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A Few More Room Frames

J. H. Dinitz
Department of Mathematics
University of Vermont
Burlington VT 05405

D. R. Stinson
Computer Science and Engineering
University of Nebraska
Lincoln NE 68588

1 Introduction

Since its introduction more than 15 years ago by Wallis [16], the notion of Room frame (and the more general notion of holes in designs) has become one of the most powerful tools in combinatorial design theory. The main purpose of this paper is to present Room frames for many new orders. We begin with the definitions.

Let S be a set, and let $\{S_1, \ldots, S_n\}$ be a partition of S. An $\{S_1, \ldots, S_n\}$*–Room frame* is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols of S,
2. the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),
3. each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$,
4. the pairs occurring in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \bigcup_{i=1}^{n}(S_i \times S_i)$.

As is usually done in the literature, we shall refer to a Room frame simply as a *frame*. The *type* of a frame F is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1}t_2^{u_2}\ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$. The *order* of the frame is $|S|$.

Frames of type $1^1 3^4$ and $1^4 3^3$ are presented in Figures 1 and 2.

Figure 1: A frame of type $1^1 3^4$

	9c		5b	2d				36		7a	48	
				8b	ad			5c			79	16
				1c	9b		5d				6a	78
59						ac	1b		7d	68		
2a	18	9d						4b	3c			
		ab	8d				4c				13	29
8c								1d	2b	49		3a
	6d		7c			3b				12		45
6b		17			2c	4d				35		
3d	7b	6c			14						25	
		58	1a	39				27	46			
			69	4a		28	37		15			
47	5a				38	19	26					

Figure 2: A frame of type $1^4 3^3$

	ad			3c		28		6b		79	45	
6d		47			ac	9b			15		38	
	14		6a	8b				2c	7d			59
	39	12			8d		5b		6c		7a	
2b		8c						4d		3a	19	
9c		ab	23				1d					48
			9d				4c		3b	18		2a
34	7b	5d				1c					26	
			7c		4b	3d				25		16
	5c		1b	2d						46		37
5a	68			49	13		27					
			58		29	4a	36	17				
78		69		1a				35	24			

The first Room frame construction was used to prove the existence of a Room square of side 257 in [16]. Room frames have been used extensively since then; see, for example, [4], [5], [8], [11], [10] and [9].

We summarize existence results for frames in the following theorem. Parts 6 – 8 are restatements of theorems concerning Room squares with subsquares in the context of frames.

Theorem 1.1 *There exist frames of the following types:*

1. t^4 *for all* t *divisible by* 4 *[5],*
2. t^5 *for all* t *divisible by* 2, 3, 5 *or* 7 *[5],*
3. t^u *for* $u \geq 6$ *and both* t *and* u *even [5],*
4. t^u *for all* t *and all odd* $u \geq 7$ *[5],*
5. $2^a 4^b$ *for all* $a + b = 6$ *[8],*
6. $1^{u-v}v^1$ *for* $v = 3, 5, 7$ *and all odd* $u \geq 3v + 2$ *[8],*
7. $1^{u-v}v^1$ *for odd* $v \geq 393$ *and all odd* $u \geq 3v + 2$ *[15],*
8. $1^{2v+2}v^1$ *for odd* $v \geq 3$ *[17].*

We also summarize nonexistence results.

Theorem 1.2 *There does not exist a frame of type* $T = t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ *in any of the following cases:*

1. $\sum_{i=1}^{k} u_i = 2$ *or* 3 *(i.e. if the number of holes is* 2 *or* 3*)*
2. $T = 2^4$ *[13],*
3. $T = 1^5$ *[12],*
4. *there exist* i *and* j *such that* $t_i \not\equiv t_j \bmod 2$*,*
5. *every* t_i *is odd, but* $\sum_{i=1}^{k} t_i u_i$ *is even,*
6. *there exist* i *and* j *such that* $3t_i + t_j > \sum_{h=1}^{k} t_h u_h$ *[14].*

The following corollary is an immediate consequence of parts 2, 5 and 6 of Theorem 1.2.

Corollary 1.3 *If a frame has exactly four holes, then its type must be* t^4 *for some even* $t > 2$.

In addition to the frames listed in Theorem 1.1 above, many frames can be constructed by recursive techniques which use smaller frames as ingredients. We will list some useful recursive constructions for frames. The first of these employs group-divisible designs. A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*,

2. $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and

3. every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. As with frames, we use an "exponential" notation to describe group-types.

We refer to the following construction as the Fundamental Frame Construction or FFC.

Construction 1.1 (Fundamental Frame Construction) *[14] Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD having type T, and let $w : X \to \mathbb{Z}^+ \cup \{0\}$ (we say that w is a weighting). For every $A \in \mathcal{A}$, suppose there is a frame having type $\{w(x) : x \in A\}$. Then there is a frame having type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

Another recursive construction is the Filling in Holes Construction which we give next.

Construction 1.2 (Filling in Holes) *Suppose there is a frame of type ${t_1}^{u_1}{t_2}^{u_2} \ldots {t_k}^{u_k}$, and for each $1 \le i \le k$ suppose there exists a frame of type ${a_{i1}}^{b_{i1}}{a_{i2}}^{b_{i2}} \ldots {a_{im_i}}^{b_{im_i}}$ where $\sum_{j=1}^{m_i} a_{ij}b_{ij} = t_i$. Then there exists a frame of type $\{{a_{ij}}^{u_i b_{ij}} : 1 \le i \le k, 1 \le j \le m_i\}$.*

If T is the type ${t_1}^{u_1}{t_2}^{u_2} \ldots {t_k}^{u_k}$ and m is an integer, then mT is defined to be the type $m{t_1}^{u_1}m{t_2}^{u_2} \ldots m{t_k}^{u_k}$. The following recursive construction is referred to as the Inflation Construction. It essentially "blows up" every filled cell into a pair of orthogonal Latin squares.

Construction 1.3 (Inflation Construction) *[14] Suppose there is a frame of type T, and suppose m is a positive integer, $m \neq 2$ or 6. Then there is a frame of type mT.*

Several newer recursive constructions for frames can be found in [2] and [10].

In this paper we will make extensive use of a direct method for making Room frames which uses the computer. The algorithm employed is called a *hill-climbing algorithm*. This algorithm was first developed for finding one-factorizations of K_n and Room squares [7]. In [8] a discussion is given describing the application of this algorithm to finding frames. We note here that this is a nondeterministic algorithm that is never guaranteed to find a given frame. However, we have found that with the experience gained from running the program, that we can determine orders for which the algorithm has a good chance of succeeding. We will discuss this further in Section 4.

The purpose of this paper is to give Room frames for many new orders. In Section 2 we will prove a new non-existence result. In Section 3 we will give the results of computer runs to construct frames of relatively small sizes. Then in Section 5 we will use some of these frames and a PBD-closure result to construct frames of type $2^a 4^b$ for all $a + b \ge 48$.

2 Non-existence of certain frames

In this section we will prove two theorems which extend Theorem 1.2.

Theorem 2.1 *If there exists a frame F of order m which contains three holes of sizes a, b and c respectively (not necessarily distinct), then $3(b+c) \leq 2(m-a)$. Further, if F contains exactly five holes and if $a=b=c$, then $3(b+c) < 2(m-a)$ (or equivalently, $m > 4a$).*

Proof: Let F be an $\{S_1, \ldots, S_n\}$-frame of order m on symbol set S, where $|S_1| = a$, $|S_2| = b$ and $|S_3| = c$. Let A denote the rows of F indexed by $S \backslash (S_1 \cup S_2 \cup S_3)$, and let B denote the columns of F indexed by $S \backslash (S_1 \cup S_2 \cup S_3)$.

Since any symbol from S_1 must be paired with all of the symbols from S_2 and S_3, and since these pairs can only occur in cells in the regions $A \cup B$, we see that $b + c \leq 2(m - (a+b+c))$. Thus, $3(b+c) \leq 2(m-a)$, which is the desired result.

Now, make the further assumptions that $a = b = c$, F contains exactly five holes and $m = 4a$. This last assumption implies that every occurrence of an element from S_i $(1 \leq i \leq 3)$ in A or B must be with an element from S_j for some j, $1 \leq j \leq 3$, $j \neq i$. Let $x \in S_4$. Now, x must occur in each column indexed by S_5. In such a cell, x cannot occur with any symbol from $S_1 \cup S_2 \cup S_3$, nor with any symbol from S_4 or S_5. Hence, no symbol can occur with x in such a cell, and we have a contradiction. □

Theorem 2.2 *If there exists a frame F of order m which contains three holes of size a and exactly six holes in total, then $m > 4a$.*

Proof: Let F be an $\{S_1, \ldots, S_6\}$-frame of order m on symbol set S, where $|S_1| = |S_2| = |S_3| = a$. By Theorem 2.1, $m \geq 4a$. Assume $m = 4a$. Let A denote the rows of F indexed by $S \backslash (S_1 \cup S_2 \cup S_3)$, and let B denote the columns of F indexed by $S \backslash (S_1 \cup S_2 \cup S_3)$. As in the proof of Theorem 2.1, every occurrence of an element from S_i $(1 \leq i \leq 3)$ in A or B must be with an element from S_j for some j, $1 \leq j \leq 3$, $j \neq i$.

Now, let $x \in S_4$. Suppose the pair $\{x, y\}$ occurs in a cell of $A \cap B$, say in cell (r, c) where (without loss of generality) $r \in S_5$ and $c \in S_6$. First, $y \notin S_1 \cup S_2 \cup S_3$ by the observation above. Also, $y \notin S_5$ since $r \in S_5$, $y \notin S_6$ since $c \in S_6$, and certainly $y \notin S_4$. It follows that x cannot occur in any cell of $A \cap B$. Then x must occur in $a - |S_4|$ cells of $A \backslash B$ and in $a - |S_4|$ cells of $B \backslash A$. As well, x must occur in at least $3a$ further cells, since it must occur with every symbol of $S_1 \cup S_2 \cup S_3$, and no such pair can occur in $A \cup B$. Since x occurs a total of $4a - |S_4|$ times in the frame, we must have $5a - 2|S_4| \leq 4a - |S_4|$, or $|S_4| \geq a$. But $|S_4| < a$, so we have a contradiction. □

In the Sections 3 and 5, we will be concerned with frames with holes of sizes 2 and 4. The following corollary rules out one particular frame of this type.

Corollary 2.3 *There does not exist a frame of type* $2^2 4^3$.

Proof: Let $a = b = c = 4$ and $m = 16$ in Theorem 2.1. □

3 Existence of some small frames

In this section we will discuss the existence of small frames of several different types. These types include t^4, $1^a 3^b$, $2^a 4^b$, $4^a 6^b$, and $1^{u-v} v^1$. All of the frames given in this section were found using the hill-climbing algorithm for frames which was discussed in [8]. In order to save space in this paper, these frames are given in the research report [3].

The first result we give concerns frames with exactly four holes. Recall from Corollary 1.3 that if a frame F has exactly four holes, then all of the holes must be of the same size. We also note from Theorem 1.2 that if F is of type t^4, then t must be even and $t \neq 2$. From Theorem 1.1 there exists a frame of type 4^4 and indeed for all types $(4t)^4$ for $t \geq 1$. The two smallest unknown types for frames with four holes are the types 6^4 and 10^4. Their existence is ensured by the next theorem, and the actual frames are presented in [3].

Theorem 3.1 *There exist frames of type* 6^4 *and* 10^4.

Using the Inflation Construction and Theorems 1.1 and 3.1, we have the following result concerning frames with exactly four holes.

Corollary 3.2 *There exists a frame of type* t^4 *if* t *is divisible by 4, 6, or 10.*

We next turn our attention to frames of types $1^a 3^b$. We note here that it was the existence of a frame of type $1^8 3^1$ which was the key ingredient in finding an exponential lower bound on the number of nonisomorphic Room squares in [6]. By Theorem 1.2, a frame of type $1^a 3^b$ must necessarily have $a + b$ odd and greater than 3. We have the following result in the cases of 5, 7 or 9 holes.

Theorem 3.3 *There exist frames of types* $1^a 3^b$ *for all* $a + b = 5, 7$ *or* 9, *except for* $(a, b) = (2, 3), (3, 2), (4, 1), (5, 0), (5, 2)$ *or* $(6, 1)$.

Proof: The exceptions follow from Theorem 1.2. Frames of type $1^1 3^4$ and $1^4 3^3$ were presented in Figures 1 and 2. The other frames are given in [3]. □

The following theorem concerns the existence of frames of type $2^a 4^b$. The existence of these frames is completely determined when the number of holes is between 5 and 14.

Theorem 3.4 *There exist frames of type* $2^a 4^b$ *for all* $5 \leq a + b \leq 14$, *except for* $(a, b) = (4, 1), (3, 2)$ *and* $(2, 3)$.

Proof: The non-existence results follow from Theorem 1.2. The frame of type $2^1 4^4$ is given in [14], and the frames of types 2^5 and 4^5 exist from Theorem 1.1. The other frames are in the research report [3]. □

By Theorem 1.2, there is no frame of type $3^3 5^2$ or $3^4 5^1$. We attempted to find frames of type $3^1 5^4$ or $3^2 5^5$ by the hill-climbing algorithm, but we were unsuccessful. We did, however, have no trouble finding frames of type $3^a 5^b$ for $a + b = 7$. The frames are given in [3] and the following theorem results.

Theorem 3.5 *There exists a frame of type $3^a 5^b$ for all $a + b = 7$.*

There is a frame of type $4^4 6^1$ [14] but no other frames of type $4^a 6^b$ where $a+b = 5$ are known. The following theorem gives the existence of some new frames of this type.

Theorem 3.6 *There exists a frame of type $4^a 6^b$ for all $a + b = 6$ or 7.*

An important problem in this area of design theory is that of finding Room squares with subsquares. In this context we note that a Room square of side u which contains a subsquare (or hole) of side v is equivalent to a frame of type $1^{u-v} v^1$. It is also a necessary condition for existence that u and v be odd and that $u \geq 3v + 2$ [1]. In an earlier paper [8], it was shown that these necessary conditions are sufficient if $v = 3, 5$ or 7. By use of the hill-climbing algorithm and with a little more persistence we can now extend these results.

Theorem 3.7 *For $v = 9, 11, 13$ and 15, there exists a Room square of side u with a subsquare of side v (i.e. a frame of type $1^{u-v} v^1$) if and only if $u \geq 3v + 2$.*

Proof: For the cases $u = 3v + 2$, see Theorem 1.1. The following cases are done in the research report [3]: $v = 9, u = 31$; $v = 11, u = 37, 39, 41$; $v = 13, 43 \leq u \leq 61$ (u odd); and $v = 15, 49 \leq u \leq 63$ (u odd). All the remaining cases can be found in [8]. □

4 Observations on the hill-climbing algorithm

In this section, we present some useful empirical observations concerning the practical use of the hill-climbing algorithm. We need to first give a general description of how the hill-climbing algorithm works. For more details, we refer the reader to [8].

The algorithm proceeds in two stages. The first stage consists of assigning each pair to a row of the frame. This is invariably very easy to do. The second stage involves trying to place each pair in a cell of the frame. Since we have already assigned each pair to a row, the second stage tries to assign each pair to a column of the frame. Note that we never change the row assigned to a pair in the second stage of the algorithm. This second stage is often very difficult, succeeding with only a small probability.

The assignments of rows or columns to pairs is done by means of suitable heuristics. In stage 2, where we are attempting to assign columns to the pairs, the application of a heuristic may lead to one more pair being assigned to a column, or the number of pairs assigned to columns may remain unchanged. We define the *deficit* to be the number of pairs that are not assigned to columns. Of course, if the deficit reaches zero, then we have succeeded in constructing the frame.

Each application of a heuristic is called an *iteration* of the algorithm. We have found it useful to define a *threshold*. If the number of iterations exceeds the threshold, we abandon the search, and start over at the beginning of stage 1. After considerable experimentation, we have found a suitable threshold for the construction of frames to be $1200m$, where m is the order of the frame. (In contrast, $100m$ seems to be a suitable threshold for the construction of Room squares of side m [8].)

In Tables 1–3, we present the results of numerous computer runs on some interesting classes of frames. We performed 100 runs for each particular frame. This gave rise to a *deficit vector* $(d_0, d_1, \ldots)$, where d_i denotes the number of runs which terminated with deficit equal to i. Also of interest is the average deficit over the 100 runs.

Table 1 concerns frames of type 1^a3^{7-a}. Note that these frames do not exist for $a = 5, 6$, but exist for all other values of a (Theorem 3.3). Table 2 summarizes results on frames of type 2^a4^{6-a}. These frames were all shown to exist in [8]. Finally, Table 3 concerns subsquares of size 9 in larger Room squares, i.e. frames of type $1^{u-9}9^1$.

Table 1: Results of hill-climbing algorithm to construct frames of type 1^a3^{7-a}

	deficit vector									
type	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	total	average deficit
3^7	1	3	21	28	18	0	0	0	100	3.34
1^13^6	0	1	14	34	38	12	1	0	100	3.49
1^23^5	0	3	15	32	32	12	4	2	100	3.55
1^33^4	0	2	13	36	31	14	3	0	100	3.52
1^43^3	0	0	1	16	33	32	13	5	100	4.55
1^7	16	0	74	10	0	0	0	0	100	1.78

The results that are most easily interpreted are those in Table 3, where we observe that the deficit decreases monotonically as the order of the frame increases. It seems reasonable that more flexibility is gained as the order increases, since the

Table 2: Results of hill-climbing algorithm to construct frames of type 2^a4^{6-a}

	deficit vector											
type	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	total	average deficit
2^6	0	4	30	41	22	3	0	0	0	0	100	2.90
2^54^1	1	1	11	23	37	20	3	1	0	0	100	3.76
2^44^2	0	0	0	1	13	25	37	10	10	0	100	5.75
2^34^3	0	0	0	2	14	39	23	20	1	1	100	5.52
2^24^4	0	0	0	12	41	30	11	4	2	0	100	4.60
2^14^5	0	0	2	11	28	38	15	5	0	0	100	4.69
4^6	0	1	3	15	33	26	16	5	0	0	100	4.49

Table 3: Results of hill-climbing algorithm to construct frames of type 1^u9^1

	deficit vector												
type	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	total	average deficit
$1^{20}9^1$	0	0	1	4	22	33	21	7	9	2	1	100	5.42
$1^{22}9^1$	0	0	9	15	29	21	17	5	2	2	0	100	4.55
$1^{24}9^1$	1	3	16	31	24	16	4	3	1	1	0	100	3.66
$1^{26}9^1$	2	13	18	25	15	15	9	3	0	0	0	100	3.37
$1^{28}9^1$	2	14	22	29	18	11	2	2	0	0	0	100	2.98
$1^{30}9^1$	2	21	34	22	14	4	2	0	0	0	0	100	2.45
$1^{32}9^1$	4	20	33	22	13	8	0	0	0	0	0	100	2.44

hole of size 9 will progressively become less significant as a proportion of the whole array. In a sense, the hole of size 9 can be thought of as $9^2 = 81$ "forbidden cells" out of the total of u^2 cells. Perhaps the ratio $81/u^2$ provides some rough measure of the difficulty of constructing such a frame by means of the hill-climbing algorithm.

It is less easy to explain the results in Tables 1 and 2. It certainly appears that it is easiest to construct frames where all the holes have the same size. Also, when one hole is "large" with respect to the order of the frame, this indicates that the frame might be very difficult to construct (e.g. type 1^43^3). However, we really do not have any adequate explanation as to why a significantly lower average deficit was observed for type 2^54^1 than 2^44^2, for example.

Even though we might be unable to *predict* the average deficit, the average deficit seems to be the most useful method of determining how difficult it will be to construct the frame. It seems that any frame where the average deficit is less than 6.5 has a reasonable chance of being found by the algorithm. As some examples, for $1 \leq a \leq 5$, the frames of types 4^a6^{6-a} all had average deficits in the range from 6.0 – 6.5, and all were ultimately found by the algorithm. Most of these frames took a long time to find; for example, the frame of type $4^4 6^2$ required 90,500 attempts.

Another possible measure of the difficulty of finding a particular frame is the proportion of runs where the deficit equals one. As an example of this, we are still searching for a frame of type $3^1 5^4$. After 47,000 attempts, the average deficit is 6.78. However, in these 47,000 runs, the deficit was always at least three. Most of the frames that we have found included numerous runs where the deficit was one or two, so we are more pessimistic regarding the possibility of finding this frame.

Here are a couple more interesting examples. The frame of type $1^{34} 15^1$ was constructed after only 5793 attempts, despite an average deficit of 7.2 (we attribute this to *very* good luck). This is the highest average deficit where we have actually found the frame. On the other hand, we have been unable to construct a frame of type $1^{38} 17^1$ after over 200,000 attempts where the average deficit is 8.4. For this frame, we have had over 30 runs with deficit equal to 1.

5 The spectrum of frames of type $2^a 4^b$

In this section we will prove a PBD-closure result which will, as a corollary, determine most of the spectrum of frames of type $2^a 4^b$. We begin with some design-theoretic background.

A *pairwise balanced design* (or PBD) is a pair $(X, \mathcal{A})$, which satisfies the following properties:

1. $\mathcal{A}$ is a set of subsets of X (called *blocks*)

2. every pair of points occurs in a unique block.

$(X, \mathcal{A})$ is a (v, K)–PBD if $|X| = v$ and $|A| \in K$ for all $A \in \mathcal{A}$. Given a set K of positive integers, let $\mathrm{B}(K)$ denote the set of positive integers v for which there exists a (v, K)–PBD. The mapping $K \to \mathrm{B}(K)$ is a *closure operation* on the set of subsets of the positive integers; that is, it satisfies the properties:

1. $K \subseteq \mathrm{B}(K)$

2. $K_1 \subseteq K_2 \Rightarrow \mathrm{B}(K_1) \subseteq \mathrm{B}(K_2)$

3. $\mathrm{B}(\mathrm{B}(K)) = \mathrm{B}(K)$.

The set $\mathrm{B}(K)$ is called the *PBD-closure* of the set K. If K is any set of positive integers, then K is *PBD-closed* if $\mathrm{B}(K) = K$.

In [4] it was determined that the set $F_t = \{u : \text{there exists a frame of type } t^u\}$ is PBD-closed. If we denote $F_{st} = \{u : \text{for every } 0 \le a \le u \text{ there is a frame of type } s^a t^{u-a}\}$, then it is an easy exercise to extend that theorem to the following.

Theorem 5.1 *F_{st} is a PBD-closed set.*

We are interested in studying the set $F_{2,4}$. We already have from Theorem 3.4 that $\{6,7,\ldots,14\} \subseteq F_{2,4}$. Let $K_0 = \{6,7,\ldots,14\}$ and observe that any element in $\mathrm{B}(K_0)$ is necessarily in $F_{2,4}$. We will spend the remainder of the section determining a bound v_0 such that $v \in \mathrm{B}(K_0)$ if $v \geq v_0$. We begin with a technical lemma which will give us most of the necessary orders.

Lemma 5.2 *If there exists a TD(n,q), such that $q \in \mathrm{B}(K_0)$ and $7 \leq n \leq 14$, then $s \in \mathrm{B}(K_0)$ for all $s = 6q + a_1 + 6a_6 + 7a_7 + \ldots + 14a_{14}$ where $a_i \geq 0$, $0 \leq \sum a_i \leq n-6$ and $a_i = 0$ if $i > n$.*

Proof: Let $(X, \mathcal{G}, \mathcal{A})$ be a TD(n,q). We will delete points from this design to obtain the PBD. Leave six groups unchanged, and from a_i groups (for $i = 1$ and for $6 \leq i \leq n$) delete all but i of the points. In the remaining groups, delete all of the points. The resulting PBD has blocks with sizes from the set $\{6,7,\ldots,n,q\}$ and so $s \in \mathrm{B}(K_0)$. □

As a corollary we have the next result. The proof follows easily by picking appropriate a_i's in Lemma 5.2.

Corollary 5.3 *If there exists a TD(n,q) such that $q \geq 15$, $q \in \mathrm{B}(K_0)$ and $11 \leq n \leq 14$, then $s \in \mathrm{B}(K_0)$ for all s such that $6q \leq s \leq 6q + 14n - 84$.*

We will now use Lemma 5.2 to show that some small values are in $\mathrm{B}(K_0)$.

Lemma 5.4 *If $s = 42, 43, 44, 48, 49, 50, 51, 54, 55, 56, 57, 58$, or if $60 \leq s \leq 182$, then $s \in \mathrm{B}(K_0)$.*

Proof: All of these values are done by appropriate use of Lemma 5.2. Use a TD(8,7) to get 42,43,44,48,49,50 $\in \mathrm{B}(K_0)$ using $(a_1, a_6, a_7) = (0,0,0)$, $(1,0,0)$, $(2,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,0,1)$, respectively. With $q = 8$, use a TD(9,8) to show that $51 \in \mathrm{B}(K_0)$, and also that $s \in \mathrm{B}(K_0)$ for all $54 \leq s \leq 72$ (except possibly for $s = 59$).

A TD(10,9) yields all orders from 73 to 90. A TD(12,11) covers all values from 91 to 132 and a TD(14,13) takes care of the values from 133 to 182, thus completing the proof. □

Other small values can also be obtained by deleting points from block designs.

Lemma 5.5 $\{31, 52, 53, 59, 183, 184, 185\} \subseteq \mathrm{B}(K_0)$.

Proof: The existence of a $(31,6,1)$–BIBD ($=$ PG(2,5)) proves that $31 \in \mathrm{B}(K_0)$. From the $(57,8,1)$–BIBD (=PG(2,7)) delete either four or five points from an oval to obtain a $(53,\{6,7,8\})$–PBD or a $(52,\{6,7,8\})$–PBD, respectively. So we have that $52, 53 \in \mathrm{B}(K_0)$.

To do order 59, begin with the $(73,9,1)$–BIBD ($= PG(2,8)$) and delete six points from a hyperoval and also delete eight points from some exterior line to get a $(59,\{6,7,8,9\})$–PBD. So $59 \in \mathbf{B}(K_0)$.

The existence of a $(183,14,1)$-BIBD ($= PG(2,13)$) implies that $183 \in \mathbf{B}(K_0)$. To do orders 184 and 185, begin with the $(256,16,1)$–BIBD ($= AG(2,16)$) and proceed as follows. First, delete all the points from two lines of one parallel class. Then consider a second parallel class. There remain 14 points on each of the 16 lines in this class. Delete all points from two lines of this class, delete eight points from a third line of this class, and delete either three or four points from a fourth line of this class. The resulting PBD has either 184 or 185 points and block sizes in K_0. Hence, $184, 185 \in \mathbf{B}(K_0)$. □

We now construct a long interval of small orders in the next lemma.

Lemma 5.6 *$s \in \mathbf{B}(K_0)$ for $186 \le s \le 406$.*

Proof: From Lemma 5.5, $31 \in \mathbf{B}(K_0)$. Use Corollary 5.3 with $q = 31$ and $n = 14$ to get that $s \in \mathbf{B}(K_0)$ for all $186 \le s \le 298$. Using the fact that $49 \in \mathbf{B}(K_0)$ (Lemma 5.4), we apply Corollary 5.3 with $q = 49$ and $n = 14$ to get $s \in \mathbf{B}(K_0)$ for all $294 \le s \le 406$, thus completing the proof. □

We are now in a position to close the spectrum.

Theorem 5.7 *If $s \in \{6,7,\ldots,14,31,42,43,44\}$ or if $s \ge 48$, then $s \in \mathbf{B}(K_0)$.*

Proof: The cases where $s \le 406$ were dealt with in the previous lemmas. Now assume that $s \ge 407$, and that $q \in \mathbf{B}(K_0)$ for all q such that $48 \le q \le s-1$. Write $s = 6q + k$, where $6 \le k \le 11$. Since $s \ge 407$, then $q \ge 66$ and so there exists a $TD(7,q)$. Delete $q-k$ points from a group of the $TD(7,q)$. This yields an $(s,\{6,7,8,9,10,11,q\})$–PBD. Clearly $q \le s-1$ and so by assumption $q \in \mathbf{B}(K_0)$. By Theorem 5.1, we get that $s \in \mathbf{B}(K_0)$. This completes the proof. □

By the fact that $F_{2,4}$ is PBD-closed and contains the set K_0, we have our main result.

Theorem 5.8 *If $s \in \{6,7,\ldots,14,31,42,43,44\}$ or if $s \ge 48$, then there exists a frame of type $2^a 4^{s-a}$ for all $0 \le a \le s$.*

As a corollary, note that for all $t \ne 2,6$, we can obtain frames of type $(2t)^a(4t)^{s-a}$ for all $0 \le a \le s$ if $s \in \{6,7,\ldots,14,31,42,43,44\}$ or if $s \ge 48$. This follows immediately from the Inflation Construction. Finally, notice that there is nothing special about hole sizes 2 and 4. Use of this PBD-closure result can close the spectrum of any $F_{s,t}$, provided the necessary initial cases are done first.

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