University of Vermont | College of Engineering and Mathematical Sciences
Mathematics and Statistics

SIXTY-NINTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 10, 2026

1. Express as a rational number in lowest terms: $\dfrac{2\frac{1}{3}}{4\frac{1}{5}}$

2. If $A = (1 + 2 \cdot (3 - 4)) \cdot (5 + 6)$ and $B = 1 + 2 \cdot 3 - 4 \cdot 5 + 6$, what is the value of $A - B$?

3. A sphere is inscribed in a cube. If the surface area of the cube is 216 square units, what is the volume of the sphere?

4. Two hours from now, it will be half as long until midnight as it will be one hour from now. What time is it now? Express your answer in the form hours:minutes.

5. Amy has 3 shirts, 4 bottoms, and 2 pairs of shoes in a box. If an outfit consists of a shirt, a bottom, and a pair of shoes, how many different outfits can Amy make using clothes from the box?

6. The sum of the digits of a two-digit number is 9. When the digits are reversed, the number increases by 63. Find the number.

7. A packet of dough is exactly enough to make a circular pizza with a diameter of 12 inches. The packet is instead used to make a circular pizza with a diameter of 8 inches, with some dough left over. What is the diameter of the largest circular pizza that can be made from the leftover dough, assuming the pizzas are all the same thickness?

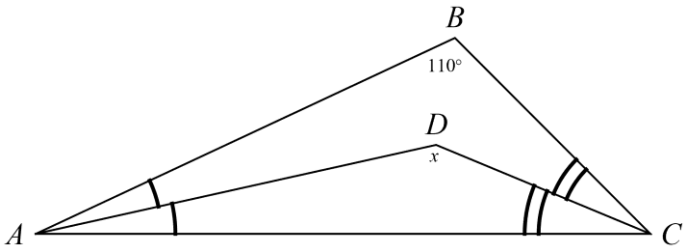
8. A bakery sells donuts for \$1.75 each and cupcakes for \$2 each. Marti buys some donuts and some cupcakes (at least one of each) for a total of exactly \$26. How many cupcakes did she buy?

9. If $\dfrac{\left(x^2y^{-12}\right)^4 (xy)^3}{\left(x^5y^2\right)^{-3}}$ is expressed in the form $x^m y^n$, what is the value of $m + n$?

10. Let $f(x) = \dfrac{1}{4x + 1}$. Find all x such that $f(2x) = 2f(x)$.

11. Anton has a certain number of chocolate bars. If he groups them into pairs, he has one left over. If he divides them into groups of three, he has two left over. If he divides them into groups of five, he has four left over. What is the smallest number of chocolate bars Anton could have?

12. In triangle ABC, angle ABC is 110°. Triangle ADC is drawn so that $\overline{AD}$ bisects angle BAC and $\overline{DC}$ bisects angle BCA as shown in the figure. Find $\angle ADC$ (labeled x in the figure) in degrees.

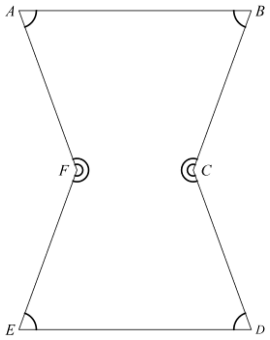


13. Find the value of $\log_2(27) \cdot \log_3(25) \cdot \log_5(32)$.

14. A real number θ satisfies $\cos\theta = \tan\theta$. Find the value of $\csc\theta + \cos^4\theta$.

15. A clock has an hour hand that is 4 inches long, a minute hand that is 5 inches long, and a second hand that is 6 inches long. When the hands are all 120° apart from each other, what is the area of the triangle whose vertices are the tips of the hands?

16. In concave hexagon $ABCDEF$, $\angle A = \angle B = \angle D = \angle E = x$ and $\angle C = \angle F = 2(x + 20)$. Find $\angle C$ in degrees.



17. Find the sum of the first 12 terms of the arithmetic sequence $10, 13, 16, 19, \ldots$

18. How many integers $x > -3$ satisfy the inequality $\dfrac{2}{x+3} < \dfrac{3}{2x+1}$?

19. Consider the set of all rectangles whose perimeter is 25 units. Make a new set of rectangles by increasing each side of each original rectangle by 1 unit. Let Δ be the difference of the area of a rectangle in the new set and its "parent" rectangle in the original set. Find $\max(\Delta) + \min(\Delta)$ where the max and min are taken over all of the rectangles.

20. If $y = x^7$ and $x^y = 196$, find x.

21. In an arbitrary triangle ABC, points D and E lie on sides $\overline{AB}$ and $\overline{BC}$ respectively, dividing these sides in the ratios $AD : DB = 1 : 2$ and $BE : EC = 3 : 4$. Find the ratio of the area of ΔDBE to that of ΔABC.

22. Express as a rational number in lowest terms: $\dfrac{1}{\sqrt{9}+\sqrt{19}} + \dfrac{1}{\sqrt{19}+\sqrt{29}} + \dfrac{1}{\sqrt{29}+\sqrt{39}} + \dfrac{1}{\sqrt{39}+\sqrt{49}}$.

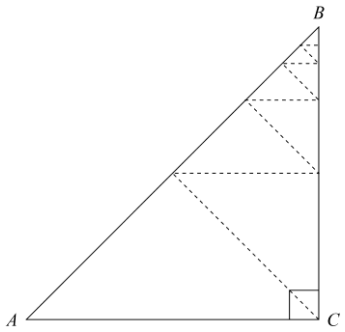
23. Find the smallest positive integer n such that $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10n$ is a perfect cube.

24. Find the value of x that satisfies the equation $\log_3(x) - \log_3\left(x^2\right) + \log_3\left(x^3\right) - \log_3\left(x^4\right) + \cdots + \log_3\left(x^{99}\right) - \log_3\left(x^{100}\right) = 200$.

25. Find the shortest distance from the origin to the circle $x^2 + y^2 - 10x + 8y + 32 = 0$.

26. Nora, Jesse, and eight of their friends sign up for a team scavenger hunt. Five of them will be randomly assigned to the Purple team, and the other five will be on the Yellow team. What is the probability that Nora and Jesse end up on the same team? Express your answer as a rational number in lowest terms.

27. Triangle ABC is an isosceles right triangle with $AC = BC = 1$. A path begins at vertex C and zig-zags between the hypotenuse and side $\overline{BC}$. The skew segments are all perpendicular to the hypotenuse, and the horizontal segments are all perpendicular to $\overline{BC}$. The first several segments of the path are shown (dashed lines) in the figure; the path continues indefinitely. Find the total length of the zig-zag path.



28. Find the sum of all the coefficients in the expansion of $\left(10x^4 - 7x^2 + 5x - 9\right)^{17}\left(x^8 + 25x - 24\right)^5\left(97x^3 - 85x^2 - 5x + 3\right)^3$.

29. A string of consecutive zeroes at the end of an integer are called *trailing zeroes*. For example, 1,096,000 has three trailing zeroes. How many trailing zeroes does 2026! have?

30. If the graph of $y = f(x)$ is a line with slope –4, then the graph of $f(f(x)) + f(2x) + 2f^{-1}(x)$ is a line with slope m. What is the value of m?

31. Alice is twice the age that Bob was when Alice was the age that Bob is now. When Bob is the age that Alice is now, the sum of Alice's age and Bob's age will be 63 years. How old are Alice and Bob now? Give your answer as an ordered pair (a, b) where a is Alice's age in years and b is Bob's age in years.

32. A tank initially contains 729 gallons of pure acid. Taras draws c gallons of acid from the tank and replaces it with c gallons of water. Then Taras draws c gallons of the resulting solution and replaces it with c gallons of water. After repeating this procedure four more times, there remains 64 gallons of acid in the tank, the rest being water. Find c.

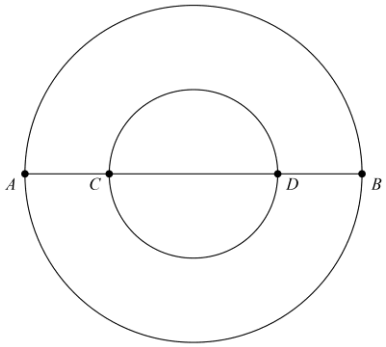
33. If t is an acute angle with $\tan(t) = \frac{3}{4}$, find the value of $\cos(3t)$. Express your answer as a rational number in lowest terms.

34. Find all real x that satisfy $|x - |3x - 4|| = 8$.

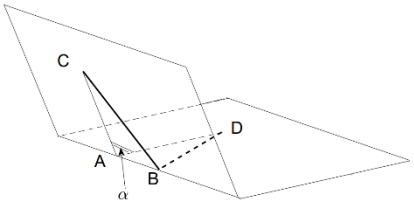
35. Let $z = w_1$ and $z = w_2$ be the two roots of the quadratic equation $z^2 + (5 - 6i)z + (4 + 5i) = 0$. Define ReIm($z$) to be the product of the real and imaginary parts of the complex number z. For example, ReIm(3 – 2i) = –6. Determine the value of ReIm(w_1) + ReIm(w_2).

36. A wire begins at the base of a circular cylinder of height 12 and radius 1, and wraps uniformly and tightly around the cylinder, completing exactly four revolutions by the time it reaches the top of the cylinder. What is the length of the wire?

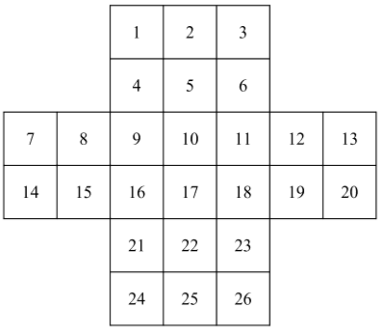
37. How many different paths are there beginning at point A and ending at point B following circular arcs or horizontal line segments in the figure if you must go around each of the two circles exactly once, and no arc or line segment can be traversed more than once? An example of going around the smaller circle: Upper arc $C \to D$, line segment $D \to C$, lower arc $C \to D$. Paths where sections are traversed in a different order are considered distinct.



38. Two planes intersect along line AB as shown in the figure. The angle between them is $\angle CAD = 120°$ (identified as α in the figure). Additionally, $AC = AD$ and $\angle CAB = \angle DAB = 90°$. Point B is chosen so that $\angle CBD = 90°$. Let $\beta = \angle CBA$. Find $\sin(\beta)$.



39. A game board consists of 26 different square spaces as shown in the figure. How many ways can a blue checker and a red checker be placed on different spaces on the board so that the two occupied spaces do not share a common edge?



40. A right triangle has legs of lengths a and b and hypotenuse of length c. What is the greatest possible value of the quantity $\frac{a+3b}{c}$?

41. What is the area of the triangle whose side lengths are the three distinct real values of x that satisfy the equation $x^3 - 10x^2 + 31x - 29 = 0$?