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## CONTINUOUS MAPS ON BLOCK DESIGNS

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Abstract

When does a block designs $D = (V,\mathcal{B})$ possess a partial function $f: V \to V$ such that the inverse image of each block is a block or the empty set? We call such maps continuous maps on $D$. Clearly if $f \in \mathrm{Aut}(D)$, then $f$ is continuous. However, there exist continuous maps on $D$ which are not one to one. The existence of such a map implies that $D$ contains an arc of size equal to the number of elements in the image of $f$. In this paper we give results on the structure of such maps and restrictions on the parameters of a design which admits a continuous map which is not an automorphism. We also present three infinite classes of examples and give some connections with the theory of semigroups.

*Introduction.*

It has long been known that the automorphisms of a block design provide an important connection between the theory of groups and combinatorics. In this paper we introduce the notion of a continuous map on a block design. We then relate algebraic properties of the monoid of continuous maps to combinatorial properties of the design.

Let $D = (V,\mathcal{B})$ be a block design with parameters $(v,b,r,k,\lambda)$. A partial function $f: V \to V$ is *continuous* on $D$ if $f^{-1}(B) \in \mathcal{B} \cup \{\emptyset\}$ for every $B \in \mathcal{B}$. Clearly a permutation is continuous if and only if it is in the automorphism group of $D$.

As an example of a continuous map which is not a permutation consider the following.

Example 1.1. Let $V = \{0,1,2,3,4,5,6\}$ and let $\mathcal{B} = \{b_i | i = 0,\ldots,6\}$ be the collection of subsets where

$b_0 = \{0,3,5,6\}$
$b_1 = \{1,4,6,0\}$
$b_2 = \{2,5,0,1\}$
$b_3 = \{3,6,1,2\}$
$b_4 = \{4,0,2,3\}$
$b_5 = \{5,1,3,4\}$
$b_6 = \{6,2,4,5\}$

Then $(V,\mathcal{B})$ is a $(7,4,2)$ design. Define the partial function $f\colon V \to V$ by

$$f(1) = f(3) = 0$$
$$f(2) = f(6) = 5$$
$$f(4) = f(5) = 4$$

Then,

$$f^{-1}(b_0) = f^{-1}(b_2) = b_3$$
$$f^{-1}(b_1) = f^{-1}(b_4) = b_5$$
$$f^{-1}(b_5) = f^{-1}(b_6) = b_6$$
$$f^{-1}(b_3) = \emptyset.$$

Therefore, $f$ is continuous.

In Section 2 we establish some preliminary results. In particular, we show that continuous maps have a natural interpretation with respect to the incident matrix of a design.

In Section 3 we show that if $r > \lambda^2$, then every continuous map $f$ is either a permutation or card $f(V) \leq 1$. This implies that in this case the computation of the monoid of continuous maps on $D$ reduces to the computation of Aut(D) and the blocks of $D$.

A partial function $f\colon V \to V$ is *homogeneous of degree* $d$ if $\text{card}(f^{-1}(x)) = d$ for every $x \in f(V)$. In Section 4 we show that every continuous function of a $(v,k,\lambda)$ design is homogeneous of degree $d$, for some integer $d$ dividing $k$. This result has many important consequences. We show that $f(V)$ is a $\frac{k}{d}$-arc in $D$. That is, every block intersects $f(V)$ in either $0$ or $\frac{k}{d}$ points [7]. This puts various arithmetic constraints on the possible values for card $f(V)$. For example, if $k$ is prime, or $k$ is relatively prime to $r - \lambda$, then card $f(V) \leq 1$ or card $f(V) = v$. We also show that a continuous map $f$ is total if and only if $f$ is an automorphism of $D$.

In Section 5 we develop three infinite classes of designs with the property that there exist continuous maps $f\colon V \to V$ and $1 < \text{card } f(V) < v$. The first example involves projective planes, the second Hadamard designs, and the third a tensor product construction involving weighing matrices.

There have been a number of results relating combinatorics and completely 0-simple semigroups. [3,5,6,9]. Here we briefly mention that the monoid of continuous maps is a familiar object in the theory of semigroups. If $D = (V,\mathcal{B})$ is a design, then there is a completely 0-simple semigroup $S(D)$ naturally associated with $D$. (See [6] Chapter 3.5 for details). It can be shown that the monoid of continuous maps is isomorphic to the translational hull of $S(D)$. Translational hulls play an important role in the theory of ideal extensions of semigroups and in various other parts of semigroup and ring theory [1,8]. In a future paper we will study the structure of the monoid of continuous maps from this point of view of the semigroup structure.

*II Preliminaries.*

An incidence system is a pair $D = (V,\mathcal{B})$ where $V$ is a finite set of points, and $\mathcal{B}$ is a collection of subsets of $V$ called blocks. A partial function $f: V \to V$ is *continuous* on $D$ if whenever $B \in \mathcal{B}$, then $f^{-1}(B) \in \mathcal{B} \cup \{\emptyset\}$. Clearly a permutation is continuous on $D$ iff it is an automorphism of $D$.

For example, if $D = (V,\mathcal{P}(V))$ then every partial function is continuous on $D$. If $D = (V,\mathcal{B})$ where $\mathcal{B}$ consists of the singletons of $V$, then $f: V \to V$ is continuous if and only if $f$ is injective. That is, for all $x,y \in V$, $f(x) = f(y) \in V$ implies $x = y$.

If $f: V \to V$ and $g: V \to V$ are continuous on $D$, then so is the composite $fg: V \to V$. Furthermore, the identity map $1: V \to V$ is continuous. Therefore, $\Omega(D) = \{f: V \to V | f \text{ is continuous on } D\}$ is a submonoid of the monoid of all partial functions acting on the left of $V$. We call $\Omega(D)$ the monoid of continuous maps on $D$.

Every $f \in \Omega(D)$ induces a partial function $\bar{f}: \mathcal{B} \to \mathcal{B}$ acting on the right of $B$ defined by

$$B\bar{f} = \begin{cases} f^{-1}(B) & \text{if } f^{-1}(B) \neq \emptyset \\ \text{undefined otherwise} \end{cases}$$

We say that $f$ and $\bar{f}$ are *linked* maps. If $f,g \in \Omega(D)$, then clearly $\overline{fg} = \bar{f}\bar{g}$. Thus the assignment $f \to \bar{f}$ is a homomorphism. We show that in certain circumstances this assignment is an isomorphism.

If $x \in V$, denote the *pencil of blocks* on $x$ by

$$P(x) = \{B \in \mathcal{B} \mid x \in B\}.$$

An incidence system $D = (V,\mathcal{B})$ is *reduced* if for all $x,w \in V$, $P(x) = P(w)$ implies $x = w$.

LEMMA 2.1. *Let $D = (V,\mathcal{B})$ be a reduced design and let $f\colon V \to V$ and $g\colon V \to V$ be continuous on $D$. Then $\bar{f} = \bar{g}$ implies $f = g$.*

*Proof.* Let $\bar{f}\colon \mathcal{B} \to \mathcal{B}$ be the map linked with $f$.

Then, for all $x \in V$ we have

$$P(x)\bar{f}^{-1} = \begin{cases} P(f(x)) & \text{if } f(x) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

For,

$$B \in P(x)\bar{f}^{-1} \Leftrightarrow B\bar{f} = f^{-1}(B) \in P(x) \Leftrightarrow x \in f^{-1}(b) \Leftrightarrow f(x) \in b.$$

Now, if $\bar{f} = \bar{g}$ it follows that $P(x)\bar{f}^{-1} = P(x)\bar{g}^{-1}$ and thus, $P(f(x)) = P(g(x))$ for all $x \in V$. Since $D$ is reduced, we also have $f(x) = g(x)$ for all $x \in V$ and the lemma is proved.

Lemma 2.1 allows us to view $\Omega(D)$ as a monoid of partial functions acting on the left of $V$ and dually as a monoid of partial functions acting on the right of $\mathcal{B}$. We will use this duality throughout this paper.

From now on we will assume that all incidence systems are balanced incomplete block designs (BIBD's). We refer the reader to [4] for background and notation. As usual the parameters of a BIBD are designated by $(v,b,r,k,\lambda)$. Since a BIBD is reduced, Lemma 2.1 holds and we have the following corollary which will be useful later.

COROLLARY 2.2. *Let* $D = (V, \mathcal{B})$ *be a design with parameters* $(v,b,r,k,\lambda)$. *Let* $f: V \to V$ *be continuous and linked with* $\bar{f}: \mathcal{B} \to \mathcal{B}$. *Then card* $P(x)\bar{f}^{-1} = r$ *for all* $x \in$ *Domain* $(f)$.

*Proof.* We have seen in the proof of Lemma 2.1 that $P(x)\bar{f}^{-1} = P(f(x))$ if $x \in$ Domain $(f)$. Since $D$ is a block design, card $P(f(x)) = r$.

Recall that a partial function $f: V \to V$ is injective if card $f^{-1}(x) \leq 1$ for all $x \in V$.

LEMMA 2.3. *Let* $D = (V, \mathcal{B})$ *be a* $(v,k,\lambda)$ *design. If* $f: V \to V$ *is a nonempty injective continuous map, then Domain* $(f) = V$. *That is,* $f$ *is a permutation.*

*Proof.* Assume that Domain $(f) = W$, a proper subset of $V$. Then card $f(V) =$ card $W < v$. Since $f$ is nonempty, there exist $x,y \in V$ such that $x \in f(V)$ and $y \notin f(V)$. But $D$ is a block design so there is a $B \in \mathcal{B}$ with $\{x,y\} \subseteq B$. Therefore, $1 \leq$ card $f^{-1}(B) =$ card $f^{-1}(B\backslash\{y\}) < k$ since $f$ is injective. It follows that $f^{-1}(B) \in \mathcal{B} \cap \{\emptyset\}$ and $f$ is not continuous. This is a contradiction and Domain $(f) = V$.

COROLLARY 2.4. *Let* $D$ *be a design. Then* $f \in \Omega(D)$ *is injective if and only if* $f \in Aut(D)$.

*Proof.* Follows from Lemma 2.3.

We will later show that $f \in \Omega(D)$ has Domain $(f) = V$ if and only if $f \in$ Aut$(D)$.

If $f: V \to V$ is a partial function, then *rank* $(f) =$ card $f(V)$. The rank function will be used to classify continuous maps. In particular the next lemma classifies continuous functions of rank 1. We will show that the only continuous functions of rank 1 are those which map a block onto a point. More formally, if $x \in V, B \in \mathcal{B}$ define a partial function $(x,B): V \to V$ by

$$(x,B)(w) = \begin{cases} x & \text{if } w \in B \\ \text{undefined otherwise} \end{cases}$$

LEMMA 2.5. *Let* $D = (V,\mathcal{B})$ *be a design. A continuous function* $f$ *has rank 1 if and only if* $f = (x,B)$ *for some* $x \in V$, $B \in \mathcal{B}$ .

*Proof.* If $f = (x,B)$ and $B' \in \mathcal{B}$, then $f^{-1}(B') = B$ if $x \in B'$, and $f^{-1}(B') = \emptyset$ if $x \notin B'$. Therefore $(x,B)$ is a continuous map. Clearly rank $(x,B) = 1$.

Conversely, assume $f \in \Omega(D)$ and that $f(V) = \{x\}$ for some $x \in V$. Choose $B' \in \mathcal{B}$ with $x \in B'$. Since $f$ is continuous it follows that $f^{-1}(B') = \text{Domain}(f) \in \mathcal{B}$. Therefore $f = (x,B)$ where $B = \text{Domain}(f)$.

The following is immediate.

COROLLARY 2.6. $f:V \to V$ *has rank 1 if and only if* $\bar{f}:\mathcal{B} \to \mathcal{B}$ *has rank 1.*

Although two nonisomorphic block designs can have isomorphic automorphism groups, the same is not true for the monoid of continuous maps.

COROLLARY 2.7. *Let* $D$ *and* $D'$ *be block designs. Then* $D$ *is isomorphic to* $D'$ *if and only if* $\Omega(D)$ *is isomorphic to* $\Omega(D')$.

*Proof.* Clearly if $D$ is isomorphic to $D'$, then $\Omega(D)$ is isomorphic to $\Omega(D')$. Conversely, let $\theta:\Omega(D) \to \Omega(D')$ be an isomorphism. It is easy to see that $f \in \Omega(D)$ has rank 1 if and only if $f\theta \in \Omega(D')$ has rank 1. Since the rank 1 maps determine the blocks of the design the result follows.

We now relate the notion of continuous map to properties of the incidence matrix of a block design $D$. Recall that the incidence matrix of $D$ is the $v$ by $b$ matrix $A$ indexed by point and blocks defined by

$$A(x_i,B_j) = \begin{cases} 1 & \text{if } x_i \in B_j \\ 0 & \text{if } x_i \notin B_j \end{cases}$$

A key property of A is the following.

LEMMA 2.8. *Let* $A$ *be the incidence matrix of a* $(v,k,\lambda)$ *design. Then* $AA^t$ *is invertible. In particular* $\mathrm{rank}(A) = v$.

*Proof.* See [4], chapter 10.

A matrix C over $\{0,1\}$ is row (column) monomial if each row (column) of C contains at most one nonzero entry. Let Q be a set and let $f:Q \to Q$ be a partial function acting on the left of Q. Define the cardQ × cardQ matrix $C_f$ by

$$C_f(p,q) = \begin{cases} 1 & \text{if } f(q) = p, \quad \text{for all } p,q \in Q. \\ 0 & \text{otherwise.} \end{cases}$$

Then $C_f$ is a column monomial matrix and the assignment $f \to C_f$ is an isomorphism between the monoid of partial functions acting on the left of Q and the monoid of cardQ × cardQ column monomial matrices. Furthermore, $\mathrm{rank} f = \mathrm{rank} C_f$. There is a dual relationship between functions acting on the right of Q and cardQ × cardQ row monomial matrices.

LEMMA 2.9. *Let* $D = (V,\mathcal{B})$ *be a block design with incidence matrix* $A$. *Then* $f:V \to V$ *is continuous if and only if there is a* $b \times b$ *row monomial matrix* $R$ *such that* $RA^t = A^tC_f$. *In this case* $R = R_{\bar{f}}$.

*Proof.* Let $f:V \to V$ and let $C_f$ be the corresponding column monomial matrix. Then direct matrix multiplication gives

$$A^tC_f(B,x) = \begin{cases} 1 & \text{if } f(x) \in B \\ 0 & \text{otherwise} \end{cases}$$

for all $B \in \mathcal{B}$, $x \in V$. Thus row B of $A^tC_f$ is the characteristic vector of $f^{-1}(B)$.

Similarly if $R_g$ is the row monomial matrix corresponding to some $g:\mathcal{B} \to \mathcal{B}$, then

$$R_g A^t(B,x) = \begin{cases} 1 & \text{if } x \in (B)g \\ 0 & \text{otherwise.} \end{cases}$$

Therefore row $B$ of $R_g A^t$ is the characteristic vector of $(B)g$. It follows that if $f:V \to V$ is continuous, then $A^t C_f = R_{\bar{f}} A^t$. Conversely, if $A^t C_f = R_g A^t$ for some $g:\mathcal{B} \to \mathcal{B}$, then the above calculation shows that $x \in f^{-1}(B) \Leftrightarrow x \in (B)g$. Thus if $f^{-1}(B) \neq \emptyset$, then $f^{-1}(B) = (B)g \in \mathcal{B}$ and $f$ is continuous.

COROLLARY 2.10. *If* $f:V \to V$ *is continuous on* $D$, *then*

$$rank(f) \le rank(\bar{f}).$$

*Proof.* By Lemma 2.9 $A^t C_f = R_{\bar{f}} A^t$. Since $AA^t$ is invertible by Lemma 2.8 we have

$$C_f = (AA^t)^{-1} A \, R_{\bar{f}} A^t$$

and thus

$$\text{rank} f = \text{rank} C_f \le \text{rank} R_{\bar{f}} = \text{rank} \bar{f} \, .$$

If $f = 1_V : V \to V$, then $\bar{f} = 1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ and thus $v = \text{rank} 1_V \le \text{rank} 1_{\mathcal{B}} =$ This is just Fisher's inequality. For symmetric designs $(v = b)$ we have even more. Let $D$ be a symmetric design and let $D'$ be the dual design.

COROLLARY 2.11. *Let* $D$ *be a symmetric design. Then* $f \in \Omega(D)$ *implies* $\bar{f} \in \Omega(D')$. *Furthermore* $rank f = rank \bar{f}$.

Let $D = (V,\mathcal{B})$ be a design. We remark that the monoid $\Omega(D)$ is a familiar object in the theory of semigroups.

By Lemma 2.5 the continuous maps of rank $\le 1$ in $\Omega(D)$ form a semigroup on the set $S(D) = (V \times \mathcal{B}) \cup \{0\}$ with product:

$$(x,B)(x',B') = \begin{cases} (x,B') & \text{if } x' \in B \\ 0 & \text{otherwise} \end{cases}$$

and

$$0 \cdot (x,B) = (x,B) \cdot 0 = 0 \cdot 0 = 0.$$

Then $S(D)$ is a completely 0-simple semigroup and Lemma 2.9 shows that $\Omega(D)$ is the translational hull of $S(D)$. See [1,6,8] for details. We will explore $\Omega(D)$ from this point of view in a future paper.

## *III A Bound on the Parameters.*

In the remainder of this paper we assume that $D = (V,\mathcal{B})$ is a BIBD with parameters $(v,b,r,k,\lambda)$ and that $f:V \to V$ is a continuous map on $D$. The main theorem of this section shows that if $r > \lambda^2$, then $\mathrm{rank}(f) \leq 1$ or $f$ is a permutation. We need two preliminary lemmata.

LEMMA 3.1. *If $f^{-1}(x) = B$ for some $B \in \mathcal{B}$ and $x \in V$, then $rank(f) = 1$.*

*Proof.* Assume that $\mathrm{rank}(f) > 1$ and let $w \neq x$ be such that $\{x,w\} \subseteq \mathrm{Im}(f)$. Let $b_1$ be a block such that $\{x,w\} \subseteq b_1$. Then

$$f^{-1}(x) \cup f^{-1}(w) = B \cup f^{-1}(w) \subseteq f^{-1}(b_1).$$

Since $\mathrm{card}(B) = k$ and $\mathrm{card} f^{-1}(w) > 0$, it follows that $\mathrm{card} f^{-1}(b_1) > k$ and thus $f^{-1}(b_1) \notin \mathcal{B}$. But $f$ is continuous so this is a contradiction.

Recall that $\bar{f}:\mathcal{B} \to \mathcal{B}$ is the map linked with $f$.

LEMMA 3.2. *If there is a block $B$ such that $card(B\bar{f}^{-1}) > \lambda$, then $rank(\bar{f}) = rank(f) = 1$.*

*Proof.* Assume that $\{B_0,\ldots,B_\lambda\} \subseteq B\bar{f}^{-1}$ for $B \in \mathcal{B}$, $0 \leq i \leq \lambda$. Thus $B_i\bar{f} = f^{-1}(B_i) = B$ and therefore $\mathrm{Im}(f) \cap B_i = \mathrm{Im}(f) \cap B_j \neq \emptyset$ for $0 \leq i,j \leq \lambda$. Furthermore $\mathrm{card}(\mathrm{Im}(f) \cap B_i) = 1$, for otherwise there would be $x,w \in V$ with $x \neq w$ and $\{x,w\} \subseteq B_i$ for all $0 \leq i \leq \lambda$, a contradiction. It follows that $B = f^{-1}(x)$ for some $x \in V$ and thus $\mathrm{rank}(f) = 1$ by Lemma 3.1. Finally $\mathrm{rank}(\bar{f}) = 1$ by Corollary 2.6.

THEOREM 3.3. *If* $r > \lambda^2$, *then* $f \in Aut(D)$ *or* $rank(f) \leq 1$.

*Proof.* Assume that $rank(f) > 1$. By Corollary 2.4, we need only show that $f$ is injective. Recall that if $x \in V$, then $P(x) = \{B \in \mathcal{B} | x \in B\}$ and that $P(x)\bar{f}^{-1} = P(f(x))$ for all $x \in Domain(f)$. Let $x, w \in Domain(f)$ be such that $f(x) = f(w)$. It follows by the above, that $P(x)\bar{f}^{-1} = P(w)\bar{f}^{-1}$ By Lemma 3.2, $card(B)\bar{f}^{-1} \leq \lambda$ for each $b \in \mathcal{B}$. Since $r > \lambda^2$, it follows from Corollary 2.2 that $card(P(x) \cap P(w)) > \lambda$. But $D$ is a block design so $x = w$ and $f$ is injective.

We remark that the following dual result also holds. Let $\mu = \max\{card(B \cap B') | B \neq B' \in \mathcal{B}\}$. If $k > \mu^2$, then $f \in Aut(D)$ or $rank(f) \leq 1$. These bounds are the best possible, since Example 1.1 presents a continuous map on the $(7,4,2)$ design which is neither rank 1 nor in the automorphism group of the design. Here $r = 4 = \lambda^2$ and $k = 4 = \mu^2$.

We note that when $r > \lambda^2$, and in particular when $\lambda = 1$, Theorem 3.3 reduces the computation of $\Omega(D)$ to the computation of $Aut(D)$ and the blocks of $D$.

## *IV The Homogeneous Lemma and Its Consequences.*

Let $D = (V, \mathcal{B})$ be a design and let $f: V \to V$ be a continuous map. In this section we show that there is a natural number $d$ such that $card(f^{-1}(x)) = d$ for all $x$ in the image of $f$. As a consequence, we will show that the image of a design under a continuous map is a design. We will also see that the parameters of the image design can be determined from the parameters of $D$ and the integer $d$.

LEMMA 4.1. *(Homogeneous lemma) Let* $f: V \to V$ *be continuous. Then there is a natural number* $d$ *such that* $card(f^{-1}(x)) = d$ *for all* $x \in f(V)$. *Moreover,* $r(k-d) = \lambda(m-d)$ *where* $m = card(Domain(f))$.

*Proof.* Let $A(V)$ be the abelian group freely generated by $V$. If $x \in V$, let $d_x = card(f^{-1}(x))$. Let $F: A(V) \to Z$ be the morphism such that $F(x) = d_x$ for each $x \in V$. If $B \in \mathcal{B}$, let $\bar{B} = \sum_{x \in B} x$ and let $\bar{V} = \sum_{x \varepsilon V} x$. Note that $F(\bar{V}) = m$, where $m = \text{card}(\text{Domain}(f))$. Furthermore

$$(1) \quad F(\bar{B}) = \begin{cases} k & \text{if } f^{-1}(B) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

since $f$ is continuous.

Now let $x \varepsilon f(V)$. It follows from the fact that $D$ is a design, that

$$\sum_{x \varepsilon B} \bar{B} = rx + \lambda(\bar{V}-x) = (r-\lambda)x + \lambda\bar{V}$$

Therefore,

$$F(\sum_{v \varepsilon b} \bar{B}) = (r-\lambda)F(x) + \lambda F(\bar{V}) = (r-\lambda)d_x + \lambda m.$$

On the otherhand, it follows from (1), that

$$F(\sum_{x \varepsilon B} \bar{B}) = \sum_{x \varepsilon B} F(\bar{B}) = \sum_{x \varepsilon B} k = rk.$$

Therefore,

$$(r-\lambda)d_x + \lambda m = rk$$

and thus,

$$(2) \quad d_x = \frac{rk - \lambda m}{r - \lambda} .$$

Since the right hand side of (2) is independent of $x \varepsilon f(V)$ we have $d = \frac{rk - \lambda m}{r - \lambda} = \text{card}(f^{-1}(x))$ for all $x \varepsilon f(V)$.

We call d the degree of the continuous map $f$ and write $d = \deg(f)$.

COROLLARY 4.2. *Let $f:V \to V$ be continuous on $D = (V,B)$. Then $\text{Domain}(f) = V$ if and only if $f \varepsilon \text{Aut}(D)$.*

*Proof.* If $f \varepsilon \text{Aut}(D)$, then certainly $\text{Domain}(f) = V$. Conversely, assume $\text{Domain}(f) = V$. It suffices by Corollary 2.4 to show that $\deg(f) = 1$. By Lemma 4.1,

$$\deg(f) = \frac{rk - \lambda v}{r - \lambda}.$$

Since $D$ is a design, $r(k-1) = \lambda(v-1)$ and it follows that $rk - \lambda v = r -\lambda$
Therefore, $\deg(f) = 1$.

COROLLARY 4.3. *Let* $f \in \Omega(D) - Aut(D)$. *Then* $rank(f) < \frac{v}{2}$.

*Proof.* If $f \in \Omega(D) - Aut(D)$, then $f$ is not injective by Corollary 2.4 and thus $\deg(f) \geq 2$. But clearly, $m = card(Domain(f)) = \deg(f) \cdot rank(f)$. Therefore, $rank(f) \leq \frac{m}{2} < \frac{v}{2}$.

We will present an example of a design $D = (V,\mathcal{B})$ and a continuous map $f:V \to V$ with $rank(f) = \frac{v-1}{2}$ in Section 5.

Let $\mathcal{B}_f$ be the collection of all $B_f = B \cap f(V)$ for all $B \in \mathcal{B}$ with $B \cap f(V) \neq \emptyset$. Let $f(D) = (f(V),\mathcal{B}_f)$, then

THEOREM 4.4. *$f(D)$ is a (possibly complete) block design.*

*Proof.* Recall that $D = (V,\mathcal{B})$ is a $(v,b,r,k,\lambda)$ BIBD. We will show that $f(D)$ is a $(v',b',r,k',\lambda)$ design where $v' = card(f(V))$, $b' = card\mathcal{B}_f$ and $k' = \frac{k}{d}$ where $d = \deg(f) = card(f^{-1}(x))$ for all $x \in f(V)$.

Clearly $v' = card f(V)$ and $b' = card\mathcal{B}_f$. For all $x \in f(V)$ if $x \in B$, then $x \in B_f = B \cap f(V) \in \mathcal{B}_f$. Thus, each point of $f(V)$ is on exactly $r$ blocks of $\mathcal{B}_f$. Similarly, if $\{x,y\} \subseteq f(V)$, then there are still $\lambda$ blocks of $\mathcal{B}_f$ which contain $\{x,y\}$.

Since $card(f^{-1}(B_f)) = k$ and since $card(f^{-1}(x)) = d$ for all $x \in B_f$, then $card B_f = \frac{k}{d}$, as required. We call $f(D)$ the *image design under* $f$.

In [7], E. Morgan defines an n-arc in a block design $D = (V,\mathcal{B})$ as a subset $W \subseteq V$ such that every block intersects $W$ in either 0 or $n$ points. The following corollary is immediate.

COROLLARY 4.5. *Let* $(V,\mathcal{B})$ *be a* $(v,k,\lambda)$ *design and let* $f:V \to V$ *be a continuous map. Then* $f(V)$ *is a* $k'$*-arc where* $k' = \frac{k}{d}$ *and* $d = \deg(f)$.

As a consequence of the two previous results it is possible to obtain many constraints on the parameters of both D and f(D).

THEOREM 4.6. *Let* $D = (V,B)$ *and* $f(D) = (f(V),B_f)$ *have parameters* $(v,b,r,k,\lambda)$ *and* $(v',b',r,k',\lambda)$, *respectively. Then*

(i) $\frac{v'-1}{k'-1} = \frac{v-1}{k-1} = \frac{r}{\lambda}$

(ii) $v' = \frac{r(k'-1)}{\lambda} + 1$

(iii) $k'$ *divides* $k$

(iv) *If* $k' < k$, *then* $k'$ *divides* $r - \lambda$.

In this case if $tk' = r - \lambda$, then

a) $\lambda k$ divides $rt(k-k')$,
b) $\lambda$ divides $r(k'-1)$,
c) $\lambda$ divides $r(k-k')$,
d) $\lambda$ divides $r(r-t)$.

*Proof.* (i) Follows since both D and f(D) are block designs with the same r and $\lambda$.

(ii) Follows from part (i).

(iii) From Theorem 4.4, $k' = \frac{k}{d}$.

(iv) [7], Corollary 3.2.

As a result of the above it is seen that the possible parameters for the image of a design compose a fairly small set. For example, if D is a (22,8,4) design and $1 < \mathrm{rank}(f) < 22$, then the only possible parameters for f(D) are (10,4,4) and (4,2,4). If D is a symmetric design we also have:

COROLLARY 4.7. *Let* D *be a symmetric design. Then* $\deg(f) = \deg(\bar{f})$. *That is for each* $B \in B_f$ *and each* $x \in f(V)$, $\mathrm{card}(f^{-1}(x)) = \mathrm{card}(B\bar{f}^{-1})$.

*Proof.* We have seen in Corollary 2.12, that $\bar{f}$ is continuous with respect to the dual design and that $\mathrm{rank} f = \mathrm{rank}\bar{f}$. It follows that the parameters of $f(D)$ are the same as the parameters of $\bar{f}(D')$. In particular $\frac{k}{\deg(f)} = \frac{k}{\deg(\bar{f})}$ and the result follows.

We conjecture that for any design $D = (V,\mathcal{B})$ and continuous map $f:V \to V$, there exists an integer $c$ such that $c = \mathrm{card}(B\bar{f}^{-1})$ for all $B \in \mathcal{B}\bar{f}$.

COROLLARY 4.8. *If* $k$ *is prime and* $f \in \Omega(D)$, *then* $f \in Aut(D)$ *or rank* $f \leq 1$.

*Proof.* Since $\deg(f)$ divides $k$ the result follows.

COROLLARY 4.9. *If* $k$ *and* $(r-\lambda)$ *are relatively prime, then* $f \in \Omega(D)$ *implies* $f \in Aut(D)$ *or* $rank(f) \leq 1$.

*Proof.* If $\mathrm{rank}(f) < v$, then $\deg(f) > 1$. It follows from Corollary 4.5 and Theorem 4.6 that $k'$ divides $k$ and $k'$ divides $r - \lambda$. Therefore $k' = 1$ and $\deg(f) = k$. Thus $\mathrm{rank}(f) = 1$.

## V *Some Examples.*

In this section we develop three different infinite classes of designs with continuous maps $f$, and $1 < \mathrm{card} f(V) < v$. The first class of designs come from the projective planes, the second from Hadamard designs, and the third from a type of tensor product construction.

For the first example, let $P = (V,\mathcal{B})$ be an $(n^2 + n + 1, n + 1, 1)$ design (a projective plane of order $n$). Let $P^c = (V,\mathcal{B}^c)$ be the complement, a $(n^2 + n + 1, n^2, n^2 - n)$ design. Let $\ell = \{v_0, v_1, \ldots, v_n\}$ be any line in $D$ and let $\ell_0, \ell_1, \ell_2, \ldots, \ell_n$ be all the lines on $y$ for some $y \in V$. Define a map $f:V \to V$ by $f(x) = v_i$ when $x \in \ell_i \backslash \{v\}$. That is, map all the points on the $i^{th}$ line, except $y$, to the point $v_i$. $f$ is well defined since $i \neq j$ implies $\ell_i \backslash \{y\} \cap \ell_j \backslash \{y\} = \emptyset$. Note that $\mathrm{card}(\mathrm{Domain}(f)) = n^2 + n$, as large as possible for $f$ to be continuous on $P^c$ but $f \notin Aut(P^c)$ by Corollary 4.2. For example, in Example 1.1 we have $\ell = \{0,5,4\}$, $y = 0$, $\ell_0 = \{0,1,3\}$, $\ell_1 = \{0,2,6\}$, $\ell_2 = \{0,4,5\}$.

PROPOSITION 5.1. *There exists a continuous map of rank $n + 1$ on the complement of the projective plane of order $n$.*

*Proof.* We claim that the map f defined above is a continuous map. We must show that if $B^c$ is a block in $P^c$, then $f^{-1}(B^c) \neq \emptyset$ implies $f^{-1}(B^c)$ is also a block. If $B = \ell$, then $f^{-1}(B^c) = \emptyset$ since $f(V) = \ell$. Assume then, that $B \neq \ell$. Then $B^c \cap f(V) = B^c \cap \ell = \ell - \{v_i\}$ for some $i$, $0 \leq i \leq n$. Therefore, $f^{-1}(B^c) = f^{-1}(\ell - \{v_i\}) = \ell_i^c \in B^c$. Therefore, f is continuous. Clearly $\text{rank}(f) = n + 1$.

We now show that these maps are the only continuous maps of rank $n + 1$ on the complement of the projective plane of order $n$. We first recall that a partial function $f:V \to V$ gives an equivalence relation <u>modf</u> on Domain (f) defined by $x(\text{modf})y$ if and only if $f(x) = f(y)$.

PROPOSITION 5.2. *Let $P^c$ be the complement projective plane of order $n$. Then a continuous map $f$ on $P^c$ has rank $n + 1$ iff the range of $f$ is a line in $P$ and modf is a parallel class in $A$, the affine plane obtained from $P$.*

*Proof.* ($\Leftarrow$) Proposition 5.1

($\Rightarrow$) Let $C = \{f \in \Omega(D) \mid \text{range}(f)$ is a line in $P$ and modf is a parallel class in $A\}$. Let $f \in \Omega(D)$ have $\text{rank}(f) = n + 1$ and assume that $f \notin C$.

First assume that range(f) is not a line in P. Let $g \in C$. It follows from Theorem 4.6 that $\text{rank}(gf) = 0, 1$, or $n + 1$. If $\text{rank}(gf) = 0$, then $\text{range}(f) \cap \text{Domain}(g) = \emptyset$. But $\text{card}(\text{Domain}(g)) = n^2+n$, and this implies that $\text{rank}(f) = 1$, a contradiction.

If $\text{rank}(gf) = 1$, then range(f) intersects exactly one equivalence class of modg. Since every equivalence class of mod(g) is a line in A and $\text{Domain}(g) = V\setminus\{x\}$ for some $x \in V$, it follows that $\text{range}(f) = \ell\setminus\{x\}$, where $\ell$ is a line in A not containing x. But modg is a parallel class in A and thus $\ell\setminus\{x\}$ is a line in P, a contradiction to the assumption on range(f).

We may therefore assume that rank(gf) = n + 1 for all $g \in C$. This implies that range(f) forms a system of distinct representatives of modg for every $g \in C$. Therefore range(f) is a system of distinct representatives for every parallel class in A, a contradiction.

We have shown that range(f) must be a line in P.

By Corollary 4.5 if f(D) has parameters $(v',k',\lambda)$ then $\frac{v'-1}{k'-1} = \frac{v-1}{k-1} = \frac{n^2+n}{n^2-1}$. By assumption $v' = n+1$, so $k' = n$ and $\frac{k}{k'} = n = \deg(f)$. So by the Homogeneous lemma, $\mathrm{card}(f^{-1}(x)) = n$ for all $x \in f(V)$. Therefore $\mathrm{card}(\mathrm{Domain}(f)) = n(n+1) = n^2+n$. So let $\mathrm{Domain}(f) = V\backslash\{v_0\}$, and let $\ell_0,\ell_1,\ldots,\ell_n$ be the lines on $v_0$ in P. Then f continuous implies $\{\ell_i\backslash\{v_0\} \mid 0 \le i \le n\}$ is the partition of Domain(f) induced by f. For let $\ell$ be the image of f and $\ell' \neq \ell$ be any other line in P. If $\ell' \cap \ell = x$, then $f^{-1}(\ell'^{c}) = f^{-1}(\ell\backslash\{x\})$. Since this must be a block in D we see that $f^{-1}(x)$ must be $\ell_i\backslash\{v_0\}$ for some $1 \le i \le n$. It follows that $\{\ell_i\backslash\{v_0\} \mid 0 \le i \le n\}$ is the partition of Domain (f) induced by f. Therefore $f \in C$.

We now show that if a map f is continuous with respect to $P^c$ then $f \in \mathrm{Aut}(D)$, $\mathrm{rank}(f) \le 1$, or $\mathrm{rank}(f) = n+1$. Together with Proposition 5.2 this gives a complete classification of continuous maps of $P^c$.

PROPOSITION 5.3. *If $D = P^c$ is the complement of the projective plane of order $n$, and $f \in \Omega(D)$, then $\mathrm{rank}(f) = 0,1,n+1$, or $n^2+n+1$.*

*Proof.* Assume $f \in \Omega(D)$ and $1 < \mathrm{rank}(f) < n^2+n+1$. Let $d = \deg(f)$. Let f(D) have parameters $(v',k',\lambda)$. Since D has parameters $(n^2+n+1, n^2, n^2-n)$ it follows from Theorem 4.6, that $k' = \frac{k}{d}$ divides n. Furthermore, $\frac{v'-1}{k'-1} = \frac{n^2+n}{n^2-1} = \frac{1}{n-1}$. It follows that if $n = tk'$, then $tv' - t = \frac{(n-t)n}{n-1}$. Therefore $n-1$ divides $n-t$ and thus $t = 1$. This implies that $k' = n$ and finally that $v' = n+1$.

If $D = (V,\mathcal{B})$ is a design, let $\rho(D) = \min\{n \mid n > 1$ and $n = \text{rank}(f)$ for some $f \in \Omega(D)\}$. We note that $\rho(D) > 2$. For suppose there is a map $f \in \Omega(D)$ with range$(f) = \{x,w\}$, $x \neq w \in V$, choose blocks $b_1, b_2 \in \mathcal{B}$ with $\{x,w\} \subseteq b_1$ and $b_2 \cap \{x,w\} = \{x\}$. Since $f^{-1}(b_1) =$ Domain$(f)$, it follows that card(Domain$(f)) = k$. But then $f^{-1}(b_2)$ is a proper nonempty subset of Domain$(f)$ and thus $f$ is not continuous.

Example 1.1 gives a design $D$ with $\rho(D) = 3$. More generally, assume $D$ is a $(v,k,\lambda)$ design with $\rho(D) = 3$. Let $f \in \Omega(D)$ have rank$(f) = 3$ and let $d = \deg(f)$. It follows that the image design $f(D)$ has parameters $(3,2,\lambda)$. Therefore, by Theorem 4.6, $\frac{r}{\lambda} = \frac{v'-1}{k'-1} = 2$. Therefore, $r = 2\lambda$. Furthermore, $k = \frac{v+1}{2}$ is even. Also $k' = 2$ divides $r - \lambda = \lambda$ and $\lambda$ is even.

We have shown that $D$ has parameters $(v, \frac{4v\lambda}{v+1}, 2\lambda, \frac{v+1}{2}, \lambda)$. If $D$ is symmetric, then $2\lambda = \frac{v+1}{2}$ and thus $v = 4\lambda - 1$. We are thus lead to study $(4\lambda-1, 2\lambda, \lambda)$ designs with $\lambda$ even. These are, of course, the parameters of the complement of a Hadamard design.

Recall that a Hadamard matrix $H$ of order $n$ is an $n \times n$ array of 1's and -1's such that $HH^T = nI$. A necessary condition for the existence of a Hadamard matrix of order $n$ is that $n = 1,2,$ or $4t$ for $t$ a natural number. It is still unknown (but strongly conjectured) that Hadamard matrices exist for all possible orders [10]. The existence of a Hadamard matrix of order $4n$ is equivalent to the existence of a $(4n-1, 2n-1, n-1)$-symmetric BIBD (called a Hadamard design of order $4n-1$). The complement of a Hadamard design or order $4n-1$ is a $(4n-1, 2n, n)$-symmetric BIBD.

We will need the following lemma.

LEMMA 5.4. *Let $D = (V,\mathcal{B})$ be a $(4n-1, 2n, n)$ BIBD. Then if there are three points $\{x_0, x_1, x_2\} \subseteq V$ such that for any block $B \in \mathcal{B}$ card$(B \cap \{x_0,x_1,x_2\}) = 0$ or $2$, and dually, if there are three lines $\{B_0, B_1, B_2\} \subseteq \mathcal{B}$ such that for any point $x \in V$ card$(P(x) \cap \{B_0,B_1,B_2\}) = 0$ or $2$, where $P(X) = \{B \in \mathcal{B} \mid x \in b\}$, then there exists a continuous map $f$ on $D$ with rank$(f) = 3$.*

*Proof*. Define a map $f : V \to V$ by

$$f(x) = \begin{cases} x_0 & \text{if } x \in B_0 \cap B_1 \\ x_1 & \text{if } x \in B_1 \cap B_2 \\ x_2 & \text{if } x \in B_0 \cap B_2 \\ \text{undefined} & \text{if } x \notin B_1 \cup B_2 \cup B_3 \end{cases}$$

It is easily checked that $f$ is continuous and $\text{rank}(f) = 3$.

PROPOSITION 5.5. *If there is a Hadamard matrix of order $n$, then there is a $(4n-1, 2n, n)$ design $D$, with a continuous map $f$ with $\text{rank}(f) = 3$.*

*Proof*. Let $H_n$ be a standardized Hadamard matrix of order $n$. Thus the first row and column of $H$ are composed entirely of $+1$'s. Let $H_{4n}$ be the $4n \times 4n$ matrix

$$H_{4n} = \begin{bmatrix} H_n & H_n & H_n & H_n \\ H_n & H_n & -H_n & -H_n \\ H_n & -H_n & H_n & -H_n \\ H_n & -H_n & -H_n & H_n \end{bmatrix}$$

which is the tensor product of $H_n$ and $H_4$, a Hadamard matrix of order 4. It is well known that $H_{4n}$ is also Hadamard.

From $H_{4n}$ construct the $4n-1 \times 4n-1$ incidence matrix $A$ of a $(4n-1, 2n, n)$ design by deleting the first row and column and changing all 1's to 0's and changing all $-1$'s to 1's. Now let $x_i$, $i = 0,1,2$ be the point corresponding to row $(i+1)n$ of the incidence matrix $A$. Analogously, let $B_i$, $i = 0,1,2$, be the block corresponding to column $(i+1)n$ of $A$.

It is easily seen that $\{x_0, x_1, x_2\}$ and $\{B_0, B_1, B_2\}$ satisfy the conditions of Lemma 5.4. Thus the result follows.

THEOREM 5.6. *Let* $n \geq 3$. *Then there is a design* $D$ *with parameters* $(2^n-1, 2^{n-1}, 2^{n-2})$ *such that* $D$ *has a continuous map of rank* $2^m-1$ *for* $1 \leq m \leq n$.

*Proof.* This is true for $n = 3$ by Example 1.1.

We assume there exists a design $D = (V, \mathcal{B})$ with parameters $(2^n-1, 2^{n-1}, 2^{n-2})$ such that there exists a continuous map of rank $2^m-1$ for $1 \leq m \leq n$. Let $\Delta = 2^n-1$ and let $V = \{v_1, v_2, \ldots, v_\Delta\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_\Delta\}$.

Construct a new design $D' = (V', \mathcal{B}')$ as follows. Let $V' = \{v_1', v_2', \ldots, v_\Delta', x_0, x_1, \ldots, x_\Delta\}$ and let $\mathcal{B}' = \{b_1', b_2', \ldots, b_\Delta', c_0, c_1, \ldots, c_\Delta\}$ where $b_i' = \{v_j' | v_j \in b_i\} \cup \{x_j | v_j \in b_i\}$ and $c_0 = \{x_0, x_1, \ldots, x_\Delta\}$ and $c_j = \{v_j' | v_j \in b_i\} \cup \{x_j | v_j \notin b_i\} \cup \{x_0\}$. It is easy to check that $D'$ is a $(2^{n+1}-1, 2^n, 2^{n-1})$ design and indeed this construction corresponds to the tensor product of $D$ with $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

If $f: V \to V$ was continuous on $D$, then define $f': V' \to V'$ by $f'(v_i') = (f(v_i))'$ and $f'(x_i) = (f(v_i))'$ and $f'(x_0) =$ undefined. Then $f'$ is continuous on $D'$ and $\mathrm{rank}(f') = \mathrm{rank} f$.

Thus there are continuous maps on $D'$ of rank $2^m-1$ for $1 \leq m \leq n$. Taking the identity map on $D'$ completes the induction.

For our final example we need the following object. A *weighing matrix* $W$ of order $n$ and weight $z$ is an $n$ by $n$ matrix with entries in $\{-1, 0, 1\}$ such that $WW^t = zI_n$. We will also say that $W$ has type $(n, z)$. We will always assume that the first row and the last column of $A$ has entries in $\{0, 1\}$.

If $W$ is a weighing matrix, let $|W|$ be the matrix obtained by changing all $-1$ entries in $W$ to $+1$. The following is our general construction.

Let $A$ be the $2k$ by $4k-2$ incidence matrix of a $(2k, k, k-1)$-BIBD and let $W$ be a weighing matrix of order $n$ and weight $z$. The $2kn$ by $(4k-2)n$ matrix $W \otimes A$ is obtained by replacing each occurence of $1$ in $W$ by $A$, each occurence of $-1$ in $W$ by $J - A$, and each occurence of $0$ in $W$ by the $2k$ by

$4k-2$ zero-matrix. Here, as usual, $J$ denotes the matrix all of whose entries equal 1. Note that $J-A$ is the incidence matrix of $A^c$ which is also a $(2k,k,k-1)$-design.

THEOREM 5.7. *If* $W$ *is a weighing matrix,* $|W|$ *is the incidence matrix of a (possibly complete)* $(n,z,\lambda)$*-design, and* $\lambda(2k-1) = 2z(k-1)$*, then* $W \otimes A$ *is the incidence matrix of a design* $D$ *with parameters* $(2kn, (4k-2)n, (2k-1)z, kz, (k-1)z)$*. Furthermore, there is a continuous map* $f$ *on* $D$ *of rank* $2k$ *and* $f(D)$ *is* $z$ *copies of the design of* $A$*.*

*Proof*. It is easy to check that every row of $W \otimes A$ has sum $(2k-1)z$ and each column of $W \otimes A$ has sum $kz$. We need only check that the inner product of any two distinct rows of $W \otimes A$ equals $(k-1)z$. We index rows of $W \otimes A$ by a pair $(i,j)$ where $1 \le i \le n$ and $1 \le j \le 2k$. Let $(i,j)$ and $(i',j')$ be distinct rows. We have three cases.

CASE 1. $i = i'$.

Since both $A$ and $A^c$ are $(2k,k,k-1)$ designs and every row in $W$ has $z$ non-zero entries, it follows that the inner product of $(i,j)$ and $(i,j')$ is $z(k-1)$.

CASE 2. $j = j'$.

Since $W$ is a weighing matrix and $|W|$ is the incidence matrix of an $(n,z,\lambda)$ design, it follows that any two distinct rows of $W$ have the same non-zero entry in exactly $\frac{\lambda}{2}$ columns. But row $j$ of $A$ inner product row $j$ of $A^c$ is 0. Therefore, the inner product of row $(i,j)$ and row $(i',j)$ is $\frac{\lambda}{2}(2k-1)$ which equals $z(k-1)$ by hypothesis.

CASE 3. $i \ne i'$ and $j \ne j'$.

Let $c$ be a column of $W$. If $W(i,c)\cdot W(i',c) = 1$, then in the columns corresponding to $c$ in $W \otimes A$, rows $i$ and $i'$ are replaced by the same matrix. Since $j \ne j'$, it follows that such a column contributes $k-1$ to the inner product of $(i,j)$ and $(i',j')$.

If $W(i,c)\cdot W(i',c) = -1$, then one of rows $i$ and $i'$ is replaced by $A$ and the other row by $J - A$. But the inner product of distinct rows in $A$ and $J - A$ is $k$. Therefore each such column contributes $k$ to the inner product of $(i,j)$ and $(i',j')$.

If $W(i,c)\cdot W(i',c) = 0$, it is clear that the columns corresponding to $c$ in $W \otimes A$ contribute $0$ to the inner product of $(i,j)$ and $(i',j')$.

Finally, the hypotheses on $W$ imply that there are $\frac{\lambda}{2}$ columns $c$ such that $W(i,c)\cdot W(i',c) = 1$ and $\frac{\lambda}{2}$ columns $c$ with $W(i,c)\cdot W(i',c) = -1$. Therefore the inner product of $(i,j)$ and $(i',j')$ equals

$$\frac{\lambda}{2}(k-1) + \frac{\lambda}{2}(k) = \frac{\lambda}{2}(2k-1) = z\cdot(k-1)$$

by hypothesis.

Therefore, $W \otimes A$ is the incidence matrix of a $(2kn,(4k-2)n,(2k-1)z,kz,(k-1)z)$-design as claimed.

Let $V = \{(i,j)\,|\,1 \le i \le n,\ 1 \le j \le 2k\}$ be the points of the design $W \otimes A$. Define $f:V \to V$ by

$$f((i,j)) = \begin{cases} (1,j) & \text{if } W(i,n) = 1 \\ \text{undefined otherwise} & \end{cases}$$

From the hypothesis that row $1$ and column $n$ of $W$ have entries in $\{0,1\}$, it easily follows that $f$ is continuous and that $f(W \otimes A)$ is $z$ copies of $A$.

We give an example of the previous construction.

EXAMPLE 5.8.

Let $W = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. Then $W$ is a weighing matrix of type $(4,3)$ and $|W|$ is the incidence matrix of a $(4,3,2)$-design.

Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$. Then A is the incidence matrix of a (4,2,1)-design and

$$A^c = J - A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$ Therefore,

$$W \otimes A = \begin{pmatrix} A & A & A & 0 \\ A & A^c & 0 & A \\ A^c & 0 & A & A \\ 0 & A & A^c & A \end{pmatrix}.$$

Since $2 \cdot 3 = 2 \cdot 3 \cdot 1$, Theorem 4.7 implies that $W \otimes A$ is the incidence matrix of a (16,23,9,6,3)-design. We let the interested reader construct the continuous map of rank 4 given by Theorem 5.7.

COROLLARY 5.9. *If there is a* $(2k,k,k-2)$ *design and a weighing matrix of order* $2k$ *and weight* $2k-1$, *then there is a* $(4k^2, 2k^2-k, 2k^2-3k+1$ *design* $D$ *with a continuous map of rank* $2k$.

*Proof*. Apply Theorem 5.7 with $n = 2k$ and $z = 2k-1$. Note that necessarily $|W|$ is the incidence matrix of a (2k,2k-1,2k-2)-design. Also, since $\lambda = 2k-2$, the arithmetic condition is satisfied.

A few remarks are in order. First, a (2k,k,k-1) design can be obtained as the residual of a (4k-1,2k-1,k-1) Hadamard design. Thus, in practice the (2k,k,k-1) designs exist. Also, weighing matrices of type (2k,2k-1) are known as conference matrices and are also known for many orders. In Corollary 5.10 we give an infinite collection of numbers 2k for which both a (2k,k,k-1) design and a weighing matrix of type (2k,2k-1) exist.

We wish to point out the $(4k^2,2k^2-k,2k^2-3k+1)$ design constructed by use of Corollary 5.9 is a quasi-residual design (of an $(8k^2-4k+1, (2k-1)^2, 2k^2-3k+1)$-design). The (16,6,3)-design constructed in Example 5.8 is indeed the residual of a (25,9,3)-design. We conjecture that all these quasi-residual designs given by Corollary 5.9 are in fact residual designs. Some possible parameters arising from Corollary 5.9 are (16,6,3) when $k = 2$, (36,15,10) when $k = 3$ and (64,28,21) when $k = 4$.

COROLLARY 5.10. *If* $p^s \equiv 1(4)$ *is a prime power, then there is a* $((p^s+1)^2, \frac{1}{2}(p^s+1)(p^s), \frac{1}{2}(p^s+1)(p^s-2)+1)$ *design with a rank* $p^s+1$ *continuous map.*

*Proof.* There exists a weighing matrix of type $(p^s+1,p^s)$ and a Hadamard matrix of order $2(p^s+1)$ ([10], Corollary 8.6). Thus there is a $(p^s+1, \frac{p^s+1}{2}, \frac{p^s+1}{2} - 1)$ design and the result follows from Corollary 5.9 with $2k = p^s + 1$.

*VI Conclusion and Further Questions.*

In this paper we have studied when there is a partial function $f:V \to V$ on a BIBD $D = (V,\mathcal{B})$ with the property that the inverse image of every block is also a block, when nonempty. This is a natural generalization of the concept of an automorphism of a design.

Indeed, we have seen that under various conditions the only such maps are either automorphisms or have $\text{rank}(f) \leq 1$. On the other hand, the existence of a continuous map with $1 < \text{rank}(f) < v$ implies that the range of $f$ is an arc in $D$, and hence is also a design. This in turn leads to various arithmetic conditions on the parameters of $D$ and the rank of $f$. For example, we have seen in Section 5 that if there is a map of rank 3 on a symmetric design $D$, then $D$ is the complement of a Hadamard design.

An important question is the following. Given a continuous map $f:V \to V$, how can $D$ be retrieved from the design $f(D)$? The second and third examples in Section 5 suggest that $D$ can be constructed by tensor product type constructions from $f(D)$. We also remark that Denniston [2] has used the concept of arc to construct new design.

Let $f:V \to V$ be a continuous map such that the conclusion of the homogeneous lemma holds for the linked map $\bar{f}:\mathcal{B} \to \mathcal{B}$, (for example, if D is symmetric). One can then define a quotient structure on V in a natural way. Furthermore, f(D) is a multiple of the quotient. We ask if this is true in general.

In a future paper, we will study more fully the algebraic structure of the monoid $\Omega(D)$ and its relation with the combinatorial structure of D. Here we briefly ask what the subgroup structure of $\Omega(D)$ implies about D.

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ADDENDUM

In the paper "Translational hulls and block designs", (to appear Semigroup Forum) we discuss continuous maps on block designs from the point of view of the semigroup structure. Also, in the paper "Continuous maps in finite projective space (to appear Congressus Numerantium) we generalize Proposition 5.3 to projective spaces of higher dimension.