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FACTORIZATIONS AND ORTHOGONAL MATCHINGS

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ABSTRACT   A matching M is orthogonal to a 1-factorization F if any two edges in M lie in distinct factors of F. We count the number of matchings with k edges which are orthogonal to F for k = 1, 2, 3, 4, and 5. If $k \leq 3$ the number is independent of F. If k = 4 or 5 it depends on $Q(F)$, the number of quadrilaterals formed by the pairwise union of factors in F.

Section 1   Introduction.

Let G be a finite simple graph. A *matching* in G is a set of nonadjacent edges. A *factor* in G (also known as a 1-factor or a perfect matching) is a matching which is incident with every vertex of G. A *factorization* of G is a partitioning of the edge set of G into factors. In this paper we restrict our attention to $G = K_{2n}$, so each factor contains n edges and each factorization contains $2n - 1$ factors.

A matching m is *orthogonal* to a factorization F if any two edges in m occur in different factors of F. Two factorizations F_1 and F_2 are *orthogonal* if each factor in F_1 is orthogonal to F_2; this implies that each factor in F_2 is orthogonal to F_1. The existence of two orthogonal factorizations of K_{2n} is equivalent to the existence of a Room square of side $2n - 1$.

We are interested in calculating the number of matchings, factors and factorizations which are orthogonal to a given factorization F by using an easily computed invariant of F. Note that the union of any two factors in F is a 2-regular graph, so it is a union of simple cycles. Let $H(F)$ denote the total number of Hamiltonian cycles formed by unioning pairs of factors in F, and let $Q(F)$ be the number of 4-cycles thus formed. In [1] we observed that in K_{10}, if $H(F_1) > H(F_2)$ then generally F_1 has more orthogonal factorizations than does F_2. This observation proved useful for finding orthogonal factorizations. Further investigation revealed that for K_{10}, the number of factors orthogonal to F was given exactly by $108 + 3 H(F)$, or equivalently by $216 - 3 Q(F)$. We will explain this phenomenon in a more general context. Define $OM_k(F)$ as the number of matchings which contain k edges and are orthogonal to F. Our main result is that if $k \leq 5$ then $OM_k(F)$ is a function only of $Q(F)$, k and n . This is not true for $k \geq 6$. The number of orthogonal factors to a factorization F of K_{12} depends on a deeper structure of F.

Section 2 The Result.

Let F be a factorization of K_{2n} and let $\bar{p} = \{p_1, p_2, \cdots, p_i\}$ be a partition of k. We say that a matching with k edges is of $F-type$ $\bar{p}$ provided that p_1 edges occur in some factor $f_1 \in F$, p_2 edges occur in some second factor f_2, etc. Thus an orthogonal matching is of F-type $\{1, 1, \cdots, 1\}$, or more succinctly, of F-type $\{1^k\}$. In order to calcuate the number of orthogonal matchings we have found it necessary to count the number of matchings of F-type $\bar{p}$ for each partition of k. Define $\#(\bar{p}, F)$ to be the number of matchings of F-type $\bar{p}$. In particular, $\#(\{1^k\}, F) = OM_k(F)$. We will use whichever notation is more convenient. When F is understood from context we will use $type$ in place of F-type and $\#(\bar{p})$ in place of $\#(\bar{p}, F)$. We will use $\binom{n}{m}$ to denote a binomial coefficient, and use $M(2n, k)$ to denote the number of matchings having k edges in K_{2n}. Notice that

$$M(2n, k) = \frac{(2n)!}{(2n-2k)!\, k!\, 2^k}.$$

We begin with the easy cases $k = 1$ and 2. The case $k = 1$ is of course trivial, $OM_1(F) = M(2n, 1) = \binom{2n}{2}$. In the case $k = 2$ there are two partitions. $\#(\{2\})$ is most easily counted by first choosing a factor in $2n - 1$ ways, then choosing the two edges in $\binom{n}{2}$ ways. Hence

$$\#(\{2\}) = (2n - 1)\binom{n}{2}. \tag{1}$$

As

$$M(2n, 2) = \#(\{2\}) + \#(\{1^2\}) \tag{2}$$

we can solve equations (1) and (2) simultaneously to get

$$OM_2(F) = M(2n, 2) - (2n - 1)\binom{n}{2}$$
$$= n\,(n-1)\,(n-2)\,(2n-1).$$

Theorem 1 *Let F be a factorization of K_{2n}. Then*

$$OM_3(F) = M(2n, 3) - (2n-1)\binom{n}{2} M(2n-4, 1) - 4\,(2n-1)\binom{n}{3}.$$

Proof : Similar to equations (1) and (2) we get

$$M(2n, 3) = \#(\{3\}) + \#(\{2, 1\}) + \#(\{1^3\}), \tag{3}$$

and

$$\#(\{3\}) = (2n - 1)\binom{n}{3}. \tag{4}$$

Next we examine the number of matchings with three edges in which a fixed pair of edges occur in the same factor $f \in F$. In other words, we consider sets of three edges containing a distinguished subset of type $\{2\}$. We can first choose a subset of type $\{2\}$ in $(2n-1)\binom{n}{2}$ ways and then choose a remaining edge in $M(2n-4,1)$ ways. These choices will triple count matchings of type $\{3\}$ and single count matchings of type $\{2,1\}$. Thus

$$(2n-1)\binom{n}{2} M(2n-4,1) = 3 \,\#(\{3\}) + \#(\{2,1\}). \tag{5}$$

This type of equation will arise repeatedly. We will refer to it as *an equation from a distinguished subset.*

We solve equations (3), (4) and (5) simultaneously to get the desired equation for $\#(\{1^3\}) = OM_3(F)$. $\square$

The cases $k = 1, 2$, and 3 illustrate the basic technique used in the cases $k = 4$ and 5. We observe that for $k \leq 3$ the number of orthogonal matchings does not depend on F. This will not be the case for $k \geq 4$.

Lemma 2 *Let F be a factorization of K_{2n}. Then*

$$\#(\{2,k-2\}) = 2\binom{2n-1}{2}\left\{ n\left[\binom{n-3}{k-2} - \binom{n-4}{k-2}\right] + \binom{n}{2}\binom{n-4}{k-2}\right\}$$
$$+ \left[\binom{n-2}{k} - 2\binom{n-3}{k} + \binom{n-4}{k}\right] 2\, Q(F)$$

except when $k = 4$, where $\#(\{2,2\})$ is ½ of the above expression.

Proof : We pick a matching of type $\{2,k-2\}$ by first selecting the ordered pair of factors (f_i, f_j) in $2\binom{2n-1}{2}$ ways. The edges in f_i will be called *red* edges, the edges in f_j are *blue*. We will choose 2 red edges and $k-2$ blue edges. Let $q_{i,j}$ denote the number of 4-cycles in $f_i \cup f_j$. Observe that $\sum_{(i,j)} q_{i,j} = 2\, Q(F)$. Consider an unordered pair of red edges. We distinguish three cases depending on the arrangement of the blue edges adjacent to these red edges. See Figure 1.
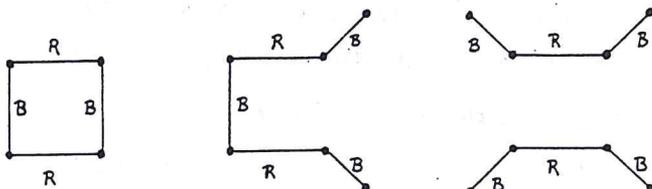
| Case 1 | Case 2 | Case 3 |

Figure 1

Case 1 will occur $q_{i,j}$ times. In this case we can choose the $k-2$ blue edges in $\binom{n-2}{k-2}$ ways. Case 2 will occur $n-2q_{i,j}$ times, since each blue edge not in a 4-cycle is in the 'middle' of the P_5. In this case we can choose the blue edges in $\binom{n-3}{k-2}$ ways. The remaining $\binom{n}{2} - q_{i,j} - (n-2q_{i,j})$ pairs of red edges occur as Case 3. The blue edges can be chosen in $\binom{n-4}{k-2}$ ways. Thus the total number of matchings of type $\{2,k-2\}$ with 2 red and $k-2$ blue edges is

$$q_{i,j}\binom{n-2}{k-2} + (n-2\,q_{i,j})\binom{n-3}{k-2} + \left[\binom{n}{2} - q_{i,j} - (n-2\,q_{i,j})\right]\binom{n-4}{k-2}$$
$$= \left[n\left(\binom{n-3}{k-2} - \binom{n-4}{k-2}\right) + \binom{n}{2}\binom{n-4}{k-2}\right] + \left[\binom{n-2}{k-2} - 2\binom{n-3}{k-2} + \binom{n-4}{k-2}\right]q_{i,j}$$

Summing over all (i,j) gives the desired $\#(\{2,k-2\})$. When $k=4$ we can no longer distinguish the two factors and hence have double counted $\#(\{2,2\})$. $\square$

Theorem 3 *Let F be a factorization of K_{2n}. Then*

$$OM_4(F) = M(2n,4) - (2n-1)\binom{n}{2}M(2n-4,2) - 3(2n-1)\binom{n}{4} + 2(2n-1)\binom{n}{3}\binom{2n-6}{2}$$
$$+ \binom{2n-1}{2}\left[\binom{n}{2}\binom{n-4}{2} - n(n-4)\right] + Q(F)$$

Proof : There are five partitions $\bar{p}$ of 4. We will find five equations in the five unknowns $\#(\{\bar{p}\})$. Clearly

$$\#(\{4\}) = (2n-1)\binom{n}{4}, \qquad\qquad (6)$$

also the sum over all partitions of 4 yields an equation

$$M(2n,4) = \#(\{4\}) + \#(\{3,1\}) + \#(\{2^2\}) + \#(\{2,1^2\}) + \#(\{1^4\}). \qquad (7)$$

Analagous to equation (5), we get two more equations by using a distinguished subset of type $\{k\}$ for $k=2$ and 3. When $k=2$ we get

$$(2n-1)\binom{n}{2}M(2n-4,2) = 6\,\#(\{4\}) + 3\,\#(\{3,1\}) + 2\,\#(\{2^2\}) + \#(\{2,1^2\}). \quad (8)$$

Similarly for $k=3$ we get

$$(2n-1)\binom{n}{3}M(2n-6,1) = 4\,\#(\{4\}) + \#(\{3,1\}) . \qquad\qquad (9)$$

Using $k=4$, Lemma 2 simplifies to

$$\#(\{2^2\}) = \binom{2n-1}{2}\left[\binom{n}{2}\binom{n-4}{2} + n(n-4)\right] + Q(F) . \qquad (10)$$

Simultaneously solving equations (6) - (10) for $\#(\{1^4\})$ gives the desired value. $\square$

Theorem 4 *Let F be a factorization of K_{2n}. Then*

$$OM_5(F) = 4\,C_1 + C_2 - 3\,C_3 + 2\,C_4 - C_5 + C_6 - 2\,C_7$$
$$+ Q(F)\left[\binom{2n-8}{2} - 4(n-4)\right]$$

for the C_i's defined below.

Proof : We proceed as in the proof of Theorem 4. There are seven partitions $\bar{p}$ of 5. We will find seven equations in the seven unknowns $\#(\{\bar{p}\})$. Set $C_1 = (2n-1)\binom{n}{5}$. Thus

$$C_1 = \#(\{5\}). \tag{11}$$

Likewise set $C_2 = M(2n,5)$. The sum over all partitions of 5 yields an equation

$$C_2 = \#(\{5\}) + \#(\{4,1\}) + \#(\{3,2\}) + \#(\{3,1^2\}) \tag{12}$$
$$+ \#(\{2^2,1\}) + \#(\{2,1^3\}) + \#(\{1^5\}).$$

We can get four more equations by using distinguished subsets. Set $C_3 = (2n-1)\binom{n}{4} M(2n-8,1)$. The matchings containing a distinguished subset of type $\{4\}$ lead to the equation

$$C_3 = 5\,\#(\{5\}) + \#(\{4,1\}). \tag{13}$$

Similarly let $C_4 = (2n-1)\binom{n}{3} M(2n-6,2)$, then a distinguished subset of type $\{3\}$ yields

$$C_4 = 10\,\#(\{5\}) + 4\,\#(\{4,1\}) + \#(\{3,2\}) + \#(\{3,1^2\}). \tag{14}$$

If $C_5 = (2n-1)\binom{n}{2} M(2n-4,3)$, then a subset of type $\{2\}$ gives

$$C_5 = 10\,\#(\{5\}) + 6\,\#(\{4,1\}) + 4\,\#(\{3,2\}) \tag{15}$$
$$+ 3\,\#(\{3,1^2\}) + 2\,\#(\{2^2,1\}) + \#(\{2,1^3\}).$$

Next, from Lemma 2, a subset of type $\{2^2\}$ leads to the equation

$$C_6 + Q(F)\,M(2n-8,1) = 3\,\#(\{3,2\}) + \#(\{2^2,1\}), \tag{16}$$

where

$$C_6 = \binom{2n-1}{2}\left[\binom{n}{2}\binom{n-4}{2} + n\,(n-4)\right] M(2n-8,1).$$

Also from Lemma 2, a subset of type $\{3,2\}$ gives

$$C_7 + 2\,Q(F)\left[\binom{n-2}{3} - 2\binom{n-3}{3} + \binom{n-4}{3}\right] = \#(\{3,2\}), \tag{17}$$

where

$$C_7 = 2\binom{2n-1}{2}\left\{n\left[\binom{n-3}{3} - \binom{n-4}{3}\right] + \binom{n}{2}\binom{n-4}{3}\right\}.$$

Simultaneously solving equations (11) - (17) for $\#(\{1^4\})$ gives the desired value. $\square$

Corollary 5 *The number of orthogonal factors to a factorization F of K_{10} is given by* $216 - 3\,Q(F)$. $\square$

Section 3 Conclusion.

The original motivation for this paper was the observation from [1] that the larger $H(F)$ is, the more orthogonal factorizations F has. We now see that more

Hamiltonian cycles in the union of two factors implies fewer 4-cycles, which in turn implies more orthogonal factors. The more orthogonal factors a factorization has, the more orthogonal factorizations it will have. The exact relationship between the number of orthogonal factors of a graph and the number of orthogonal factorizations is unclear, and accounts for the fact that in [1] we were only able to discern a general trend between $H(F)$ and the number of orthogonal factorizations.

We had hoped to be able to find the number of orthogonal factors for a given factorization of K_{2n}. Our techniques involve calculating $\#(\bar{p})$ for each partition $\bar{p}$ of n. The number of equations involved grows rather rapidly with n. We have found examples in K_{12} of two factorizations where the union of pairs of factors gives the same 'cycle vector', but which have different numbers of orthogonal factors. In this case, it appears a new lemma similar to Lemma 2 is needed. In particular, we need to count $\#(\{2^3\})$. This in turn will involve the structure of $f_1 \cup f_2 \cup f_3$, where the f_i are factors of F. This more general analysis now appears difficult.

In Table 1, we list various values of $OM_5(F)$, where F is a factorization of K_{2n}. Observe that for say, K_{40}, the number of orthogonal matchings with five edges is relatively unchanged for different factorizations. Does the same hold for the number of orthogonal factors? Some asymptotic information on the number of orthogonal factors and factorizations could be more tractable than an exact count, and would be of great interest.

We close by mentioning that a direct computer search has verified all of the results in this paper for $2n \leq 16$ and $k \leq 5$.

TABLE 1

$2n$	$OM_5(F)$
10	216 - 3 Q(F)
12	17754 - 2 Q(F)
14	342888 + 3 Q(F)
16	3240720 + 12 Q(F)
18	19978128 + 25 Q(F)
20	92391642 + 42 Q(F)
40	604809582620 + 432 Q(F)

REFERENCE

[1] Dan Archdeacon, J.H. Dinitz and W.D. Wallis, *Sets of pairwise orthogonal 1-factorizations of K_{10}*, Congressus Numerantium **43** (1984), pp. 45-79.