

## Four Orthogonal One-Factorizations on Ten Points

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### Abstract

Using list processing techniques, an exhaustive search was made for orthogonal one-factorizations of  $K_{10}$ . As a result we have found that, up to isomorphism, there is exactly one set of four mutually orthogonal one-factorizations of  $K_{10}$ , and exactly 267 sets of three mutually orthogonal one-factorizations of  $K_{10}$ .

### 1. Introduction

Let  $G$  be a graph with an even number of vertices. A *one-factor* in  $G$  is a set of (pairwise disjoint) edges which between them contain each vertex exactly once. A *one-factorization* is a way of decomposing the edges of  $G$  into pairwise disjoint one-factors. In particular it is well-known that the complete graph  $K_{2n}$  on  $2n$  vertices has a one-factorization, which consists of  $2n-1$  factors. If  $F = \{F_1, F_2, \dots, F_k\}$  and  $G = \{G_1, G_2, \dots, G_k\}$  are one-factorizations of the same graph  $G$ , we say that  $F$  and  $G$  are *isomorphic* if there exists a map  $\phi$  which permutes the vertices of  $G$  and a map  $\psi$  which permutes the integers  $\{1, 2, \dots, k\}$  such that for all  $i$ ,  $F_i\phi = G_{i\psi}$  ( $F_i\phi$  is the graph derived from  $F_i$  by replacing each edge  $(x, y)$  by  $(x\phi, y\phi)$ ). We usually refer to  $\phi$  as "the isomorphism", the existence of a suitable map  $\psi$  being implicit.

Two one-factorizations  $\{F_1, F_2, \dots, F_k\}$  and  $\{G_1, G_2, \dots, G_k\}$  of  $G$  are called *orthogonal* if, for every  $i$  and  $j$ ,  $F_i$  and  $G_j$  have at most one edge in common. Orthogonal one-factorizations of complete graphs correspond to Room Squares

(see [6, 7]), orthogonal one-factorizations of regular complete bipartite graphs correspond to Latin squares (see [8]). In this paper we discuss orthogonal one-factorizations of  $K_4$ ,  $K_6$ ,  $K_8$  and especially of  $K_{10}$ , where we have run an exhaustive search for sets of orthogonal one-factorizations.

If  $F$ ,  $G$ ,  $F'$  and  $G'$  are factorizations of the same graph,  $F$  is orthogonal to  $G$ , and there is an isomorphism which takes  $F$  to  $F'$  and simultaneously takes  $G$  to  $G'$ , we can consider the pairs  $\{F, G\}$  and  $\{F', G'\}$  to be the same up to isomorphism. If  $G$  is a complete graph, this means that the corresponding Room squares would be isomorphic also.

Let  $\nu(r)$  be the maximum possible number of mutually orthogonal one-factorizations of  $K_r$ ,  $r$  even. It is easy to see that  $\nu(r) \leq r - 3$ , but no better upper bound has been found in general. However, no case is known where  $\nu(r)$  is greater than  $\frac{r}{2} - 1$ , and some authors believe this is an upper bound for all  $r$ . It is known that  $\nu(r) \geq \frac{r}{2} - 1$  when  $r - 1$  is a prime power congruent to 3(modulo 4). However  $\nu(6) = 1$  (see the next section), so  $\nu(r) = \frac{r}{2} - 1$  cannot always be achieved, even when  $r - 1$  is a prime. The main result of this paper, that  $\nu(10) = \frac{10}{2} - 1 = 4$ , does however lend support to the conjecture that  $\nu(r) \leq \frac{r}{2} - 1$  and that this bound can almost always be attained.

For further information on  $\nu(r)$ , see [2, 3, 6, 7]. In particular, it is shown in [6] that  $\nu(r)$  approaches infinity with  $r$ .

Our aim here is to study orthogonal factorizations of  $K_{10}$  fully. Not only do we wish to evaluate  $\nu(10)$ , but we hope that a full study will aid understanding of the behaviour of orthogonal one-factorizations in general.

## 2. Small Orders

The one-factorizations of small complete graphs are easily studied. For  $K_2$  and  $K_4$  there is only one factorization.  $K_6$  admits fifteen one-factors and six one-factorizations; each factor lies in exactly two factorizations and any two factorizations have exactly one factor in common; the six factorizations are isomorphic. So there is one factorization up to isomorphism, and there are no pairs of orthogonal factorizations, up to order 6:  $\nu(2) = \nu(4) = \nu(6) = 1$ .

For  $K_8$  the situation is more interesting. A complete analysis is given in [9]. There are six non-isomorphic one-factorizations, which we list in Table 1. We shall call them  $F_1, F_2, F_3, F_4, F_5$  and  $F_6$ . Up to isomorphism there are four factorizations orthogonal to  $F_1$ , three to  $F_4$ , two to  $F_5$  and one to  $F_6$ ; those orthogonal to  $F_1$  are isomorphic to  $F_1, F_4, F_5$  and  $F_6$  respectively. Those orthogonal to  $F_4$  are isomorphic to  $F_1, F_4$  and  $F_5$  respectively; those orthogonal

to  $F_5$  are isomorphic to  $F_1$  and  $F_4$  respectively; the one orthogonal to  $F_6$  is isomorphic to  $F_1$ . Allowing for double counting (since  $\{F, G\}$  and  $\{G, F\}$  are the same pair) we have six pairs up to isomorphism. Interestingly, there are no cases of non-isomorphic pairs  $\{F, G\}$  and  $\{F, H\}$  where  $G$  is isomorphic to  $H$ , but such pairs appear for higher orders. There is precisely one set of three mutually orthogonal factorizations up to isomorphism (isomorphic to  $F_1, F_1$  and  $F_6$ ), and no set of four. So  $\nu(8) = 3$ .

All One-Factorizations of $K_8$							
Table 1							
01	23	45	67	01	23	45	67
02	13	46	57	02	13	46	57
03	12	47	56	03	12	47	56
04	15	26	37	04	15	26	37
05	14	27	36	05	14	27	36
06	17	24	35	06	17	25	34
07	16	25	34	07	16	24	35
	$F_1$				$F_2$		
01	23	45	67	01	23	45	67
02	13	46	57	02	13	46	57
03	12	47	56	03	12	47	56
04	16	25	37	04	16	27	35
05	17	26	34	05	17	26	34
06	14	27	35	06	14	25	37
07	15	24	36	07	15	24	36
	$F_3$				$F_4$		
01	23	45	67	01	23	45	67
02	13	46	57	02	14	36	57
03	14	27	56	03	16	25	47
04	16	25	37	04	17	26	35
05	17	26	34	05	12	37	46
06	12	35	47	06	15	27	34
07	15	24	36	07	13	24	56
	$F_5$				$F_6$		

### 3. Order Ten

An exhaustive search for orthogonal one-factorizations of  $K_{10}$  was made, with the main result being that  $\nu(10) = 4$ . In this section we shall discuss the method employed in this search and some of the findings.

In his thesis [4], Gelling determined the complete set of non-isomorphic one-factorizations of  $K_{10}$  (see also [5]). These are 396 in all, which we shall denote as  $G_1, G_2, \dots, G_{396}$  (in Gelling's order). Our search begins by choosing a one-factorization,  $G_n$  say. We find all one-factorizations orthogonal to  $G_n$ , and then check this list for mutual orthogonality. If any set of orthogonal one-factorizations contains a factorization isomorphic to  $G_n$ , then applying the inverse isomorphism to all the factorizations will produce an isomorphic set which contains  $G_n$  itself; so if we let  $n$  range from 1 to 396 we shall obtain a complete list of all isomorphism classes of orthogonal one-factorizations of  $K_{10}$ . (The list could contain some repetitions, as no isomorph-rejection has been carried out after the selection of  $G_n$ ; but the number of repetitions should be very small, since the one-factorization of  $K_{10}$  mostly have small automorphism groups - 298 of them have the identity group [4]).

We used a Fortran program which employed three subroutines: WINNOW, RS9S and ORTHOG. Let  $G_n = \{g_1, g_2, \dots, g_9\}$  be the  $n$ th one-factorization on Gelling's list. The subroutine WINNOW reads in all 945 one-factors of  $K_{10}$  and outputs those which could possibly be contained in a one-factorization orthogonal to  $G_n$ . That is, if  $W$  is a one-factor of  $K_{10}$ ,  $W$  will be output if and only if  $W$  and  $g_i$  have at most one edge in common for  $i = 1, 2, \dots, 9$ .

The subroutine RS9S reads in the one-factors supplied by WINNOW. From this list it constructs all possible one-factorizations using only these one-factors. So it constructs the one-factorizations orthogonal to  $G_n$ . At this point some duplication could occur - RS9S might produce two factorizations,  $K$  and  $L$  say such that some isomorphism  $\chi$  exists which maps  $G_n$  to itself and also maps  $K$  to  $L$ . As explained above, the number of such occurrences should be small, and it is much cheaper (in terms of CPU time) to allow such duplications to occur than to conduct isomorph-rejection at this stage.

Finally, the subroutine ORTHOG checks pairs of one-factorizations from RS9S for orthogonality. Then if two one-factorizations  $K$  and  $L$  are found to be orthogonal to each other,  $G_n$  and  $K$  and  $L$  form a set of three mutually orthogonal one-factorizations, and they are output.

The number of triples is sufficiently small for further work to be done most efficiently by hand. We did this and found our main theorem.

**Theorem 1:** There is exactly one set of four mutually orthogonal one-factorizations of  $K_{10}$ , up to isomorphism. This set does not extend to a set of five mutually orthogonal one-factorizations.

The set of four factorizations is shown in Table 2.

01	23	45	67	89	01	29	36	48	57
02	13	46	58	79	02	15	34	69	78
03	12	47	59	68	03	16	28	45	79
04	16	25	39	78	04	17	26	35	89
05	18	24	37	69	05	14	27	39	68
06	19	27	35	48	06	12	37	49	58
07	15	28	36	49	07	19	25	33	46
08	17	29	34	56	08	13	24	59	67
09	14	26	38	57	09	18	23	47	56
		$F_1$					$F_2$		
01	26	39	47	58	01	25	34	68	79
02	14	37	56	89	02	18	35	49	67
03	17	25	48	69	03	15	27	46	89
04	18	27	36	59	04	13	28	57	69
05	19	28	34	67	05	16	29	38	47
06	15	24	38	79	06	14	23	59	78
07	13	29	45	68	07	12	39	48	56
08	16	23	49	67	08	19	26	37	45
09	12	35	46	78	09	17	24	36	58
		$F_3$					$F_4$		

The uniqueness may be checked by computer (in about 32 hours CPU time).

The four factorizations have an interesting structure.  $F_1$  is isomorphic to  $G_{380}$  in Gelling's list, while  $F_2$ ,  $F_3$  and  $F_4$  are all isomorphic to  $G_{377}$ . The set has automorphism group of order 3, generated by  $\sigma = (013)(476)(598)$  which is an automorphism of  $F_1$  and swaps  $F_4 \rightarrow F_3 \rightarrow F_2 \rightarrow F_4$ .

We note again that our result says that  $\nu(10) = 4$ . This is significant in that it is the first known example of a number  $r$  with  $r \equiv 2 \pmod{4}$  and  $\nu(r) \geq \frac{r}{2} - 1$ .

By use of invariants of one-factorizations it is possible to compute the exact number of non-isomorphic sets of three mutually orthogonal one-factorizations of  $K_{10}$ . This number is 267. A listing of these triples and a description of the method will appear in a later paper. Beaman [1] determined that there are exactly 511,562 distinct ordered pairs of orthogonal 1-factorizations (non-isomorphic Room squares) and exactly 257,630 unordered pairs (inequivalent Room squares).

Notice that our computational approach was essentially a list-processing one. Backtrack methods were tried experimentally, but are slower by a considerable margin (by a factor of over 100 in the WINNOWER process).

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### Addendum

For a complete list of all 267 sets of three mutually orthogonal one factorizations of  $K_{10}$ , see

D.S. Archdeacon, J.H. Dinitz, and W.D. Wallis, "Sets of pairwise orthogonal 1-factorizations of  $K_{10}$ ", *Congressus Numerantium*, to appear.

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