CONGRESSUS NUMERANTIUM

VOLUME 35

DECEMBER, 1982

WINNIPEG, CANADA

Continuous Maps in Finite Projective Space

J. H. Dinitz   S. W. Margolis

Department of Mathematics
University of Vermont

1. Introduction

Let $D = (X,A)$ be a (v, k, λ) - block design. A partial function $f: X \to X$ is continuous on D if $f^{-1}(b) \in A \cup \{ \phi \}$ for every $b \in A$. In Dinitz and Margolis [2], general properties of continuous maps were determined and several examples were given. In this paper, we will characterize all continuous maps on designs arising from the point - hyperplane incidence in PG(n,q).

Let V be the n + 1 dimensional vector space over GF(q). Assuming $n \geq 2$, let P(V) = PG(n,q) be the Desargeusian projective geometry of dimension n over GF(q). For the remainder of this paper, let P be the classical projective design obtained from P(V). That is, P is the block design with points, the points of PG(n,q); blocks, the hyperplanes of PG(n,q) and incidence induced by the incidence in PG(n,q). It is well known that P is a $(q^{n+1} - 1/q - 1,\ q^n - 1/q - 1,\ q^{n-1} - 1/q - 1)$ - design. Denote by P^c the complement of P, a $(q^{n+1} - 1/q - 1,\ q^n,\ q^{n-1}(q - 1))$ - design.

If K is a subspace of V, let [K] be the induced subspace of P(V). Given any semilinear map $g: V \to V$ define a partial function $\bar{g} : P \to P$ by $\bar{g}([v]) = [g(v)]$ for all [v] in domain($\bar{g}$) = P - [Ker g].

Let h be any partial function. Then mod(h) is the equivalence

CONGRESSUS NUMERANTIUM, Vol. 35 (1982), pp. 239-244

relation on domain(h) defined by x mod(h) y <=> h(x) = h(y). Given h define $\tilde{h}$: domain(h)/ mod(h) $\to$ Range(h) to be the bijection induced by h.

The following is our main result.

<u>Theorem 1.1</u> The following are equivalent:

1) f: PG(n,q) $\nrightarrow$ PG(n,q) is a continuous map on P^c.

2) There exists subspaces $K \subseteq P(V)$ and $S \subseteq P(V)$ with dim(K) = k and dim(S) = n - k + 1, such that domain(f) = $P \setminus K$ and Range(f) = S, and, such that $\tilde{f}$: P/K $\to$ S is an isomorphism (of n - k + 1 dimensional projective spaces).

3) There exists some (not necessarily 1 - 1) semilinear map g: V $\to$ V such that f = $\bar{g}$.

2. Proof of Theorem 1.1

We will prove the equivalences by proving the implications 1 => 2 => 3 => 1.

(1 => 2). Assume f is any continuous map on P^c. Let D = domain(f), R = range(f), and for $S \subseteq$ PG(n,q) let $b(S) = f^{-1}(S) \cup D^c$.

<u>Proposition 2.1</u>. Let H be a hyperplane of P(V) such that $R \not\subseteq H$. Then b(H) is also a hyperplane.

<u>Proof</u>. Since $R \not\subseteq H$, $H \cap R \neq \phi$. Thus $f^{-1}(H^c) = f^{-1}(H^c \cap R)$ is the complement of some hyperplane in P. Now, $b(H) = (f^{-1}(H^c))^c$ and so b(H) is a hyperplane.

<u>Proposition 2.2</u>. For every $x \in R$, b(x) is a k-dimensional subspace of P(V), for some k.

<u>Proof</u>. Since $b(x) = \bigcap_{x \in H} b(H)$ we have that b(x) is a subspace. It was

shown in [2] that f continuous implies $|f^{-1}(x)| = |f^{-1}(y)|$ for all $x,y \in$ range(f). Thus $|b(x)| = |b(y)|$ for all $x,y \in$ range(f). Therefore, $\dim(b(x)) = k$, independent of x.

<u>Proposition 2.3</u>. D^c is a subspace of P(V).

<u>Proof</u>. If $|R| \geq 2$, then for any $x \neq y \in R$, $D^c = b(x) \cap b(y)$. Thus D^c is a subspace. If $R = \{x\}$, it follows from Proposition 2.1 that D is the complement of some hyperplane.

<u>Proposition 2.4</u>. R = range(f) is a subspace of P(V).

<u>Proof</u>. Let $x \neq y \in R$ and let $\ell = x \vee y$ be the line joining x and y. We must show that $\ell \subseteq R$. Let $H_1, H_2, \ldots, H_s$ be the hyperplanes containing both x and y. Then $b(\ell) = b(\bigcap_i H_i) = \bigcap_i b(H_i)$ and thus $b(\ell)$ is a subspace of P(V) containing $D^c = (\text{domain}(f))^c$.

Let $z \in f^{-1}(x)$, $w \in f^{-1}(y)$ and let $\ell' = z \vee w$ be the line on z and w. We wish to show that f is one-to-one from ℓ' onto ℓ. This would imply $\ell \subseteq R$. First, note that $\ell' \subseteq D$, since otherwise there exists $p \in \ell' \cap D^c$ such that $w \vee z = p \vee z$. But $p \vee z \subseteq b(x)$, and since b(x) is a subspace this implies that $w \in b(x)$, a contradiction. Now let $t \neq t' \in \ell'$ be such that $f(t) = f(t')$. Then $\ell' = t \vee t' \subseteq b(f(t))$, and thus $f(\ell') = f(t)$ a point. However, since $x \neq y$ and $x \in f(\ell')$ and $y \in f(\ell')$ we have a contradiction. Therefore, f is one-to-one on ℓ'. It follows that $|f(\ell')| = |\ell|$ and since $f(\ell') \subseteq \ell$, we have $f(\ell') = \ell \subseteq R$. Thus, R is a subspace of P(V).

The following proposition is verified algebraically. We omit the proof.

Proposition 2.5. If $\dim(b(x)) = k + 1$ and if $\dim(P(V)) = n$ then $\dim(D^c) = k$, $\dim(R) = n - k + 1$ and for every $x \in R$, $|f^{-1}(x)| = q^{k+1}$.

It only remains to show that f induces an isomorphism $\tilde{f}: P/D^c \to R$. It is well known that P/D^c is isomorphic to R. We need only show that $\tilde{f}$ maps lines to lines.

Let K' be a k + 2 dimensional subspace of P(V) contains D^c. Then K'/D^c is a line in P/D^c. Clearly $|f(K')| = q + 1$. Let $x \in f(K')$ and $y \in f(K')$ and let $z \in f^{-1}(x)$ and $w \in f^{-1}(y)$. Then as in Proposition 2.4 $f(z \vee w) = f(K') \subseteq x \vee y$.

By cardinality it follows that $f(K') = x \vee y$ a line in R. So we have that if K'/D^c is a line in P/D^c then $\tilde{f}(K'/D^c)$ is a line in R. This completes the proof of (1 => 2).

(2 => 3). This result is merely a generalization of the Fundamental Theorem of Projective Geometry (see[1]) which states that every automorphism $f: P(V) \to P(V)$ is induced by a one-to-one semilinear map $\overline{f}: V \to V$. Here the semilinear map is not necessarily required to be one-to-one. The proof is straightforward and is omitted.

(3 => 1). Let $f = \overline{g}$ where $g: V \to V$ is a (not necessarily one-to-one) semilinear map. Let $H \subseteq V$ be a hyperplane and let R = range(g) and K = kernel(g). Then $g^{-1}(H) = g^{-1}(H \cap R)$ so $\dim(g^{-1}(H)) = \dim(H \cap R) + \dim(K)$.

If $\dim(H \cap R) = \dim(R)$, the $g^{-1}(H) = V$ and so $\overline{g}^{-1}[H] = P(V)$. Otherwise, $\dim(H \cap R) = \dim(R) - 1$ then $\dim(g^{-1}(H)) = \dim(V) - 1$. Thus $g^{-1}(H) = H'$ a hyperplane of V and so $\overline{g}^{-1}([H]) = [H']$.

So we have that $\overline{g}^{-1}$ acting on the complement of a hyperplane in P(V) is either ϕ or the complement of some hyperplane. Thus $\overline{g} = f$ is continuous on P^c.

This completes the proof of our main theorem.

We shall note that if P is a non Desarguesian projective plane then it is still true that 1 <=> 2 in Theorem 1.1. This result was proven in our earlier paper on this topic ([2], Prop. 5.2 and Prop. 5.3).

3. Conclusion

In our earlier paper on this topic [2] we characterized continuous maps on the complements of projective planes and on the complement of some Hadamard designs. The main theorem of this paper generalizes both of these results to the case where the design is the complement of any projective space.

This theorem characterizes these maps in three different ways; 1) in terms of designs, 2) geometrically, and 3) algebraically. Indeed, we have shown that the continuous maps on the complement of a projective space are a natural generalization of the collineations of the projective space. Since the set of all continuous maps on P^c forms a monoid $\Omega(P^c)$ under composition we have shown that Aut(P) is the group of units of $\Omega(P^c)$.

Since continuous maps are defined on arbitrary block designs, it is encouraging to see that the continuous maps on this particular class of designs have such a natural interpretation.

REFERENCES

Artin, E., Geometric Algebra. New York - London: Interscience, 1957.

Dinitz, J. H., and S. W. Margolis. "Continuous Maps in Block Designs." submitted.