THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-FOURTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
WEEK OF MARCH 8, 2021

1. What is the value of the expression $\dfrac{1^2+2^3+3^4}{2^1+3^2+4^3}$? Express your answer as a rational number in lowest terms.

2. If 20% of 30% of 40% of x equals 120, what is the value of x?

3. The exterior of a solid white cube is painted black. After it is painted, the cube is cut into 64 smaller congruent cubes. How many of the smaller cubes have at least one face painted black?

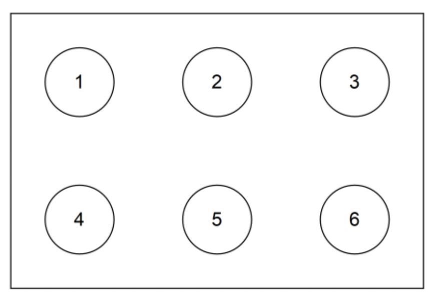
4. If 8 woodcutters can chop 12 cords of wood in 7.5 hours, how many hours will it take 4 woodcutters to chop 4 cords of wood, assuming that everyone works at the same rate?

5. Express as a rational number in lowest terms: $\dfrac{3}{4-\dfrac{5}{6-\frac{7}{8}}}$

6. A fish tank with a rectangular base and rectangular sides is 15 inches long, 12 inches wide and 14 inches high. It is filled with water to a line that is 2 inches below the rim. When 100 identical marbles are dropped into the tank, the water level rises one-quarter of an inch. What is the volume of one marble? Express your answer as a rational number in lowest terms.

7. It snowed for 4 hours one day. During hour 2 it snowed twice as much as during hour 1. During hour 3, it snowed one-third more than it did in hour 2. During hour 4 it snowed only 20% of what it snowed in the previous hour. How much did it snow during hour 1, if the total snowfall was 10 inches? Express your answer as a rational number in lowest terms.

8. Auggie has 6 marbles, numbered 1 through 6. He drops the marbles onto a board with six circular indentations that are also numbered 1 through 6. The marbles roll around until each of them randomly settles into a different one of the circular indentations. What is the probability that every marble ends up on the circle whose number matches the number of the marble? Express your answer as a rational number in lowest terms.



9. Bottle A contains one quart of liquid that is 10% maple syrup. Bottle B contains three quarts of liquid that is 30% maple syrup. The contents of the two bottles are emptied into a larger container and mixed together. What is the percentage of maple syrup in the mixture?

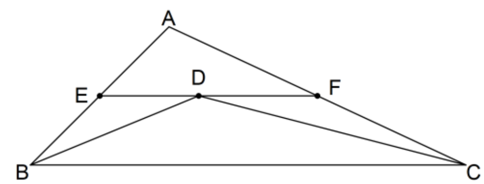
10. A bus has enough seats for at most 45 passengers. Bus fare is \$8 per adult and \$5 per child. If the total fare paid by all the passengers is exactly \$250, what is the maximum number of children that could be on the bus?

11. The average of a list of n numbers is 30. The average of a different list of $2n$ numbers is 42. If the two lists are combined, what is the average of all of the numbers in the combined list?

12. If $2021_5 + 2021_6 = (x)_{11}$, find x.

13. Find all real x that satisfy $|x - 50| = |x - 27| + 9$.

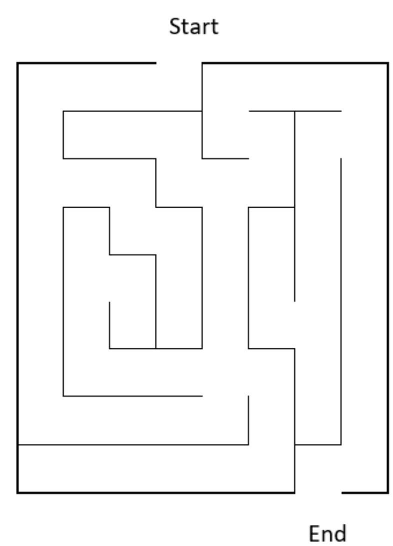
14. Triangle ABC has side lengths $AB = 10$, $BC = 22$ and $AC = 16$. The bisectors of angles B and C intersect at point D. A line is drawn through D and parallel to $\overline{BC}$, intersecting $\overline{AB}$ and $\overline{AC}$ at points E and F respectively. What is the perimeter of triangle AEF?



15. How many ways can a 2×6 rectangle be covered with six identical black 2×1 dominoes?

16. Your robot, Gort, is a bit unreliable. Twenty percent of Gort's statements are lies. Suppose for each $n = 1, 2, 3, \cdots, 100$, you ask Gort ten times "Are $n\%$ of your statements lies?" How many of Gort's responses would you expect to be yes?

17. In the maze shown in the figure, lines represent walls and the spaces between the lines represent walking paths. Alberto begins at Start and walks along the paths. At each junction, he randomly chooses a path and continues walking, without ever turning around and retracing his steps. If he reaches a dead end, he loses the game and stops. What is the probability that Alberto successfully traverses the maze from Start to End? Express your answer as a rational number in lowest terms.



18. The ratio of children to adults at a party is 2 to 3. A bus arrives and brings 30 more children to the party, making the ratio of children to adults 3 to 2. How many people were at the party before the bus arrived?

19. Five stones are weighed two at a time, giving combined weights (in ounces) of: 60, 62, 63, 64, 65, 66, 67, 68, 70, 71. What is the difference between the weights of the heaviest stone and the lightest stone?

20. Jesse walks up a flight of 12 stairs, going up 1 or 2 steps with each stride. The eighth step is broken and cannot be used. How many ways can Jesse go up the stairs?

21. A circle of radius 10 and a circle of radius 17 intersect at two points. The line segment connecting the points of intersection has length 16. What are the two possible values for the distance between the centers of the circles?

22. Express as a rational number in lowest terms: $\dfrac{1000! + 999! + 998!}{1001! - 1000!}$

23. Maggie's work-from-home outfit consists of either a fleece top with or without a t-shirt underneath it, or a hoodie with a t-shirt underneath; and either a pair of leggings or a pair of sweatpants; and a pair of slippers. How many different work-from-home outfits are possible if Maggie chooses from 8 t-shirts, 6 fleece tops, 2 hoodies, 3 pairs of leggings, 2 pairs of sweatpants and 3 pairs of slippers?

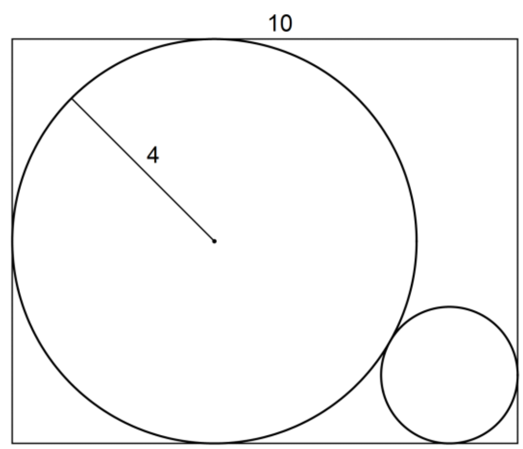
24. Find all complex numbers $z = a + bi$ such that $\dfrac{\overline{z}}{5} + \dfrac{-1+3i}{z} = 1$ where $\overline{z} = a - bi$ is the complex conjugate of z.

25. Find the area of the region in the plane that satisfies $|4x - 16| \leq y \leq 2x - 2$.

26. For how many positive integers n is the remainder equal to 5 when 2021 is divided by n?

27. Let $a_1 = 10$, $a_2 = -1$ and $a_n = \dfrac{a_{n-1}}{a_{n-2}}$ for $n \geq 3$. Find $\displaystyle\sum_{n=1}^{2021} a_n$

28. A large circle of radius 4 is tangent to three sides of a rectangle whose width is 10. A smaller circle is tangent to two sides of the rectangle and to the large circle as in the figure. Find the radius of the smaller circle.



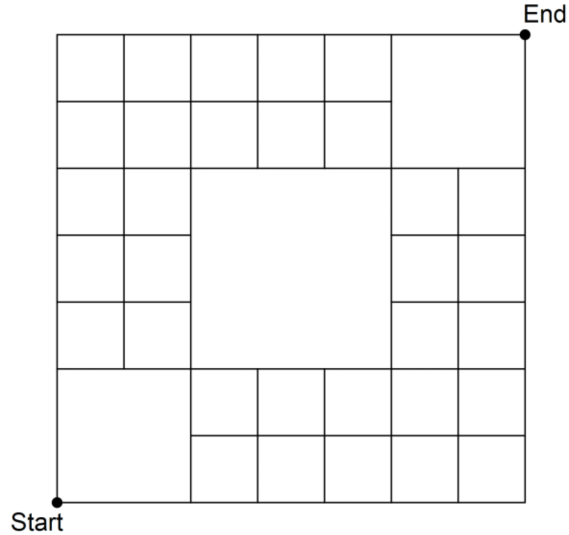
29. Find the smallest positive integer k such that k^3 ends in 77.

30. Convert $(1.\overline{20})_3$ to base ten. Express your answer as a rational number (in base ten) in lowest terms.

31. If a, b and c are positive numbers with $\log_a(b) = -6$ and $\log_b(c) = \frac{4}{9}$, find the value of $\log_c(\sqrt{ab})$.

32. If x is a real number, define $f(x) = \sqrt{x^2 - 12x} - \sqrt{21x - x^2 - 54}$. Find the largest real value of $f(x)$.

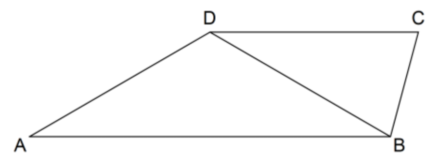
33. How many different paths are there from Start to End traveling upward or to the right along lines in the figure?



34. If $\sin(t) = \frac{1}{4}$, find the value of $\sin(3t)$.

35. Twenty-five balls, numbered 1 through 25, are placed in a container. Nora randomly takes two balls out of the container. What is the probability that the sum of the numbers on the two balls is a multiple of 6? Express your answer as a rational number in lowest terms.

36. In trapezoid $ABCD$, sides $\overline{AB}$ and $\overline{DC}$ are parallel, and $AD = DB = DC = 2 \cdot CB$. Find $\cos A$.

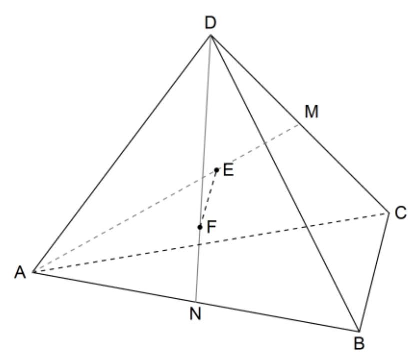


37. If $S = \{1,2,3,4,5\}$, let T be the set of all <u>unordered</u> triples of elements selected from S (repetition permitted). How many <u>unordered</u> pairs of elements selected from T (repetition permitted) are there?

38. How many ordered pairs (m, n) of integers are there such that $m^4 - n^4 = 2021(m^2 + n^2)$?

39. Find the smallest positive integer n for which $\sqrt{30 + \sqrt{n}} + \sqrt{30 - \sqrt{n}}$ is an integer.

40. Let $ABCD$ be an arbitrary pyramid. Let $\overline{AM}$ be a median of ΔACD and $\overline{DN}$ be a median of ΔDAB. Points E and F are taken on $\overline{AM}$ and $\overline{DN}$ so that line $\overline{EF}$ is parallel (not skew!) to $\overline{BC}$. Find the ratio $EF : BC$.



41. Let R be the region consisting of all points (x, y) in the plane such that there exist real numbers $a, b,$ and c with $x = \cos(a) + \frac{1}{2}\cos(b) + \frac{1}{3}\cos(c)$ and $y = \sin(a) + \frac{1}{2}\sin(b) + \frac{1}{3}\sin(c)$. What is the area of R?