THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-FIFTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 8, 2022

1. Express as a rational number in lowest terms: $\dfrac{2^{-2} - 3^{-2}}{2^{-2} + 3^{-2}}$

2. If A is one-third of one-fourth of one-fifth, and B is two-thirds of two-fourths of two-fifths, what is the value of $\frac{A}{B}$? Express your answer as a rational number in lowest terms.

3. At a school book fair, 3/5 of the books were sold in the morning and 1/4 of the remaining books were sold in the afternoon. If 250 more books were sold in the morning than in the afternoon, how many books were there at the start?

4. One school had 10% more students than another school. After 18 students moved from one school to the other, both schools had the same number of students. How many students are in the two schools combined?

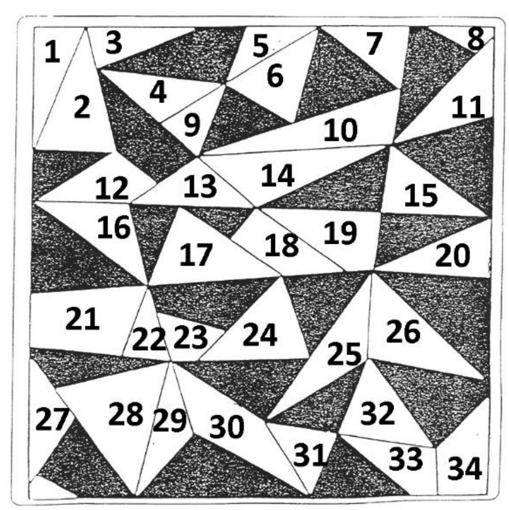
5. Express as a rational number in lowest terms: $\sqrt{\dfrac{4^7 + 2^5}{8^3 + 64^3}}$

6. Helen has three solid steel cubes: one of side length 3cm, another of side length 4cm, and a third of side length 5cm. Helen melts all three cubes down and uses all the material to form a single cube. What is the side length of this new cube?

7. How many ways are there to distribute 18 identical cupcakes into three bakery boxes of different sizes so that no box is empty, and the smallest box has fewer cupcakes than the medium-sized box, which has fewer cupcakes than the largest box?

8. This shape  is called a regular five-pointed star.

   There is a regular five-pointed star (possibly in a different orientation than the example above) in the figure. Find the sum of all the numbers found within this star.

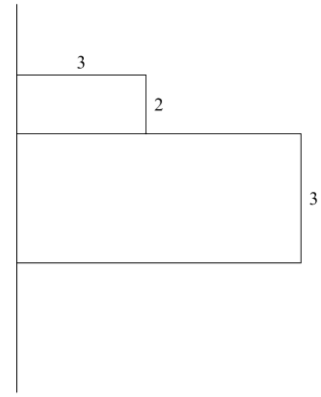


9. The width of a rectangle decreases by 15% while its length increases by 40%. By what percentage does the area of the rectangle increase?

10. The operation * is defined by $a * b = b^2\pi^{a-1}$. Find the value of $2 * (4 * 3)$.

11. A segment $\overline{AB}$ has length 4. Points K and M lie on $\overline{AB}$, with $AK:KM:MB = 1:2:3$. Find the length of $\overline{KM}$.

12. Two flags on a flagpole are similar rectangles, with dimensions as shown in the figure. Find the volume of the 3-dimensional shape formed by rotating the flags around the pole once (360 degrees).



13. What is the first (leftmost) digit of the hexadecimal representation of the binary number 1 1 0 1 1 0 1 1 0 1 0 0 0 0 1 0 1 0 1 1 1 1 0 1 0 0 1 ?

14. Let x and y be positive integer solutions to the equation $8x - 3y = 19$. What is the smallest possible sum of x and y?

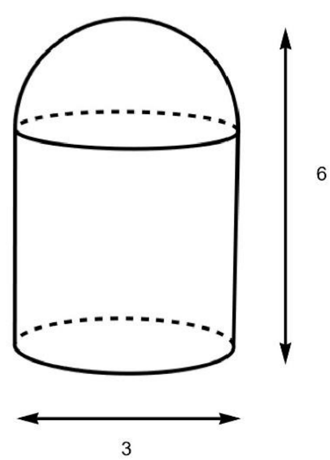
15. If $f(x) = ax^6 + bx^4 + x^3 - 6x + 2$ and $f(-2) = 5$, find $f(2)$.

16. Find the area of a triangle with side lengths 4, 6, and 8.

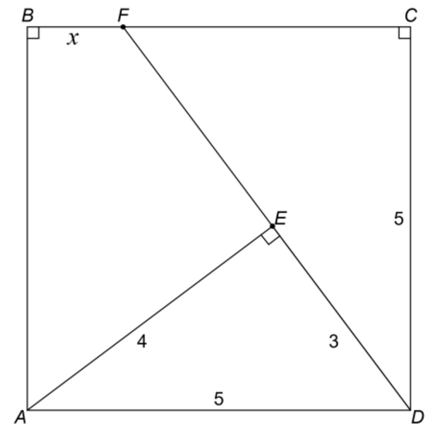
17. How many positive integers n satisfy the inequality: $-2 \leq \log_{\frac{1}{n}}(9) \leq -\frac{1}{3}$?

18. It takes 4 adults working together 20 minutes to clear the snow off an ice rink. It takes 2 adults and 2 teenagers working together 15 minutes to clear the snow off the same ice rink. How long will it take 3 teenagers working together to clear the snow off the ice rink?

19. Miff has a recipe for muffins that are 3 inches in diameter with a height of 6 inches. The shape of the muffin is a circular cylinder with a hemisphere on top. Miff modifies the recipe so that the volume of each muffin is reduced by 25%, decreasing the height of the muffin without altering the diameter or overall shape. What is the new height of the muffin?



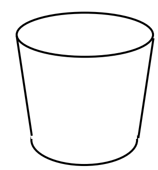
20. $ABCD$ is a square of width 5. Point F lies on $\overline{BC}$ and point E lies on $\overline{DF}$, with DEA forming a 3-4-5 right triangle as shown in the figure. Find the length BF (labeled x in the figure).



21. Nora and her brother Jesse travel toward each other on a bike path, beginning one mile apart. Jesse rides his skateboard at a constant speed of 6 miles per hour, and Nora rides her bike at a constant speed of 10 miles per hour. Their dog Auggie begins by Jesse's side and runs toward Nora at a speed of 12 miles per hour. When Auggie reaches Nora, he instantly turns around and runs to Jesse, then instantly turns around again and runs to Nora. Auggie continues running back and forth between Nora and Jesse at 12 miles per hour until Nora and Jesse meet. How far does Auggie run?

22. Let x be any real number and let $f(x) = x^2 - |6x - 12| + 5$. What is the smallest possible value of $f(x)$?

23. The top and bottom of a paper cup are circular, and its sides taper at a constant rate from a diameter of 3 inches at the top to 2 inches at the bottom. The height of the cup is 4 inches. Find the volume of the cup.



24. How many positive integer factors does 888,888 have?

25. The ages of Alice and Bob (in years) are given by some two-digit numbers. The four-digit number made by concatenating the ages of Alice and Bob (in that order) is a square of an integer. The same statement will also hold true 13 years from now. Find the ages of Alice and Bob.

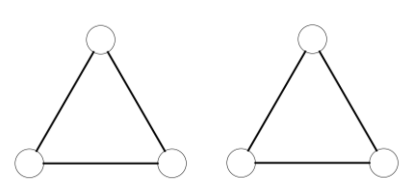
26. Find a positive integer n satisfying the system: $x + y = 4, \quad x(n+2) = 80, \quad yn = 45$

27. Find all real numbers m such that the equation $|x+5| - |x-3| + |x-6| = m$ has exactly three real solutions.

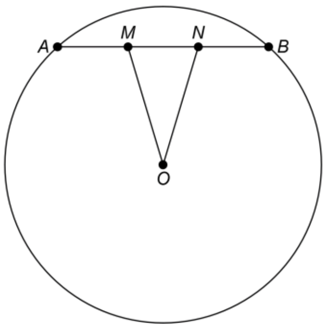
28. A 6-sided die with sides labeled A, B, C, D, E, and F is weighted so that it lands on B, C, D, or E equally often as each other, lands on A four times as often as it lands on B, and lands on F only one-fourth as often as it lands on B. If the die is rolled once, what is the probability that it lands on A? Express your answer as a rational number in lowest terms.

29. If $f(x) = ax + b$ is a linear function, and $f(7x+6) = 5x + f(4x+3) + f(2x+1)$ for every value of x, what is the ordered pair (a, b)?

30. Dave has three white balls, three black balls, three red sticks, and three blue sticks. All sticks are the same length, and all balls are the same size. He uses sticks and balls to create two equilateral triangles as shown. How many different configurations of two triangles could Dave make? (Two configurations are the same if one can be obtained from the other by rotating or moving triangle(s) in space.)



31. A chord is drawn between points A and B on a circle with radius 9 and center O. Points M and N lie on chord $\overline{AB}$ with $AM = MN = NB$. Find the perimeter of triangle MNO if $AB = 12$.



32. A square is inscribed in a circle of radius 5. This square has a circle inscribed in it and there is a square inscribed in that circle, with this pattern repeating indefinitely. What is the sum of the areas of all the circles?

33. In a dice game, the player can place any amount of money on squares labeled 1,2,3,4,5,6. Three fair 6-sided dice whose sides are also labeled 1,2,3,4,5,6 are then thrown. If the player's square's number appears on n dice ($n = 1, 2,$ or 3), the player receives $(n + 1)$ times the money that they placed on the square. If $n = 0$, the house keeps the money. One day the total amount of money placed is $27,000. What is the expected gain of the house on this amount?

34. Find the value of the expression: $\log_8((1024!)^{15}) - \log_2((1023!)^5)$

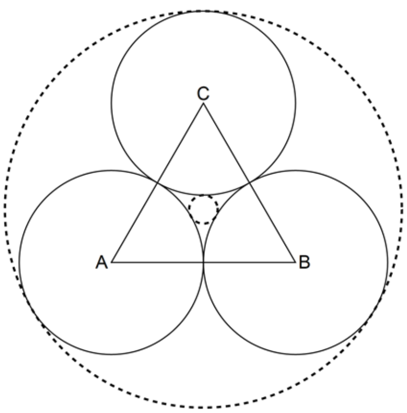
35. Tilly is counting in base 9, from 1_9 up to 2022_9. She lists the numbers on paper as she counts: $1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, \cdots, 2022$. How many times will the digit 2 appear on Tilly's paper?

36. How many positive integers less than or equal to 2022 are relatively prime to 1001?

37. Find the smallest positive x satisfying the equation $\cos 4x \cos 5x = \cos 6x \cos 7x$.

38. Let x be a positive integer with the property that x is 43 more than the sum of the squares of its digits. For example, one such x is $129 = 43 + 1^2 + 2^2 + 9^2$. Find the difference between the largest and smallest such x.

39. Triangle ABC is equilateral. Three circles of equal radius are drawn with centers at A, B and C respectively and so that the circles are tangent to one another as in the figure. These three circles are tangent externally to a large circle of radius R and tangent internally to a small circle of radius r (the large and small dashed circles in the figure). Find $\frac{R}{r}$.



40. Find the smallest positive integer n such that 2^n, when written in base 10, has a leading digit of 7.

41. Express as a rational number in lowest terms: $\cos\left(\frac{\pi}{15}\right)\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{3\pi}{15}\right)\cdots\cos\left(\frac{7\pi}{15}\right)$