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THE EXISTENCE OF CERTAIN ROOM GRAPHS

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Abstract

Recently, Kramer, Kreher, Magliveras, and Mesner [4] introduced a new combinatorial object called a Room graph. These graphs present an interesting generalization of Room squares. However, their more general nature allows the existence of a Room graph of order 5 (on the Peterson graph!) whereas the corresponding Room square of order 5 does not exist. In this paper we show the existence of a certain class of Room graphs.

1. Introduction.

A Room square of order s is a square array R of side s , satisfying the following properties:

- (i) each cell of R is either empty or contains an unordered pair of elements (*symbols*) chosen from a set S of size $s + 1$,
- (ii) each symbol occurs precisely once in each row and column of R ,
- (iii) every unordered pair of symbols occurs in a unique cell of R .

The following is shown in [5].

THEOREM 1.1. *A Room square of side s exists if and only if s is an odd positive integer other than 3 or 5.*

Given a Room square R on symbol set S , one can construct the corresponding Room graph G as follows. Let $V(G)$ be the rows and columns of R (thus $|V(G)| = 2s$). Call the vertices $r_1, \dots, r_s, c_1, \dots, c_s$. Construct an edge from vertex r_i to vertex c_j if and only if cell (r_i, c_j) of R is non-empty. There are no edges from r_i to r_j or c_i to c_j . Now define a bijection between a set of unordered pairs of elements of S into $E(G)$ by

$\{x,y\} \rightarrow \text{edge } \{r_i, c_j\} \text{ if } R(r_i, c_j) = \{x,y\}.$

We isolate the properties of a Room graph. Formally, a *Room graph* RG of order s (or $RG(s)$) is a connected simple graph G with $2s$ vertices together with a mapping of the set of all unordered pairs of elements in $S(|S| = s + 1)$ onto the edges of G such that:

- (i) an edge of G holds exactly one unordered pair of elements in S ,
- (ii) the edges adjacent with each vertex hold each symbol in S exactly once.
- (iii) each unordered pair of elements in S appears once in G .

The above construction together with Theorem 1.1 and this definition give the following result.

THEOREM 1.2. *There exists a $RG(n)$ for all odd n , $n \neq 3, 5$.*

This definition is a special case of the more general Room graphs defined in [4]. As has previously been noted, the existence of a Room square of order s implies the existence of a Room graph of order s . However, the converse is not true. The following (first given in [4]) is an example of a Room graph of order 5 (which, of course, does not correspond to any Room square).

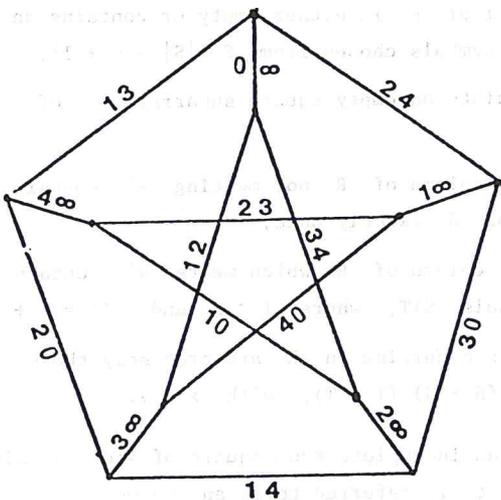


Figure 1. A Room graph of order 5

The above example suggests the following question, for what orders n can *nonbipartite* Room graphs of order n be constructed? It was conjectured in [4] that nonbipartite Room graphs of order n exist for all $n \geq 5$. In this paper we prove that this conjecture is true.

In the next section we will develop the two major constructions necessary to solve this problem. In Section 3, we use these constructions to obtain a preliminary bound, and in Section 4 we investigate smaller cases of n .

2. Useful Constructions.

Our main construction uses the following combinatorial object.

An *incomplete Room square* of side s missing a subsquare of side t is a square array R of side s which satisfies the following:

- (i) each cell of R is either empty or contains an unordered pair of symbols chosen from S ($|S| = s + 1$),
- (ii) there exists an empty square subarray R' of R having side t ,
- (iii) a row or column of R not meeting R' contains each symbol of S exactly once,
- (iv) a row or column of R which meets R' contains precisely the symbols $S \setminus T$, where $T \subseteq S$ and $|T| = t + 1$, and
- (v) the pairs occurring in R are precisely those $\{x, y\} \in (S \times S) \setminus (T \times T)$, with $x \neq y$.

For brevity, an incomplete Room square of side s missing a subsquare of side t is referred to as an *incomplete* (s, t) -Room square. Our first construction starts with an incomplete (s, t) -Room square and "fills in" a Room graph of order t to get a Room graph of order s .

THEOREM 2.1. *If there exists an incomplete (s, t) -Room square and an $RG(t)$, then there exists an $RG(s)$. Furthermore, if the $RG(t)$ is nonbipartite, then so is the $RG(s)$.*

Proof. From the incomplete (s, t) -Room square, construct an incomplete Room graph in the manner already described. Then, on the $2t$ vertices corresponding to the "missing" subsquare, put the $RG(t)$ on symbol set T . The resultant graph is an $RG(s)$. Since the $RG(t)$ is a subgraph of the $RG(s)$ if it is nonbipartite, then the whole graph is also nonbipartite.

Suppose G is a Room graph of order s . A subgraph of G is said to be a Room subgraph of order t provided it is itself a Room graph (of order t). Note that any Room graph has Room subgraphs of orders 1 and 0 (by convention).

The following theorem is a Moore-type construction and will be useful in constructing Room graphs of many small orders.

sub-RG(w), then if $(v - w) \neq 6$, there exists an $RG(u(v - w) + w)$.
 Furthermore, if the $RG(v)$ is nonbipartite, then so is the
 $RG(u(v - w) + w)$.

Proof. We will construct a Room graph G of order $u(v - w) + w$.

Let G be an $RG(u)$ on the symbols $U = \{1, 2, \dots, u, \infty\}$. There exists a perfect matching of the $2u$ vertices of G determined by those edges containing ∞ . Let $V(G) = \{s_1, s_2, \dots, s_u, s_{u+1}, \dots, s_{2u}\}$ where pair $\{\infty, i\}$ is on edge $\{s_i, s_{i+u}\}$, $1 \leq i \leq u$. Now, for each vertex s_i in $V(G)$ construct $v - w$ new vertices s_i^j , $1 \leq j \leq v - w$. Also, construct $2w$ new vertices $S_1, S_2, \dots, S_w, T_1, \dots, T_w$. Note that we have now constructed $2u(v - w) + 2w = 2(u(v - w) + w)$ total vertices the required number.

Let H be an $RG(v)$ on the symbol set $V = \{\alpha, 1, 2, \dots, v\}$ and H' be the sub $RG(w)$ on the symbol set $W = \{\alpha, v - w + 1, \dots, v\}$. Also, let R and L be two orthogonal Latin squares of order $v - w$ on the symbols of $V \setminus W$ and indexed by that set.

Now, let s_i and s_j be any two adjacent vertices in G with $i \neq j \pm u$ and with $\{a, b\}$ on edge $\{s_i, s_j\}$. Connect all vertices s_i^k to s_j^t with an edge for $1 \leq t, k \leq v - w$, and put the unordered pair $\{(R(k, t), a), (L(k, t), b)\} \subseteq V \setminus W \times (U \setminus \infty)$ on the edge connecting s_i^k to s_j^t .

For each i , $1 \leq i \leq u$, let H_i be the Room graph H with each symbol $a \in V \setminus W$ replaced by the symbol $(a, i) \in V \setminus W \times (U \setminus \infty)$, each symbol $a \in W$ left unchanged, and with the edges of the sub Room graph H' deleted. Now, on the vertices $s_i^j, s_{i+u}^k, S_1, \dots, S_w, T_1, \dots, T_w$, with $1 \leq i \leq u$, $1 \leq j, k \leq v - w$, put the Room graph H_i with the (deleted) sub-Room graph on the vertices $S_1, S_2, \dots, S_w, T_1, \dots, T_w$. Finally, on the vertices $S_1, \dots, S_w, T_1, \dots, T_w$ put the Room graph H' .

That the resultant graph G is indeed an $RG(u(v - w) + w)$ on the symbol set $(V \setminus W \times (U \setminus \infty)) \cup W$ is easily checked. Also, if H is nonbipartite, then so is G .

completely analogous to the Moore-type construction for Room squares (see [3]).

For the sake of completeness we state but do not prove the following theorem.

THEOREM 2.3. *If (V, \mathcal{B}) is a Pairwise Balanced Design (PBD) such that for every $B \in \mathcal{B}$ there is an $RG(b)$, then there is an $RG(V)$. Furthermore, if one of the $RG(B)$ is not bipartite, then the resultant $RG(V)$ is also not bipartite.*

3. A Preliminary Bound.

The following two results are proven in [6].

LEMMA 3.1 (Stinson). *Suppose there exist 5 mutually orthogonal Latin squares (MOLS) of order n . Let $0 \leq t_1, t \leq n$, and $t_1 \geq 3$. Then there exists an incomplete $(s, 2t+1)$ -Room square where $s = 2(5n + t_1 + t) + 1$.*

LEMMA 3.2. (Stinson). *If $s \geq \max\{u + 644, 6u + 9\}$ with s, u odd, then there exists an incomplete (s, u) -Room square.*

We are now in a position to obtain our first bound.

THEOREM 3.3. *If $s \geq 649$, s odd, then there exists a nonbipartite $RG(s)$.*

Proof. A nonbipartite Room graph of order 5 is given in Figure 1. The theorem then follows from Theorem 2.1 and Lemma 3.2 with $u = 5$.

At this time some small values are needed.

LEMMA 3.4. *There exist nonbipartite $RG(n)$ for $n = 5, 7, 9, 11$.*

Pf. [4].

THEOREM 3.5. *If there exist 5 MOLS of side n , then there exists a nonbipartite $RG(s)$ where*

$$a) \quad s = 2(5n + t_1) + 5 \quad 3 \leq t_1 \leq n \quad n \geq 5$$

$$b) \quad s = 2(5n + t_1) + 11 \quad 3 \leq t_1 \leq n \quad n \geq 11$$

Proof. In Lemma 3.1 let $t = 2$ for part a and $t = 5$ for part b. Then apply Theorem 2.1.

COROLLARY 3.6. *There exists a nonbipartite $RG(s)$ if there are 5 MOLS of order n and*

$$a) \quad 10n + 11 \leq s \leq 12n + 5 \quad \text{or}$$

$$b) \quad 10n + 17 \leq s \leq 12n + 11$$

We use Corollary 3.6 to establish the following theorem.

THEOREM 3.7. *If there does not exist a nonbipartite $RG(s)$ for $s \geq 81$, s odd, then $s \in \{115, 117, 119, 169\}$.*

Proof. In the table below we list values of n for which there exist 5 MOLS ([1]) and give the intervals which are covered by use of Corollary 3.5 a or b. The only orders not covered are those listed in the statement of this theorem. Those greater than 737 are covered by Theorem 3.3.

<u>n</u>	<u>Interval Covered</u>	<u>Corollary</u>
61	$621 \leq s \leq 737$	3.5a
53	$541 \leq s \leq 641$	3.5a
47	$481 \leq s \leq 569$	3.5a
41	$421 \leq s \leq 497$	3.5a
37	$381 \leq s \leq 449$	3.5a
32	$331 \leq s \leq 389$	3.5a
27	$281 \leq s \leq 329$	3.5a
23	$241 \leq s \leq 281$	3.5a
19	$207 \leq s \leq 239$	3.5b
17	$181 \leq s \leq 209$	3.5a
16	$171 \leq s \leq 197$	3.5a
13	$147 \leq s \leq 167$	3.5b
13	$141 \leq s \leq 161$	3.5a
11	$127 \leq s \leq 143$	3.5b
11	$121 \leq s \leq 137$	3.5a
9	$101 \leq s \leq 113$	3.5a
8	$91 \leq s \leq 101$	3.5a
7	$81 \leq s \leq 89$	3.5a

We conclude this section with the following theorem.

THEOREM 3.8. *If $s \geq 81$, s odd, then there exists a nonbipartite $RG(s)$.*

Proof. By Theorem 3.7 we need only consider the cases $s = 115, 117, 119$, and 169 .

Let $s = 115 = 23 \cdot 5$. By Theorem 1.2 there exists an $RG(23)$, and by Lemma 3.4 there is a nonbipartite $RG(5)$. Thus, by Theorem

4.2 there exists a nonbipartite $RG(115)$.

For $117 = 13 \cdot 9$, $119 = 17 \cdot 7$, and $169 = 21(9 - 1) + 1$ the result again follows from Lemma 3.4 and Theorems 1.2 and 2.2.

4. Low Orders.

We have shown that if $s \geq 81$, s odd, then there exists a nonbipartite $RG(s)$. In this section we complete the problem by showing that if $s \geq 5$, s odd, then there exists a nonbipartite $RG(s)$. To construct many of these low orders it is necessary to use Room graph starters. The starters which suffice for our purposes are a very special case of the more general Room graph starters defined in [4, Prop. 6].

Let $n = 2k + 1$. A Room graph starter of order n (in \mathbb{Z}_n) is composed of two sets

$$A = \{\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_k, y_k\}\} \text{ and}$$

$$B = \{\{u_1, v_1\}, \{u_2, v_2\}, \dots, \{u_k, v_k\}\}$$

which satisfy the properties

- (i) $\bigcup_{i=1}^k \{x_i, y_i\} = \mathbb{Z}_n \setminus \{0\}$,
- $\bigcup_{i=1}^k \{u_i, v_i\} = \mathbb{Z}_n \setminus \{0\}$,
- (ii) for $i \geq 3$, $y_i - x_i \equiv i \pmod{n}$,
- for $i \geq 3$, $v_i - u_i \equiv i \pmod{n}$,
- (iii) $y_1 - x_1 \equiv 1 \pmod{n}$, $y_2 - x_2 \equiv 1 \pmod{n}$,
- $v_1 - u_1 \equiv 2 \pmod{n}$, $v_2 - u_2 \equiv 2 \pmod{n}$;
- (iv) for all $i \neq j \geq 3$, $u_i - x_i \not\equiv u_j - x_j$.

PROPOSITION 4.1. If there exists a Room graph starter of order n , then there exists a Room graph of order n . Furthermore, this graph is nonbipartite.

Proof. For $m \in \mathbb{Z}_n$, let $A + m = \{m, \infty, \{x_i + m, y_i + m\} \mid i = 1, 2, \dots, k\}$ and let $B + m$ be defined analogously. A Room graph G on the symbols $S = \mathbb{Z}_n \cup \{\infty\}$ is constructed as follows. Construct a bijection between the unordered pairs in S and the edges of G in such a way that the edge containing $\{i, j\}$ is adjacent to the edge containing $\{a, b\}$ if and only if there exists some m such that $\{\{i, j\}, \{a, b\}\} \subset A + m$ or $B + m$.

This graph is well defined since properties (ii) and (iii) above insure that each unordered pair of elements in S occurs in precisely two of the sets $A + m_0$ and $B + m_1$, $m_0, m_1 \in \mathbb{Z}_n$. Therefore each edge has two disjoint end vertices. By property (iv) G has no parallel edges. Thus G is a simple graph.

By construction each edge holds exactly one unordered pair of elements in S and each unordered pair of elements in S appears once in G . Only the elements of some $A + m$ or $B + m$ are mapped to the edges adjacent to any vertex. Thus by property (i) above, the edges adjacent to each vertex hold each symbol of S exactly once.

By the nature of the construction there is a cycle C_1 in G consisting of all edges which hold pairs of difference 1. This cycle is always of length n and since n is odd this implies the graph is nonbipartite. Finally, the graph is connected since there is also a cycle C_2 of length n consisting of edges which hold pairs of difference 2. Since C_2 is vertex disjoint from C_1 and since there is an edge from C_1 to C_2 , the graph is connected.

LEMMA 4.2. *If $13 \leq s \leq 79$, s odd, then there exists a non-bipartite $RG(s)$.*

Proof. Room graph starters are given in the appendix for $s \in \{13, 15, 17, 19, 23, 27, 31, 33, 39, 43, 47, 59, 67, 79\}$. These starters were found by use of an algorithm similar to the one described in [2]. For the remaining orders we apply Theorem 2.2 with the indicated constructions.

<u>Order</u>	<u>Construction</u>	<u>Order</u>	<u>Construction</u>
21	= $5(5-1) + 1$	55	= $5 \cdot 11$
25	= $5 \cdot 5$	57	= $7(9-1) + 1$
29	= $7(5-1) + 1$	61	= $5(13-1) + 1$
35	= $5 \cdot 7$	63	= $7 \cdot 9$
37	= $9(5-1) + 1$	65	= $5 \cdot 13$
41	= $5(9-1) + 1$	69	= $17(5-1) + 1$
45	= $5 \cdot 9$	71	= $5(15-1) + 1$
49	= $7 \cdot 7$	73	= $9(9-1) + 1$
51	= $5(11-1) + 1$	75	= $5 \cdot 15$
53	= $13(5-1) + 1$	77	= $7 \cdot 11$

THEOREM 4.3. *If $s \geq 5$, s odd, then there exists a nonbipartite Room graph of order s .*

Proof. Lemmata 3.4 and 4.2 and Theorem 3.8.

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Room Graph Starter of Order 13

2, 3 7, 8 6, 9 10, 1 12, 4 5, 11
 12, 1 4, 6 7, 10 5, 9 11, 3 2, 8

Room Graph Starter of Order 15

12, 13 7, 8 6, 9 1, 5 14, 4 11, 2 3, 10
 12, 14 11, 13 3, 6 4, 8 5, 10 1, 7 2, 9

Room Graph Starter of Order 17

15, 16 13, 14 4, 7 6, 10 3, 8 5, 11 12, 2 1, 9
 10, 12 9, 11 2, 5 16, 3 8, 13 1, 7 14, 4 15, 6

Room Graph Starter of Order 19

17, 18 15, 16 3, 6 5, 9 7, 12 8, 14 4, 11 13, 2 1, 10
 14, 16 4, 6 9, 12 13, 17 5, 10 2, 8 15, 3 18, 7 11, 1

Room Graph Starter of Order 23

19, 20 17, 18 9, 12 6, 10 3, 8 7, 13 21, 5 14, 22 16, 2 1, 11
 4, 15
 2, 4 22, 1 7, 10 16, 20 14, 19 12, 18 6, 13 3, 11 8, 17 5, 15
 21, 9

Room Graph Starter of Order 27

17, 18 13, 14 4, 7 1, 5 21, 26 16, 22 8, 15 25, 6 2, 11 10, 20
 19, 3 12, 24 23, 9
 11, 13 6, 8 16, 19 18, 22 15, 20 23, 2 21, 1 4, 12 25, 7 26, 9
 3, 14 5, 17 24, 10

Room Graph Starter of Order 31

16, 17 2, 3 4, 7 24, 28 6, 11 9, 15 25, 1 19, 27 20, 29 12, 22
 10, 21 14, 26 5, 18 30, 13 8, 23
 20, 22 5, 7 30, 2 12, 16 9, 14 17, 23 18, 25 26, 3 1, 10 11, 21
 24, 4 27, 8 6, 19 15, 29 13, 28

Room Graph Starter of Order 33

17, 18 24, 25 5, 8 16, 20 32, 4 30, 3 2, 9 6, 14 19, 28 21, 31
 1, 12 11, 23 13, 26 15, 29 7, 22 27, 10

28, 30 6, 8 23, 26 21, 25 10, 15 13, 19 29, 3 16, 24 5, 14 12, 22
 31, 9 32, 11 27, 7 4, 18 20, 2 1, 17

Room Graph Starter of Order 39

33, 34 15, 16 22, 25 38, 3 26, 31 23, 29 7, 14 13, 21 2, 11 10, 20
 36, 8 12, 24 32, 6 30, 5 28, 4 19, 35 1, 18 9, 27 37, 17

29, 31 26, 28 10, 13 37, 2 35, 1 36, 3 17, 24 15, 23 16, 25 4, 14
 34, 6 8, 20 5, 18 19, 33 7, 22 11, 27 21, 38 30, 9 32, 12

Room Graph Starter of Order 43

39, 40 37, 38 6, 9 19, 23 42, 4 35, 41 8, 15 16, 24 13, 22 18, 28
 33, 1 20, 32 17, 30 7, 21 14, 29 10, 26 31, 5 36, 11 27, 3 25, 2
 34, 12

27, 29 16, 18 4, 7 41, 2 6, 11 28, 34 23, 30 13, 21 15, 24 32, 42
 3, 14 40, 9 20, 33 39, 10 22, 37 35, 8 19, 36 26, 1 12, 31 5, 25
 17, 38

Room Graph Starter of Order 47

11, 12 42, 43 46, 2 14, 18 31, 36 15, 21 44, 4 24, 32 25, 34 27, 37
 30, 41 45, 10 22, 35 3, 17 13, 28 7, 23 39, 9 8, 26 1, 20 33, 6
 19, 40 16, 38 29, 5

1, 3 34, 36 41, 44 25, 29 27, 32 14, 20 5, 12 37, 45 8, 17 13, 23
 28, 39 7, 19 11, 24 4, 18 31, 46 22, 38 9, 26 35, 6 30, 2 43, 16
 21, 42 40, 15 10, 33

Room Graph Starter of Order 59

56, 57 41, 42 3, 6 22, 26 18, 23 27, 33 17, 24 7, 15 34, 43 54, 5
 21, 32 25, 37 48, 2 36, 50 45, 1 28, 44 14, 31 53, 12 11, 30 52, 13
 47, 9 29, 51 35, 58 55, 20 38, 4 49, 16 19, 46 39, 8 40, 10

25, 27 53, 55 51, 54 2, 6 16, 21 12, 18 29, 36 9, 17 24, 33 40, 50
 41, 52 31, 43 56, 10 28, 42 32, 47 57, 14 46, 4 20, 38 30, 49 15, 35
 1, 22 45, 8 3, 26 58, 23 39, 5 11, 37 7, 34 44, 13 19, 48

Room Graph Starter of Order 67

30, 31	27, 28	65, 1	46, 50	42, 47	64, 3	29, 36	17, 25	15, 24	61, 4
5, 16	22, 34	43, 56	62, 9	48, 63	19, 35	49, 66	2, 20	33, 52	18, 38
59, 13	57, 12	37, 60	53, 10	26, 51	32, 58	14, 41	45, 6	11, 40	44, 7
8, 39	23, 55	21, 54							
56, 58	59, 61	33, 36	9, 13	45, 50	40, 46	24, 31	65, 6	51, 60	28, 38
4, 15	66, 11	16, 29	18, 32	7, 22	52, 1	20, 37	54, 5	44, 63	23, 43
49, 3	55, 10	12, 35	17, 41	14, 39	21, 47	30, 57	25, 53	19, 48	64, 27
62, 26	2, 34	42, 8							

Room Graph Starter of Order 79

76, 77	65, 66	57, 60	63, 67	9, 14	46, 52	18, 25	33, 41	34, 43	12, 22
26, 37	5, 17	61, 74	24, 38	13, 28	39, 55	53, 70	1, 19	21, 40	15, 35
78, 20	49, 71	27, 50	48, 72	58, 4	42, 68	2, 29	8, 36	73, 23	59, 10
44, 75	30, 62	31, 64	51, 6	47, 3	54, 11	32, 69	7, 45	56, 16	
37, 39	38, 40	28, 31	74, 78	30, 35	11, 17	52, 59	24, 32	51, 60	63, 73
16, 27	21, 33	69, 3	44, 58	55, 70	77, 14	50, 67	66, 5	53, 72	68, 9
4, 25	64, 7	34, 57	12, 36	18, 43	75, 22	15, 42	20, 48	76, 26	41, 71
54, 6	49, 2	56, 10	13, 47	45, 1	29, 65	61, 19	8, 46	23, 62	