University of Vermont | College of Engineering and Mathematical Sciences
Mathematics and Statistics

SIXTY-EIGHTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION

MARCH 11, 2025

1. Express as a rational number in lowest terms: $\dfrac{\frac{4}{9}-\frac{9}{4}}{\frac{3}{8}-\frac{8}{3}}$

2. Find the value of the expression: $\sqrt{\dfrac{1^1 2^2 3^3 4^4 5^5}{1^5 2^4 3^3 4^2 5^1}}$

3. The price of a pair of shoes is reduced by 20%, and the new price is later reduced by 25%. If the final price is $36, what was the original price of the shoes?

4. Let $f(x) = |x|$. Find $f\left(2 - f\left(-5 - f(-1)\right)\right)$.

5. The sum of a perfect square and a positive perfect cube is 241. What is the value of their product?

6. A sphere is inscribed in a cube that has a volume of $64\ \text{cm}^3$. What is the volume of the sphere?

7. Circles A and B are concentric, and the radius of circle A is three times the radius of circle B. What is the probability that a point chosen randomly within circle A will lie outside circle B? Express your answer as a rational number in lowest terms.

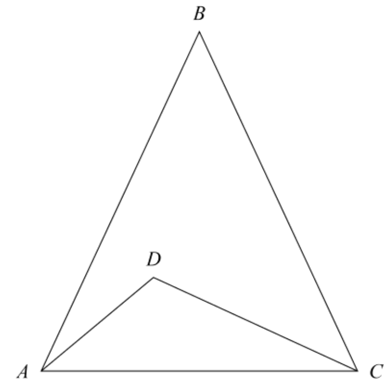
8. A 12-hour clock is slow. It loses 5 minutes per day. Assuming that the clock is now set to the correct time, how many hours will it be before it next shows the correct time?

9. The product of three consecutive positive integers is 8 times their sum. What is the sum of the squares of the three integers?

10. A quadratic polynomial $p(x)$ satisfies $p(0) = 3$, $p(1) = 5$, and $p(2) = 8$. Find the value of $p(5)$.

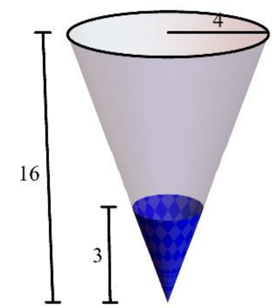
11. A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face of the cube. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume of the remaining solid?

12. In the given figure, $AB = BC$, $\angle DAC = \angle BCD$, and $\angle ABC = 50°$.
Find $\angle ADC$.



13. Find the slope of the line that passes through the centers of the two circles given by the equations $x^2 + y^2 - 4x + 6y + 4 = 0$ and $x^2 + y^2 + 6x + 4y + 9 = 0$.

14. Water is poured into a container in the shape of a right circular cone with radius 4 feet and height 16 feet. Find the volume of the water in the cone when the height of the water is 3 feet.



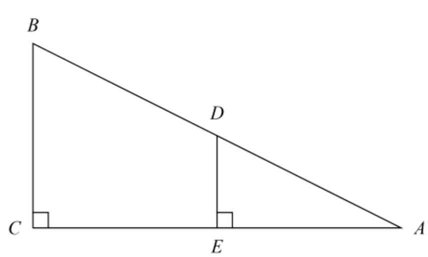
15. For a party that Alice, Bob, and Chris hosted, Alice bought two pies and Bob bought four, with each of the pies costing the same. Alice and Bob told Chris how much they had spent on the pies. Chris then contributed $18 in cash and asked Alice and Bob to split this money so that the monetary contribution of each of the three of them would be the same. If Alice keeps $A and Bob keeps $B of Chris's money, find the ordered pair (A,B).

16. Express in simplest form: $\left(\sqrt{50} - \sqrt{49}\right)^{1/3}$.

17. Find all solutions of $\big||x + 1| - 2\big| = |x|$

18. Find the value of $\log_2\left(\log_2\left(\log_2 4^{8^5}\right)\right)$.

19. In the given figure, $DE = 4$, $AC = 20$, and $EC = BC$.
Find all possible values of BC.



20. If $x = 10$ radians, find $\arcsin(\sin x)$.

21. Express as a rational number in lowest terms: $2^0 - 2^{-1} + 2^{-2} - 2^{-3} + 2^{-4} - 2^{-5} + 2^{-6} - 2^{-7} + 2^{-8} - 2^{-9} + 2^{-10}$

22. Two integers are relatively prime to each other if their greatest common divisor is 1. How many integers between 1 and 2025 inclusive are relatively prime to 2025?

23. Find the largest number b such that the line $y = 18x + b$ intersects the parabola $y = -x^2 + 6x + 12$ at least once.

24. Find the shortest distance between the circles $(x-4)^2 + (y+3)^2 = 36$ and $(x+1)^2 + (y-5)^2 = 4$.

25. The integer 121,000 can be written in the form $20^a 22^b 25^c$ for rational numbers a, b, and c. Find $a + b + c$.

26. The two diagonals of a convex quadrilateral have lengths 10 and 14, and intersect at a $60°$ angle. What is the area of the quadrilateral?

27. The year 2025, considered as an integer, satisfies $2025 = (1 + 2 + 3 + \cdots + n)^2$ for a certain positive integer n. What is the next year that will have this property for some positive integer n?

28. Two positive numbers a and b satisfy: $2 + \log_2(a) = 3 + \log_3(b) = \log_6(a+b)$. Find $\frac{1}{a} + \frac{1}{b}$.

29. If $\sin\alpha = -\frac{1}{\sqrt{2}}$ and $\cos(\alpha - \beta) = \frac{1}{2}$ with $\beta > 0$, what is the minimum possible value of β?

30. Let R be the region in three-dimensional space consisting of all points (x, y, z) such that the distance from (x, y, z) to the nearest point on the line segment $\overline{AB}$ is at most 3 units. If the volume of R is 216π cubic units, what is the length of $\overline{AB}$?

31. Jim, Melissa, Nora, and Jesse each get on a sled and they all race to the bottom of a hill. How many ways can the race turn out if ties are possible?

32. What are the last two digits of 6^{260} when it is multiplied out?

33. How many ordered pairs of integers (m, n) satisfy $|m - n| \leq |m + n| \leq 16$?

34. If the letters in **ABRACADABRA** are randomly arranged, what is the probability that all of the A's are next to each other in a single grouping? Express your answer as a rational number in lowest terms.

35. Let a, b, c, and d be the roots of the polynomial $p(x) = x^4 - 14x^3 + 41x^2 - 74x - 182$. Find $a^2 + b^2 + c^2 + d^2$.

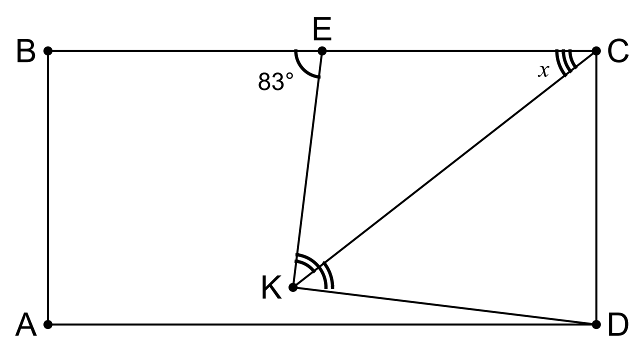
36. On April 8, 2024, the Burlington, Vermont area experienced a total solar eclipse, which motivated the following question. Suppose you have two disks of radius 1 in the plane (representing the Sun and the Moon), and the distance between their centers is d. One of the disks is moving across the other disk so that they overlap. What is the area of overlap when $d = \sqrt{2}$?

37. Let h and k be positive integers with $\frac{1}{h} + \frac{1}{k} = \frac{1}{13}$. Find the maximum value of $h + k$.

38. Find the value of the expression: $$\frac{1}{\cos\left(\frac{2\pi}{5}\right)} + \frac{1}{\cos\left(\frac{4\pi}{5}\right)} + \frac{1}{\cos\left(\frac{6\pi}{5}\right)} + \frac{1}{\cos\left(\frac{8\pi}{5}\right)}$$

39. What is the smallest positive integer whose digits consist only of 6s and 7s that is divisible by both 6 and 7?

40. In rectangle $ABCD$, side $BC = 2CD$. Point E is the midpoint of $\overline{BC}$, and point K is taken inside the rectangle so that $\angle BEK = 83°$ and $\angle EKC = \angle DKC$. Find $\angle ECK$ (labeled x in the figure).



41. If x and y are real numbers such that $x^4 + x^2y^2 + y^4 = 2025$ and $x^2 + xy + y^2 = 75$ and $x > y > 0$, find the value of x.