid $ABCD$ into two regions of equal area. Find the length EF.



37. Find the sum of all integers n for which $\frac{936}{n-5}$ is an integer.

38. $\{S_n\}$ is a sequence of smaller and smaller concentric squares. The largest square, S_0, has width 1. Each subsequent square is three-fourths the width of the previous square. Let A_n be the area of the region that is inside square S_{2n} and outside square S_{2n+1} for $n \geq 0$. The first four such regions are shown shaded in the figure.

Find $\sum_{n=0}^{\infty} A_n$



39. Determine the number of positive real solutions x to the equation $\log_{20}(x) = \sin(20\pi x)$.

40. If x and y are positive real numbers with $x^2y = 1024$, then the greatest possible value of $x^{\log_8 y}$ is 4^z. What is the value of z?

41. Let S be the sum of all real numbers x such that $(2^x - 64)^3 + (3^x - 81)^3 = (2^x + 3^x - 145)^3$. Find the greatest integer less than or equal to S.