Wielandt's Theorem on Automorphism Towers

Notes by Richard Foote

I have always wondered whether Wielandt's Theorem on the finiteness of automorphism towers could be made more transparent by couching a proof in the more modern terminology of components, etc. (One classic book containing a proof is [**Zas**], Appendix G.) Here is my attempt. I make no claim that this proof is better, shorter or even significantly different from either the original or proofs in various books; but rather this is a record of my musings on the subject.

Notation is standard, as may be found in [**DF**] or [**Asc**]. All groups in this note are finite. Throughout the notes, "acts" means "acts by conjugation." For completeness, the cited exercises from [**DF**] are included in the Appendix at the end of this note.

Recall that a group G acts on itself by conjugation resulting in a homomorphism

$$G \longrightarrow \mathrm{Aut}(G)$$

whose kernel is the center of G (see [**DF**], Section 4.4 and its exercises for definitions and basic facts). In particular, if $Z(G) = 1$, then this is an embedding, and we may identify G with its group of inner automorphisms, which, by Exercise 4.4.1, forms a normal subgroup of $\mathrm{Aut}(G)$. It is immediate too from this exercise that if G has trivial center, then the center of $\mathrm{Aut}(G)$ is likewise trivial, so we may repeat the process. Replacing the homomorphisms, as above, by containments, we then iteratively obtain a tower:

$$\begin{gathered}\text{If } Z(G)=1 \text{ we have}\\ G = A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \cdots \trianglelefteq A_n = A \qquad \text{where} \qquad A_{i+1} = \mathrm{Aut}(A_i), \quad 0 \le i \le n-1.\end{gathered} \tag{$*$}$$

Wielandt's beautiful result asserts that this automorphism tower eventually stabilizes:

Theorem (*H. Wielandt, 1939)* Under the hypotheses of $(*)$, there is some N such that $A_i = A_N$ for all $i \ge N$. (Equivalently, in any such tower, the order of A_n is bounded by some function of $|G|$, independent of n.)

Note that the order of the automorphism group of any centerless group X is bounded by $|X|!$ because $\mathrm{Aut}(X)$ faithfully permutes the elements of X. Therefore, to prove Wielandt's Theorem it suffices to find certain (finitely many) specific subgroups X_i of G such that $|A_n|$ is bounded in terms of $|\mathrm{Aut}(X_i)|$. At the heart of this process lies the *generalized Fitting subgroup* of G, and its fundamental property, due to Helmut Bender. For convenience we review this subject area.

B. Background Preliminaries

Let p be a prime and let X, H be groups. The results in this section, labeled **B.i** for integers **i**, might be considered as part of the "bread-and-butter" toolkit for finite group theorists! Experts may skip this section.

Recall that H is a *subnormal subgroup* of X if there is a chain

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_m = X \tag{$**$}$$

where each H_i is normal in H_{i+1} (but not necessarily normal in X). This is usually denoted by $H \trianglelefteq\trianglelefteq X$. For example, by [**DF**] Section 6.1, every subgroup of a nilpotent group is subnormal (and conversely). Note that "subnormality" is transitive. Also, such H is subnormal in every subgroup of X containing it.

B.1 If P is a Sylow p-subgroup of X and H is subnormal in X, then $P \cap H$ is a Sylow p-subgroup of H.
(*Proof:* Use induction and Exercise 4.5.34).

A subgroup L of X is called a *component* of X if the following hold:

(i) $L \trianglelefteq\trianglelefteq X$,

(ii) L is perfect, i.e., $L = [L, L]$, and

(iii) $L/Z(L)$ is a (non-abelian) simple group.

A group satisfying only (ii) and (iii) is called *quasisimple*; so a component of X is a quasisimple subnormal subgroup of X. Non-abelian simple groups are, a fortiori, quasisimple. $SL_2(\mathbb{F}_5)$ is a quasisimple group of order 120 with a center of order 2.

Define $E(X)$ *to be the group generated by all components of* X*.*

B.2 $E(X) = L_1 * L_2 * \cdots * L_r$ is a (commuting) central product of all the components L_i of X.
(*Proof:* See [**Asc**], 31.7.)

B.3 The only normal subgroups of $E(X)$ are products of some of the L_i together with some subgroup of the center of $E(X)$.
(*Proof:* Ibid. This follows also from [**DF**], Exercise 5.4.18 applied to $E(X)/Z(E(X))$.)

B.4 The *Fitting subgroup* of X, denoted by $F(X)$, is the largest normal nilpotent subgroup of X. Also,

$$F(X) = O_{p_1}(X) \times O_{p_2}(X) \times \cdots \times O_{p_s}(X)$$

where $p_1, \ldots, p_s$ are all the distinct primes dividing $|X|$.
(*Proof:* Easy exercise. See [**Asc**], 31.8.)

B.5 $E(X)$ and $F(X)$ commute and are both characteristic in X.
(*Proof:* This follows from B.3 and B.4; see also [**Asc**], 31.12.)

Define the *generalized Fitting subgroup* of X, denoted by $F^*(X)$, to be $F^*(X) := F(X)E(X)$.

B.6 *(The Generalized Fitting Subgroup Theorem)* $C_X(F^*(X)) = Z(F^*(X))$.
(*Proof:* See [**Asc**], 31.13. Note that $F^*(X)$ is characteristic in X.)

B.7 As a direct consequence of B.6 and the opening remarks of this note we get:

$$\textit{For any finite group } X, \qquad |X| \leq |Z(F^*(X))| \cdot |\mathrm{Aut}(F^*(X))|.$$

In particular, if $M = |F^*(X)|$, then an upper bound for $|X|$ is $M \cdot M!$. This is crude because, in particular, automorphisms of a group cannot be arbitrary permutations of that group: they must all fix the identity, preserve the orders of elements, etc.

B.8 *(Three Subgroups Lemma)* Let H, E be any subgroups of X with E perfect. If $[H, E, E] = 1$, then $[H, E] = 1$.
(*Proof:* See [**Asc**], 8.9. Strictly speaking this is just a consequence of the more generally formulated Three Subgroups Lemma in [**Asc**].)

Recall that $O^p(X)$ is the *smallest* normal subgroup of X for which $X/O^p(X)$ is a p-group. It is easy to see that $O^p(X)$ is the subgroup generated by all elements of X of order prime to p.

B.9 If $H \trianglelefteq\trianglelefteq X$ and $|X : H|$ is a power of p, then $O^p(X) = O^p(H)$.
(*Proof:* This is an easy induction on m in $(**)$.)

C. Proof of Wielandt's Theorem

Henceforth let G be a finite group with $Z(G) = 1$. Adopt the notation of $(*)$. By the Generalized Fitting Subgroup Theorem (or, more precisely, B.7) it suffices to bound $|F^*(A)|$ in terms of $|G|$. We do this by bounding $|E(A)|$ and then $|F(A)|$ separately, in a series of steps.

Step 1: $C_A(G) = 1$.

Proof: Proceed by induction on the length, n, of the chain in $(*)$. If $n = 0$ then $G = A$ and $C_G(G) = Z(G) = 1$. If $n = 1$ then the result is the observation that $A_1 = \mathrm{Aut}(G)$ acts faithfully on G, made earlier. For $n \geq 2$, by induction we have $C_{A_{n-1}}(G) = 1$. Let $C = C_A(G)$. Then since $G \trianglelefteq A_1$, we see that A_1 normalizes C; and since $A_1 \leq A_{n-1} \trianglelefteq A$ we have

$$[C, A_1] \leq C \cap A_{n-1} = C_{A_{n-1}}(G) = 1,$$

i.e., $C \leq C_A(A_1)$. By definition of C the reverse containment is obvious, so $C = C_A(A_1)$. Now again use induction with G replaced by A_1, so that the chain starting at A_1 has length $n - 1$, to get $C = 1$.

Step 2: $E(A) = E(G)$.

Proof: By induction on n we have $E(A_{n-1}) = E(G)$. Since $E(G)$ is subnormal in A, each component of $E(G)$ is likewise a component of A, hence is a component of $E(A)$. Let E_2 be the central product of all components of A that are *not* contained in G, so by B.2:

$$E(A) = E(G) * E_2$$

where $*$ denotes central product; and both factors are normalized by G. Since $G \leq A_{n-1} \trianglelefteq A$ we have

$$[G, E_2] \leq E_2 \cap A_{n-1}.$$

Since the latter subgroup is normal in E_2, by definition of E_2 and B.3 we have $[G, E_2] \leq Z(E_2)$. Thus $[G, E_2, E_2] = 1$. Since E_2 is perfect, by B.8 we have $[G, E_2] = 1$, i.e., $E_2 \leq C_A(G)$. The result now follows from Step 1.

Step 2 accomplishes the first stage of our proof of Wielandt. We next bound $|F(A)|$ in terms of $|G|$. It suffices to bound the number of distinct primes p dividing $|F(A)|$ and each $|O_p(A)|$.

Step 3: For every prime p, $O_p(A)$ normalizes and acts faithfully by conjugation on $O^p(G)$. In particular, if p does not divide $|G|$ then $O_p(A)$ acts faithfully on G.

Proof: For an arbitrary prime p let $X = O_p(A)G$, and let P be a Sylow p-subgroup of X. By B.9,

$$O^p(G) = O^p(X) \trianglelefteq X,$$

as needed for the first assertion. Let $P_0 = C_P(O^p(G)) \trianglelefteq P$. Since $P_0 \trianglelefteq P$, if $P_0 \neq 1$ then $P_1 := P_0 \cap Z(P) \neq 1$. By B.1, P contains a Sylow p-subgroup of G. Thus P_1 centralizes every p'-element of G as well as a Sylow p-subgroup of G. This forces $P_1 \leq C_A(G)$, which is trivial by Step 1; and so $P_0 = 1$ as well. This proves Step 3.

Wielandt's Theorem now follows easily: For all primes p not dividing $|G|$ we have $O^p(G) = G$; and so from Step 3, each corresponding $O_p(A)$ is bounded by the order of a Sylow p-subgroup of

Aut(G). Since these $|O_p(A)|$ are relatively prime for different p, the direct factor of $F(A)$ consisting of all $O_p(A)$ for p not dividing $|G|$ is bounded in order by $|\mathrm{Aut}(G)|$. By Step 3 we thus have

$$|F(A)| \leq |\mathrm{Aut}(G)| \cdot \prod_{p \,\big|\, |G|} |\mathrm{Aut}(O^p(G))|.$$

Combining this with Step 2 gives

$$|F^*(A)| \leq |E(G)| \cdot |\mathrm{Aut}(G)| \cdot \prod_{p \,\big|\, |G|} |\mathrm{Aut}(O^p(G))|.$$

So $|A_n|$ is (crudely) bounded above, independent of n in $(*)$, by B.7 applied to the above.

Exercises

Let G be a finite group with $Z(G) = 1$. Adopt the notation of $(*)$.

1. Prove that $N_A(G) = A_1$. Deduce that if $G \trianglelefteq A$ then $A_2 = A_1$, i.e., the tower terminates in at most one step.
2. Let G be a direct product of non-abelian simple groups. Prove that $A_2 = A_1$ in $(*)$.

References

[Asc] *Finite Group Theory*, M. Aschbacher, Cambridge U. Press, 1986.

[DF] *Abstract Algebra, third edition*, D. Dummit and R. Foote, Wiley, 2003.

[Zas] *The Theory of Groups*, H. Zassenhaus, Chelsea, 1958.

Appendix — Cited exercises from [DF]

4.1.1. If $\sigma \in \mathrm{Aut}(G)$ and ϕ_g is (left) conjugation by g prove that $\sigma\phi_g\sigma^{-1} = \phi_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. (The group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G.)

4.5.34. Let $P \in Syl_p(G)$ and assume $N \trianglelefteq G$. Use the conjugacy part of Sylow's Theorem to prove that $P \cap N$ is a Sylow p-subgroup of N. Deduce that PN/N is a Sylow p-subgroup of G/N. (This may also be done by the Second Isomorphism Theorem—cf. Exercise 9, Section 3.3.)

5.4.18. Let $K_1, K_2, \ldots, K_n$ be non-abelian simple groups and let $G = K_1 \times K_2 \times \cdots \times K_n$. Prove that every normal subgroup of G is of the form G_I for some subset I of $\{1, 2, \ldots, n\}$ (where G_I is the direct product of the K_i for $i \in I$).
[Hint: If $N \trianglelefteq G$ and $x = (a_1, \ldots, a_n) \in N$ with some $a_i \neq 1$, then show that there is some $g_i \in G_i$ not commuting with a_i. Show $[(1, \ldots, g_i, \ldots, 1), x] \in K_i \cap N$ and deduce $K_i \leq N$.]