THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-SIXTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 14, 2023

1. Express as a rational number in lowest terms: $\dfrac{\frac{1}{4}-\frac{2}{5}}{\frac{1}{2}-\frac{2}{3}}$

2. Express as a rational number in lowest terms: $\dfrac{1^8+2^4+4^2+8^1}{1^{-8}+2^{-4}+4^{-2}+8^{-1}}$

3. Express as a rational number in lowest terms: $\dfrac{\sqrt{0.56}}{\sqrt{1.8}\cdot\sqrt{6.3}}$

4. A circular above-ground swimming pool 18 feet in diameter and 4 feet deep springs a leak in the side. The leak is 1 foot above the bottom of the pool, and leaks at a steady rate of 2 cubic feet per minute. How long will it take for the water level to reach the level of the leak?

5. A farm has chickens and cows. Lily counts their heads and legs, for a total of 26 heads and 74 legs. How many cows are there?

6. A collection of n marbles numbered 1 through n are arranged sequentially in a circle. Marble 14 is directly across the circle from marble 67. What is n?

7. If $\frac{5}{4}$ of $\frac{1}{2}$ of $\frac{6}{7}$ of a number is 15, what is the number?

8. Simplify and express the result as a rational number in lowest terms: $\dfrac{2023!}{2023!-2022!}$

9. Line L passes through the points $(3,-2)$ and $(7,5)$. What is the slope of the line perpendicular to L?

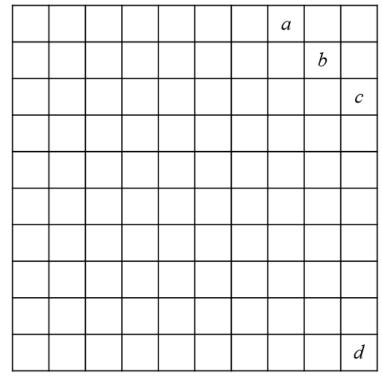
10. Find the value of $787\cdot 789-786\cdot 788$.

11. The ratio of a rectangle's width to its length is 3:4. If the area of the rectangle is 192 square units, what is its perimeter?

12. Let $f(x)=2+\dfrac{1}{x}$. For what x is $f\big(f(x)\big)=8$?

13. In a random arrangement of the letters in **FALAFEL**, what is the probability that the letters are listed in alphabetical order? Express your answer as a rational number in lowest terms.

14. Given a 10 by 10 grid of squares, what is the greatest number of squares that can be shaded in, so that there is no more than one shaded square on each diagonal, row, and column of the grid? A diagonal consists of a group of one or more squares that can be connected by a line drawn at a 45° angle through opposite corners of each square. There are 19 diagonals with positive slope and 19 diagonals with negative slope. For example, the squares labeled a, b, and c lie on one of the negative slope diagonals, and the square labeled d lies on a one-square diagonal with positive slope.



15. A clock with a second-hand is laid flat on a merry-go-round, which rotates counterclockwise at a rate of three rotations per minute. The second-hand initially points North. How many times does the second-hand pass South in 20 minutes?

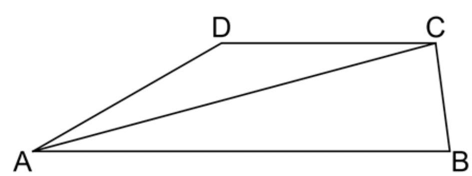
16. Express in simplest form: $\dfrac{3\sqrt{2}+2\sqrt{3}}{\sqrt{3}+\sqrt{2}}$

17. A polynomial $p(x)$ has remainder 3 when divided by $x-1$ and remainder 19 when divided by $x-5$. What is the remainder when $p(x)$ is divided by $(x-1)(x-5)$?

18. Rectangle *ABCD* has vertices at the points (5,5), (5,10), (20,10) and (20,5). Find m such that the line $y = mx$ divides rectangle *ABCD* into two regions of equal area.

19. If $\sin x = 5\cos x$, what is the value of $\sin(2x)$? Express your answer as a rational number in lowest terms.

20. In trapezoid $ABCD$, sides $\overline{AB}$ and $\overline{DC}$ are parallel, $AD = DC$, and $AC = AB$. If $\angle ABC = 77°$, what is $\angle DAC$?



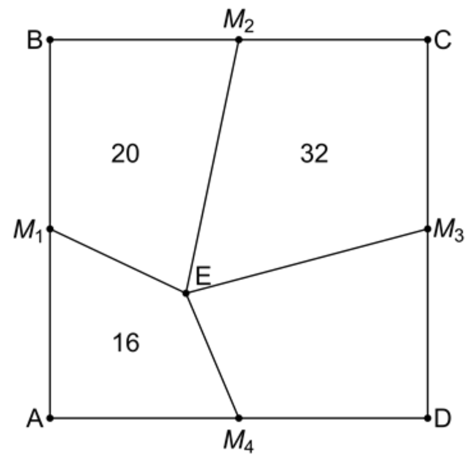
21. Find all real solutions of $|x+1| - |3x-1| + |5x+1| = x+2$.

22. Solve the system of equations for x and y.

$6751x + 3249y = 31751$ and $3249x + 6751y = 28249$

23. The numbers $x-3$, $\sqrt{5x}$, and $x+16$, taken in this order, are terms of a real-valued geometric progression. Find the third of these numbers.

24. In square $ABCD$, points M_1 through M_4 are the midpoints of sides AB through AD as shown. Point E is chosen inside $ABCD$ so that the areas of quadrilaterals $EM_1\text{A}M_4$, $EM_2\text{B}M_1$, and $EM_3\text{C}M_2$ are 16, 20, and 32 square units respectively. Find the area of quadrilateral EM_4DM_3.



25. Bob picks up his partner Melanie daily after work at 5pm and drives her home. One beautiful sunny day Melanie finishes work at 4pm and decides to walk home along the same route that Bob drives. On the way, she meets Bob, who stops, picks her up, and drives her home. This day Melanie arrives home 10 minutes earlier than usual. How long did Melanie walk? (Disregard the time for stopping and getting in and out of the car, and assume the trip from home to Melanie's work takes exactly the same time as the trip from work to home and occurs at a constant speed.)

26. Express the result of $2023_4 + 2023_5$ in base 9.

27. Nora makes a list of 12 consecutive integers, erases one of them, then adds up the remaining 11 integers. If the resulting sum is 6862, what number was erased from Nora's list?

28. Adira can rake the lawn in one and a half hours. It takes Gray two hours to rake the same lawn. One day Adira goes outside to rake the lawn and works alone for 27 minutes before Gray comes out to help. If Adira and Gray work together, how long will it take them to rake the rest of the lawn?

29. Find the largest positive integer n such that 2023! is divisible by 23^n.

30. How many integers n satisfy the inequality $n^2 - 6n < 17$?

31. If $2^x = \frac{1}{81}$, $3^y = \sqrt[3]{49}$, and 7^z=32, find the value of xyz. Express your answer as a rational number in lowest terms.

32. A lattice point is a point (x, y) for which both x and y are integers. How many lattice points with $x > 0$ and $y > 0$ lie on the line $7x + 6y = 500$?

33. If $x^2 + y^2 = 4x + 8y - 7$, what is the largest possible value of $2x + 3y$?

34. Find all real $x > -1$ that satisfy $\sqrt{3}\left(\sqrt{3x+4} - \sqrt{x+1}\right) = 2x + 3$.

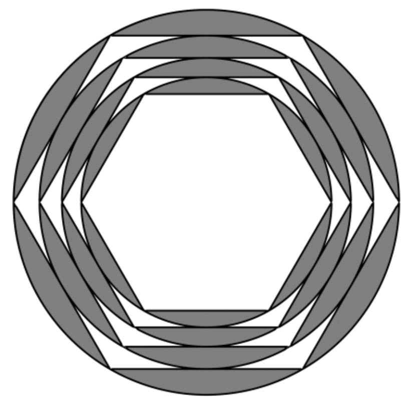
35. A number of nuts $N < 5000$ is such that after giving one nut to a dog, the remainder can be divided equally between five people A, B, C, D, and E. That is, $N = 5k + 1$ for some natural number k. Person A goes to the original pile of nuts secretly, takes their share, gives one nut to the dog, and leaves. Person B, unaware of what A has done, does the same; they do not notice the change because the equation still holds for a different k after A's visit. Persons C, D, and E follow the pattern. Find N.

36. Let P be a randomly chosen point (x, y) in the triangular region of the plane with vertices $(0,0)$, $(0,10)$ and $(6,0)$. What is the probability that the distance from P to $(0,0)$ is less than the distance from P to $(6,2)$? Express your answer as a rational number in lowest terms.

37. The roots of $x^2 + ax + b = 0$ are the squares of the roots of $x^2 + 3x + 12 = 0$. Find $a + b$.

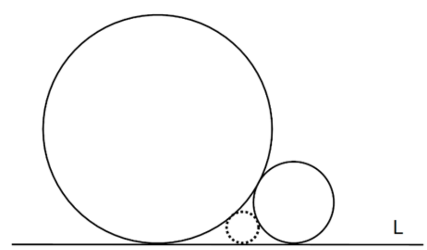
38. A sequence of smaller and smaller circles and regular hexagons are inscribed in one another in an alternating pattern beginning with a circle. Let A_1 be the area of the region between the largest circle and its inscribed hexagon, A_2 be the area of the region between the next circle and its inscribed hexagon, and so forth. The first four such regions are shown shaded in the figure.

If the largest circle has radius 2, find $\sum_{n=1}^{\infty} A_n$



39. Find the value of the sum: $\sum_{n=1}^{1000} \sin^2\left(\frac{n\pi}{25}\right)$

40. Two circles (shown with solid lines in the figure) have radii 4 cm and 1 cm respectively and are tangent to each other and to the line *L*. A third circle (shown with the dotted line) is tangent to each of the first two circles and to *L*. Find the radius of the third circle.



41. Suppose $p(x)$ is a nonzero polynomial such that $\dfrac{p(x)}{p(2x+1)} = \dfrac{x-7}{16x+8}$ for all x where $p(2x+1) \neq 0$.

Find the value of $\dfrac{p(4)}{p(5)}$.