

Howell Designs with Sub-designs

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A Howell design of side s and order $2n$, or more briefly an $H(s, 2n)$, is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements from some $(2n)$ -set V such that (1) every element of V occurs in precisely one cell of each row and each column, and (2) every unordered pair of elements from V is in at most one cell of the array. It follows immediately from the definition of an $H(s, 2n)$ that $n \leq s \leq 2n - 1$. The two boundary cases are well known designs: an $H(2n - 1, 2n)$ is a Room square of side $2n - 1$ and the existence of a pair of mutually orthogonal Latin squares of order n implies the existence of an $H(n, 2n)$. We are interested in the existence of Howell designs which contain as a subarray another Howell design. The existence of Room squares with Room square sub-designs and a pair of mutually orthogonal Latin squares with Latin square sub-designs has been investigated. In this paper, we consider the general problem of constructing $H(s, 2n)$ which contain as sub-designs $H(t, 2m)$. We describe some bounds on the parameters and several constructions for the general case, then we concentrate on determining the spectrum for Howell designs where $t = m$ or $t = 2m - 1$. © 1994 Academic Press, Inc.

1. INTRODUCTION

A *Howell design* of side s and order $2n$, or more briefly an $H(s, 2n)$, is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements from some $(2n)$ -set V such that

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- (1) every element of V occurs in precisely one cell of each row and each column, and
- (2) every unordered pair of elements from V is in at most one cell of the array.

It follows immediately from the definition of an $H(s, 2n)$ that $n \leq s \leq 2n - 1$.

An $H(2n - 1, 2n)$ is also called a *Room square* of order $2n - 1$ or a $RS(2n - 1)$. The spectrum of Room squares was completed in [26]: there exists a $RS(2n - 1)$ for all positive integers n , $n \neq 2$ or 3 . There is an extensive literature available on Room squares, see [26] and a recent survey [14]. At the other boundary, the existence of a pair of mutually orthogonal Latin squares of order n implies the existence of an $H(n, 2n)$. Thus, there is an $H(n, 2n)$ for n a positive integer, $n \neq 2$ or 6 [3]. (An $H(6, 12)$ is displayed in [17].)

Several other special cases of $H(s, 2n)$ s were investigated and a summary of these results can be found in [27, 2]. The spectrum for $H(s, 2n)$ was completed in two papers, [27] and [2].

THEOREM 1.1. [27, 2] *There exists an $H(s, 2n)$ for all positive integers s and n except when $(s, 2n) \in \{(2, 4), (3, 4), (5, 6), (5, 8)\}$.*

An $H^*(s, 2n)$ is an $H(s, 2n)$ in which there is a subset W of V , $|W| = 2n - s$, such that no pair of elements from W appears in the design. *-designs are useful in recursive constructions. We note that there exist $H^*(s, 2n)$ for s even with two exceptions: there is no $H^*(2, 4)$ and there is no $H^*(6, 12)$ [2]. Information on *-designs for s odd can be found in [27]. Many of the recursive constructions also use Howell designs in standard form. Suppose H is an $H^*(s, 2n)$ defined on $\mathbb{Z}_s \cup W$. H is said to be in standard form if there is an element of W , say ∞ , so that $\{i, \infty\}$ occurs in cell (i, i) for $i = 0, 1, \dots, s - 1$.

Suppose that H is an $H(s, 2n)$ defined on the symbol set V . A $t \times t$ subarray G of H is said to be a *Howell sub-design* $H(t, 2m)$ if it is itself a Howell design of side t defined on a symbol set $U \subseteq V$ of size $2m$. In view of Theorem 1.1, no Howell design can contain a Howell sub-design $H(t, 2m)$ when $(t, 2m) \in \{(2, 4), (3, 4), (5, 6), (5, 8)\}$. However, we can construct Howell designs which are *missing* sub-designs of these orders. In this case we can still speak of the pairs of elements which would occur in such sub-designs (if they existed). We give the following formal definition.

Let V be a set of $2n$ symbols and let U be a subset of V of cardinality $2m$. An *incomplete Howell design* $H(s, 2n) - H(t, 2m)$ is an $s \times s$ array H which satisfies the following conditions.

- (1) Every cell in H is either empty or contains an unordered pair of elements of V .

(2) Every unordered pair of elements from V is in at most one cell of the array.

(3) There is an empty t by t subarray G of H .

(4) Every element of $V - U$ occurs in precisely one cell of each row and each column of H .

(5) Every element of U occurs once in each row and column not meeting G , but not in any row or column meeting G .

(6) The pairs of elements in $H - G$ plus the pairs of elements occurring in some $H(t, 2m)$ (defined on the elements of U) are the pairs which occur in an $H(s, 2n)$ defined on the elements of V .

The empty subarray G is often referred to as the *hole*. Observe that G can be filled in with an appropriate $H(t, 2m)$ defined on the elements of U (provided such a design exists), thereby constructing an $H(s, 2n)$. By appropriate, we mean that there is no pair of elements occurring together in the $H(t, 2m)$ which already appears together in the large array H . We must therefore be careful at times to note the pairs which are thought to be contained in the subarray.

We also refer to an $H(s, 2n) - H(t, 2m)$ incomplete Howell design as an $H(s, 2n)$ which *contains* an $H(t, 2m)$ sub-design or as an $H(s, 2n)$ which is *missing* an $H(t, 2m)$ sub-design if the $H(t, 2m)$ does not exist. An example of an incomplete Howell design with the sub-design filled in is the

| | | | | | | | | | | | | |
|--------|--------|--------|-------|-------|--------|--------|-------|--------|-------|-------|--------|--------|
| 1, 4 | 2, 5 | 3, 6 | 9, 14 | | | 7, 8 | | 10, 13 | | | | 11, 12 |
| 3, 5 | 1, 6 | 2, 4 | | 7, 14 | | | 9, 10 | | 8, 11 | | 12, 13 | |
| 2, 6 | 3, 4 | 1, 5 | | | | 12, 14 | 8, 13 | 9, 11 | | 7, 10 | | |
| | | 10, 14 | 7, 12 | 5, 13 | 3, 8 | 4, 6 | | | | 2, 11 | 1, 9 | |
| 9, 12 | 7, 11 | | 4, 8 | 6, 10 | | | | | 5, 14 | | 2, 3 | 1, 13 |
| | | 8, 12 | 5, 10 | | 4, 9 | | 1, 11 | 3, 7 | 6, 13 | | | 2, 14 |
| | | 11, 13 | 1, 2 | 8, 9 | | | 4, 14 | 6, 12 | 3, 10 | | | 5, 7 |
| | 9, 13 | | | 2, 12 | 5, 11 | 1, 10 | | | | 3, 14 | 4, 7 | 6, 8 |
| 11, 14 | | | 3, 13 | | 1, 12 | | 2, 7 | | | 6, 9 | 5, 8 | 4, 10 |
| | | 7, 9 | | 1, 3 | 13, 14 | | 5, 6 | 2, 8 | 4, 12 | | 10, 11 | |
| | 8, 14 | | 6, 11 | | 2, 10 | 5, 9 | 3, 12 | | 1, 7 | 4, 13 | | |
| 7, 13 | 10, 12 | | | | | 3, 11 | | 4, 5 | 2, 9 | 1, 8 | 6, 14 | |
| 8, 10 | | | | 4, 11 | 6, 7 | 2, 13 | | 1, 14 | | 5, 12 | | 3, 9 |

FIG. 1. An $H(13, 14)$ design with an $H^*(3, 6)$ sub-design.

$H(13, 14)$ displayed in Fig. 1 which contains an $H(3, 6)$ in the upper left hand corner.

There has been much interest in two special cases of the problem of finding Howell designs with sub-designs. The cases where the large square and the subsquare are both Room squares or are both orthogonal Latin squares have been studied. The following are the best results in these cases. They are stated in terms of Howell designs.

THEOREM 1.2 (Room Squares with Room Subsquares).

- (1) [5] *If there exists an $H(v, v + 1) - H(w, w + 1)$, then $v \geq 3w + 2$.*
- (2) [9, 10] *For $w = 3, 5, 7, 9, 11, 13, 15$, and for all odd $v \geq 3w + 2$, there is an $H(v, v + 1) - H(w, w + 1)$.*
- (3) [9] *For all odd $w \geq 37$ and for all odd $v \geq (7w - 5)/2$, there is an $H(v, v + 1) - H(w, w + 1)$.*
- (4) [9] *For all odd $w \geq 127$ and for all odd $v \geq 3w + 240$, there is an $H(v, v + 1) - H(w, w + 1)$.*
- (5) [33] *For all odd $w \geq 503$ and all odd $v \geq 3w + 2$, there is an $H(v, v + 1) - H(w, w + 1)$.*

THEOREM 1.3 [16] (Orthogonal Latin Squares with Orthogonal Latin Subsquares). *An $H^*(v, 2v) - H^*(w, 2w)$ exists if and only if $v \geq 3w$ and $(v, w) \neq (6, 1)$.*

In this paper, we investigate the existence of incomplete Howell designs, $H(s, 2n) - H(t, 2m)$. This is the first study of Howell designs with sub-designs where both of the designs are not Room squares or are not a pair of orthogonal Latin squares.

The paper is organized as follows. In the next section, we provide some bounds on the parameters for incomplete Howell designs. In Section 3 we describe some useful recursive constructions for Howell designs with sub-designs. We concentrate on Howell designs which contain Room squares as sub-designs in Section 4. In addition to providing some general results, in this section we completely determine the spectra of $H(n, n + 2) - RS(3)$, $H(n, n + 2) - RS(5)$ and $H(n, n + 2)$ which contain as subarrays $RS(7)$. Section 5 discusses Howell designs where the sub-design is a pair of orthogonal Latin squares. In this section, we determine the spectra completely for $H(n, n + 1) - H^*(m, 2m)$ and $H(n, n + 2) - H^*(m, 2m)$ for $m = 2, 3$, and 4. Some general results are also provided in Section 5. Finally, in Section 6 we prove a general result for $H(s, s + \alpha) - H(t, s + \alpha)$ where $\alpha \equiv 0 \pmod{2}$.

2. BOUNDS

As we have seen from the special cases discussed in Theorems 1.2 and 1.3, in order for an $H(n, n + \alpha) - H(m, m + \beta)$ to exist, n must necessarily be greater than some lower bound which is a function of m (and probably α and β). In this section, we will give such a general lower bound. We will also use this general bound to provide bounds for specific values of α and β .

We begin with some notation which will be used throughout this section.

Let M be a set of cardinality $m + \beta$ and let N be a set of cardinality $n + \alpha - \beta$. Suppose H is an $H(n + m, n + m + \alpha)$ defined on $M \cup N$ which contains as a subarray an $H(m, m + \beta)$ (called H_1) defined on M . We write H in the following form.

$$H = \begin{array}{|c|c|} \hline H_1 & A \\ \hline C & B \\ \hline \end{array} \left. \begin{array}{l} \} m \\ \} n \end{array} \right.$$

The number of filled cells in H_1 is $\frac{1}{2}(m + \beta)m$, thus the rectangles A and C both contain $\frac{1}{2}(n + \alpha - \beta)m$ filled cells. Therefore the square B contains $\frac{1}{2}(n^2 + n\alpha - m\alpha + m\beta)$ filled cells. Now let m_B denote the number of pairs in B with both elements from the set M , n_B denote the number of pairs in B with both elements from N , and p denote the number of pairs in B with one element from M and one from N . We have that

$$n_B + m_B + p = \frac{1}{2}(n^2 + n\alpha - m\alpha + m\beta). \tag{1}$$

Also, by counting occurrences of the symbols from M that occur in B we get

$$2m_B + p = (m + \beta)n. \tag{2}$$

(We note that since $(m + \beta)$ is even, the equation above says that p must be even. Thus, in B the number of pairs where one element is from M and one is from N is always an even number.)

Now since H_1 and B are the only cells which contain pairs where both elements are from M , we get that $\frac{1}{2}(m + \beta)m + m_B \leq \binom{m + \beta}{2}$. This simplifies to

$$m_B \leq \frac{1}{2}(m + \beta)(\beta - 1). \tag{3}$$

After solving Eqs. (1) and (2) for p and setting them equal, we have

$$\frac{1}{2}(n^2 + n\alpha - m\alpha + m\beta) - m_B - n_B = (m + \beta)n - 2m_B$$

or

$$n_B = \frac{1}{2}(n^2 + n\alpha - m\alpha + m\beta) - (m + \beta)n + m_B.$$

Now use the bound for m_B from Eq. (3) to get

$$0 \leq n_B \leq \frac{1}{2}(n^2 + n\alpha - m\alpha + m\beta) - (m + \beta)n + \frac{1}{2}(m + \beta)(\beta - 1)$$

After solving this inequality for m we get the following general bound for incomplete Howell designs.

THEOREM 2.1. *If there exists an $H(n + m, n + m + \alpha) - H(m, m + \beta)$, then*

$$m \leq \frac{n^2 + n\alpha - 2n\beta + \beta^2 - \beta}{2n + \alpha - 2\beta + 1}. \tag{4}$$

The next bound on the parameters comes from comparing the number of empty cells in a row of the sub-design to that of the full Howell design. Clearly, there must be at least as many empty cells in a row of the full Howell design as in a row of the sub-design. Thus, $m + n - \frac{1}{2}(m + n + \alpha) \geq m - \frac{1}{2}(m + \beta)$, which gives the next theorem.

THEOREM 2.2. *If there exists an $H(n + m, n + m + \alpha) - H(m, m + \beta)$, then $n \geq \alpha - \beta$.*

We remark that Theorem 2.1 gives a much better bound when α is small while the bound from Theorem 2.2 may be better for large values of α .

Another easy bound to prove is the following:

LEMMA 2.3. *If there exists an $H(n + m, n + m + \alpha) - H(m, m + \beta)$, then $n \geq m$.*

Proof. A symbol from the set N must occur in every row of A and every column of C . Thus, the total number of occurrences $m + n$ must be at least $2m$ and $n \geq m$. ■

We now proceed to determine more specific bounds for special classes of Howell designs with sub-designs. We will give bounds for the classes which are discussed in Sections 4, 5, and 6 of this paper.

In Section 4, we will look at the case of a Howell design with a Room square sub-design (an $H(n + m, n + m + \alpha) - H(m, m + 1)$). The next theorem provides a lower bound for $n + m$ in this case.

THEOREM 2.4. *If there exists an $H(n+m, n+m+\alpha) - H(m, m+1)$, then $n+m \geq 3m - \alpha + 2$ whenever $\alpha \leq m+1$, and $n+m \geq m + \alpha - 1$ whenever $\alpha \geq m+2$.*

Proof. First assume that $\alpha \leq 2m$. Plug $\beta = 1$ into Eq. (4) and write the resulting inequality as $0 \leq n^2 + (\alpha - 2 - 2m)n + m(1 - \alpha)$. Apply the quadratic formula and take the positive root to obtain

$$\begin{aligned} n &\geq m+1 - \frac{\alpha}{2} + \frac{1}{2} \sqrt{4m^2 + \alpha^2 - 4\alpha + 4 + 4m} \\ &= m+1 - \frac{\alpha}{2} + \frac{1}{2} \sqrt{4m^2 - 4m\alpha + \alpha^2 - 4\alpha + 4 + 4m + 4m\alpha} \\ &> m+1 - \frac{\alpha}{2} + \frac{1}{2} \sqrt{(2m - \alpha)^2}. \end{aligned}$$

Thus $n \geq 2m + 2 - \alpha$ and so it follows that $n+m \geq 3m - \alpha + 2$ when $\alpha \leq 2m$.

From Theorem 2.2 we see that we must also have $m+n \geq m + \alpha - 1$. The result now follows since $3m - \alpha + 2 \geq m + \alpha - 1$ whenever $\alpha \leq m+1$. ■

In Section 5, we will concern ourselves with the case of $H(n+m, n+m+\alpha) - H(m, 2m)$ where $\alpha = 1$ or 2. The next theorem gives a lower bound for $n+m$ in this case.

THEOREM 2.5. *If there exists an $H(n+m, n+m+\alpha) - H(m, 2m)$, then $n+m \geq \max\{4m - \alpha + 2, 2m\}$.*

Proof. This proof is similar to that of the previous theorem. First use $\beta = m$ in Eq. (4) and write the resulting inequality as $0 \leq n^2 - (4m - \alpha)n + 3m^2 - m(\alpha + 2)$. Apply the quadratic formula and take the positive root to obtain

$$n \geq 2m - \frac{\alpha}{2} + \sqrt{m^2 - m(\alpha - 2) + \frac{\alpha^2}{4}}.$$

The expression under the radical sign can be simplified to $(m - (\alpha - 2)/2)^2 + \alpha - 1$. So if $\alpha \geq 1$ we get

$$n > 2m - \frac{\alpha}{2} + \sqrt{\left(m - \frac{\alpha - 2}{2}\right)^2}.$$

Therefore $n \geq 3m - \alpha + 2$, if $1 \leq \alpha \leq 2m + 2$. If $\alpha = 1$, then we have that $n \geq 2m - \frac{1}{2} + (m + \frac{1}{2}) = 3m$. This says that $m+n \geq 4m$. But since $\alpha = 1$, we are in the Room square case and $m+n$ must be odd. Thus $n+m \geq 4m+1$.

If $\alpha \geq 2m+2$ the result follows from Theorem 2.3. ■

One other bound is for the specific case when $\alpha = \beta$. This case will be discussed in Section 6. We have the following theorem.

THEOREM 2.6. *If there exists an $H(n + m, n + m + \alpha) - H(m, m + \alpha)$, then $n + m \geq 3m + 2$.*

Proof. Let $\beta = \alpha$ in Theorem 2.1 and proceed as in Theorem 2.4 to get

$$\begin{aligned} n &\geq m + \frac{\alpha}{2} + \frac{1}{2} \sqrt{4m^2 - 3\alpha^2 + 4\alpha + 4m} \\ &= m + \frac{\alpha}{2} + \sqrt{m^2 - \frac{3}{4}\alpha^2 + \alpha + m} \end{aligned} \tag{5}$$

It is not difficult to show that since $m \geq \alpha \geq 1$, then $m^2 - (3/4)\alpha^2 + \alpha + m > (m - (\alpha/2 - 1))^2$. Therefore, from Eq. (5) we have that $n \geq m + \alpha/2 + m - (\alpha/2 - 1)$ or $m + n \geq 3m + 1$. Now, if α is even, then n and $m + n$ are both even and therefore $m + n \geq 3m + 2$. If α is odd, then n and $m + n$ are both odd and again we have that $m + n \geq 3m + 2$. Note that if $\alpha = 1$, this is Theorem 1.2. ■

3. CONSTRUCTIONS

In this section, we describe several constructions, both direct and recursive, for Howell designs with sub-designs. The main recursive techniques for constructing Howell designs with sub-designs use frames. All of the frames described in this paper are a special type of frame with block size 2, called Room frames, and we restrict our definitions and results to these frames. We refer the interested reader to [22] for more general definitions and results on frames.

Let V be a set of v elements. Let V_1, V_2, \dots, V_n be a partition of V . A $\{V_1, V_2, \dots, V_n\}$ -Room frame F is a square array of side v which satisfies the properties listed below. We index the rows and columns of F by the elements of V .

(1) Each cell is either empty or contains an unordered pair of symbols of V .

(2) The subarrays indexed by $V_i \times V_i$ are empty for $i = 1, 2, \dots, n$. (These subsquares are often referred to as the *holes of F*).

(3) Row (or column) x contains each element of $V - V_i$ for $x \in V_i$.

(4) The pairs occurring in F are precisely those $\{u, v\}$, where $\{u, v\} \in (V \times V) - \bigcup_{1 \leq i \leq n} (V_i \times V_i)$.

A $\{V_1, V_2, \dots, V_n\}$ -Room frame F is said to be *skew* if at most one of the cells (i, j) and (j, i) ($i \neq j$) is nonempty.

The type of a $\{V_1, V_2, \dots, V_n\}$ -Room frame is the multiset $\{|V_1|, |V_2|, \dots, |V_n|\}$. We will say that a Room frame has type $t_1^{u_1} t_2^{u_2} \dots t_k^{u_k}$ if there are u_i V_j 's of cardinality t_i , $1 \leq i \leq k$. The term frame was originally used for Room frames, see for example [9, 12, 29]; for convenience, in this paper we will use the terms frame and Room frame interchangeably.

The basic frame construction for Howell designs is stated below. It is often referred to as the "filling in the holes" construction.

THEOREM 3.1 [29]. *If there exists a $\{G_1, G_2, \dots, G_m\}$ -frame and $H^*(|G_i|, |G_i| + \alpha)$ for $i = 1, 2, \dots, m$, then there exists an $H^*(\sum_{i=1}^m |G_i|, \sum_{i=1}^m |G_i| + \alpha)$ which contains an $H^*(|G_i|, |G_i| + \alpha)$ sub-design for all $i = 1, 2, \dots, n$.*

COROLLARY 3.2. *If there exists a $\{G_1, G_2, \dots, G_m\}$ -frame and $H^*(|G_i| + w, |G_i| + w + \alpha) - H^*(w, w + \alpha)$ for $i = 1, 2, \dots, n$, then there exists an $H^*(\sum_{i=1}^m |G_i| + w, \sum_{i=1}^m |G_i| + w + \alpha) - H^*(w, w + \alpha)$.*

It is clear that this construction will always produce $H(n, n + \alpha)$ which contains as subarrays $H(m, m + \alpha)$ for some m and n . There are also two ways to use the basic frame construction to produce $H(n, n + \alpha)$ which contain as sub-designs $H(m, m + \beta)$ for $\alpha \neq \beta$. The first method is to use a sub-design $H(|G_i|, |G_i| + \alpha)$ which contains as a sub-design an $H(m, m + \beta)$ in one of the holes. The second method is to construct frames which already contain as subarrays the desired sub-design (an $H(m, m + \beta)$ or an $H(m, m + \beta)$ in standard form with the main diagonal deleted).

We describe several constructions for Room frames. The constructions are applied in later sections to construct Howell designs with sub-designs. The following construction provides some of the most useful Room frames; it was originally used to determine the spectrum of skew Room squares in [29, 31]. We will use it by filling in the holes with at least one sub-design which contains the desired sub-design.

THEOREM 3.3 [29, 31]. *Let $0 \leq t \leq 3m$. If there exists a $TD(5, m)$, then there is a skew frame of type $(4m)^4 (2t)$.*

It is easy to see how this theorem can be used to produce incomplete Howell designs and Howell designs with Howell sub-designs.

COROLLARY 3.4. *Suppose $m \geq 4$, $m \neq 6, 10$, $0 \leq t \leq 3m$, and $\alpha \equiv 0 \pmod{2}$. Then there exists an $H^*(16m + 2t, 16m + 2t + \alpha) - H^*(2t, 2t + \alpha)$ and an $H^*(16m + 2t, 16m + 2t + \alpha)$ which contains an $H^*(4m, 4m + \alpha)$.*

Proof. We apply Theorems 3.1 and 3.3. $TD(5, m)$ exist for all $m \geq 4$, $m \neq 6, 10$, see [18, 4]. We use $H^*(4m, 4m + \alpha)$ to fill in the holes of the frames in Theorem 3.3; these designs exist by Theorem 1.1 (see [2]). ■

The next two frame constructions are standard constructions for frames. The first is a direct product construction and the second is a special case of Wilson’s Fundamental Construction [35] applied to frames. Both of these constructions can be used to construct skew frames by using skew frames as input designs.

THEOREM 3.5 (Direct Product) [30]. *If there exists a frame of type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$ and a pair of mutually orthogonal Latin squares of side m , then there is a frame of type $(mt_1)^{u_1} (mt_2)^{u_2} \dots (mt_n)^{u_n}$.*

The Fundamental Construction is stated in terms of group divisible designs. A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$ where (i) X is a set (called *points*), (ii) \mathcal{G} is a partition of X into subsets (called *groups*), (iii) \mathcal{A} is a family of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and (iv) every pair of points from distinct groups occurs in exactly one block. The *group-type* (or *type*) of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. Usually an “exponential notation” is used to describe the type of a GDD: a GDD of type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$ is a GDD where there are u_i groups of size t_i for $1 \leq i \leq n$. A *transversal design* $TD(k, n)$ is a GDD of type n^k where every block has cardinality k . (We note that the pairs in a Room frame form a GDD with block size 2 and groups V_1, V_2, \dots, V_n .)

THEOREM 3.6 (Fundamental Construction) [35, 30]. *Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD, and let $w : X \rightarrow \mathbb{Z}^+ \cup \{0\}$ (w is called a *weighting*). For every $A \in \mathcal{A}$, suppose there is a Room frame of type $\{w(x) : x \in A\}$, then there is a Room frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

It is clear that the frame produced by the direct product construction contains as a subarray an $H^*(m, 2m)$. A frame of type 1^n exists if and only if there exists a $RS(n)$. If we use frames of type 1^n in either of these two constructions and fill in the holes of the frames with designs in standard form, then the resulting design contains as a subarray a $RS(n)$.

In order to describe the next construction, we need some definitions. Let V be a finite set of size n and let K be a subset of V of size k . An *incomplete orthogonal array* $IA(n, k, s)$ is an $(n^2 - k^2) \times s$ array written on the symbol set V such that every ordered pair of $(V \times V) - (K \times K)$ occurs in every pair of columns of the array. An $IA(n, k, s)$ is equivalent to a set of $s - 2$ mutually orthogonal Latin squares of order n which are missing a subsquare of order k . We need not be able to fill in the $k \times k$ missing

subsquares with Latin squares of order k . Note that an $IA(n, k, 4)$ can be used to construct an $H(n, 2n) - H(k, 2k)$.

Let F be a Room frame of type $(2t)^n$ defined on $V \cup W$ where $|W| = 2t$. F has a set S of l ordered partitionable transversals of the $|V| \times |V|$ array indexed by the elements of V if the transversals in S satisfy the following properties. Let $S = \{S^1, S^2, \dots, S^l\}$.

(1) S^j contains $2t(n-1)$ pairs which can be partitioned into two sets S^j_1 and S^j_2 , $|S^j_1| = |S^j_2| = t(n-1)$, where every element of V occurs precisely once in S^j_i , $i = 1, 2$.

(2) The pairs in F can be ordered so that every element in V occurs precisely once as a first coordinate and precisely once as a second coordinate in the pairs of S^j .

The next two constructions for frames first appeared in [21] and were used to construct skew frames. They can be generalized and, for completeness, we include the proof of one of the more general constructions.

THEOREM 3.7. *Suppose there exists*

(1) *a frame of type $(2t)^n$ with a set S of l ordered partitionable transversals, $S = \{S^1, S^2, \dots, S^l\}$,*

(2) *a pair of orthogonal Latin squares of side m , and*

(3) *$IA(m + k_j, k_j, 4)$ where $\sum_{j=1}^l k_j = k$.*

Then there exists a frame of type $(2tm)^{n-1} (2tm + 2k)$.

Proof. Let $V = \bigcup_{i=1}^{n-1} V_i$ where $|V_i| = 2t$ for all i . Let $M = \{1, 2, \dots, m\}$. Let $\beta_j = \{\alpha_{i,k_j} \mid i = 1, 2, \dots, k_j\}$ and let $\gamma_j = \{\infty_{i,k_j} \mid i = 1, 2, \dots, k_j\}$. Let $\beta = \bigcup_{j=1}^l \beta_j$ and let $\gamma = \bigcup_{j=1}^l \gamma_j$.

Let N_1 and N_2 be a pair of orthogonal Latin squares of side m defined on M . N will be the array of pairs formed by the superposition of N_1 and N_2 , $N = N_1 \circ N_2$. N_{xy} is the array of pairs formed by replacing each pair (a, b) in N with the pair $((a, x), (b, y))$.

We use an $IA(m + k_j, k_j, 4)$ to construct a pair of orthogonal Latin squares of side $m + k_j$ which is missing a pair of orthogonal Latin squares of side k_j . (The smaller Latin squares need not exist.) Let I denote the $m + k_j$ square array of pairs formed by superimposing the pair of Latin squares. I^j_{xy} will denote I defined on the symbols $(M \times \{x, y\}) \cup (\beta_j \cup \gamma_j)$. More precisely, if (x, y) is an ordered pair, then the pair of Latin squares used to construct I^j_{xy} will be defined on $(M \times \{x\}) \cup \beta_j$ and $(M \times \{y\}) \cup \gamma_j$

respectively, where the missing subarrays are defined on β_j and γ_j . I^j_{xy} can be written in the following form.

$$I^j_{xy} = \begin{array}{|c|c|} \hline A^j_{xy} & B^j_{xy} \\ \hline D^j_{xy} & E \\ \hline \end{array}$$

Let F be a frame of type $(2t)^n$ defined on $V \cup W$ where $|W| = 2t$ and F has a set S of l ordered partitionable transversals, $S = \{S^1, S^2, \dots, S^l\}$. We construct a frame of type $(2tm)^{n-1} (2tm + 2k)$ from F as follows. We first construct a $2tmn \times 2tmn$ array from F . If (x, y) is a pair in $F - S$, replace (x, y) by the $m \times m$ array N_{xy} . If (x, y) is a pair in S^j , replace (x, y) by the $m \times m$ array A^j_{xy} . All empty cells of F are replaced by $m \times m$ empty arrays. The resulting array F_1 has a diagonal of empty square arrays of side $2tm$.

We add $2k$ new rows and $2k$ new columns to F_1 . Suppose S^i_j contains the pair (x, y) in row r of F . Then we place (x, y) in row r of a column vector C^i_j . C^i_j will contain $t(n-1)$ pairs; the other entries will be zero. Next suppose S^i_j will contain the pair (x, y) in column r of F . Then we place (x, y) in column r of a row vector R^i_j . We now expand these vectors. Replace each pair (x, y) in C^i_j with the $m \times k_j$ array B^j_{xy} and replace each pair (x, y) in R^i_j with the $k_j \times m$ array D^j_{xy} . Label the resulting arrays C^{i*}_j and R^{i*}_j , respectively. Let $\mathcal{C} = [C^{1*}_1, C^{1*}_2, \dots, C^{l*}_1, C^{l*}_2]$ and let $\mathcal{R} = [R^{1*}_1, R^{1*}_2, \dots, R^{l*}_1, R^{l*}_2]^T$. \mathcal{C} is a $2tmn \times 2k$ array and \mathcal{R} is a $2k \times 2tmn$ array.

We add the arrays \mathcal{C} and \mathcal{R} to F_1 as follows.

$$F_2 = \begin{array}{|c|c|} \hline F_1 & \mathcal{C} \\ \hline \mathcal{R} & E \\ \hline \end{array}$$

The resulting array F_2 is a $2tmn + 2k$ square array. E is an empty array of side $2k$. F_2 is a frame of type $(2tm)^{n-1} (2tm + 2k)$. The hole of size $2tm + 2k$ is defined on $(M \times W) \cup (\beta \cup \gamma)$; the other holes of the frame are defined on $M \times V_i$ for $i = 1, 2, \dots, n - 1$. ■

The next theorem shows that this construction can also be used with frames which contain a set of complete ordered partitionable transversals, see [21] for the proof.

THEOREM 3.8 [21]. *Suppose there exists*

- (1) *a frame of type $(2t)^n$ with a set S of l complete ordered partitionable transversals, $S = \{S^1, S^2, \dots, S^l\}$,*
- (2) *a pair of orthogonal Latin squares of side m , and*
- (3) *$IA(m + k_j, k_j, 4)$ where $\sum_{j=1}^l k_j = k$.*

Then there exists a frame of type $(2tm)^n (2k)$.

Again, it is clear that these frames contain as subarrays $H^*(m, 2m)$.

We recall that the existence question for Room squares with sub-Room squares has been almost completely settled, and that the spectrum of $IA(n, k, 4)$ s is known. The next case to consider is $H(n, n+2)$ with $H(m; m+2)$ sub-designs, and we describe a special construction for this case.

This construction is a generalization of a construction for Room squares which uses houses to construct Room squares containing Latin square and Room square sub-designs. (It was first used to construct Room squares with sub-Room squares [32].) Let V be a set of $2n$ elements, and let F be a partition of V into n pairs. A *house* of order n is a $2n \times 2n$ array H which satisfies the properties listed below.

- (1) Each cell of H is either empty or contains a pair of distinct elements of V .
- (2) Each element of V occurs once in each row and once in each column of H .
- (3) Each pair in F occurs twice in H , once in the first row of H and once in the second row of H . Every other (unordered) pair of distinct elements of V occurs once in H .
- (4) Each column of H contains one pair from F .

The spectrum of houses is known: there exists a house of order n for n a positive integer and $n \neq 2$ [32].

THEOREM 3.9 [32]. *Suppose there exists a house of order n , a $RS(r)$, a $RS(2n-1)$, and a pair of mutually orthogonal Latin squares of order $(r+1)/2$. Then there exists a $RS(nr+n-1)$ which contains as subarrays a $RS(r)$, a $RS(2n-1)$, and an $H^*((r+1)/2, r+1)$.*

We modify this construction to produce $H(2n, 2n+2)$ which contain Latin square and $H(m, m+2)$ sub-designs.

THEOREM 3.10. *If there exists an $H(2n-2, 2n)$, an $H(2m, 2m+2)$, a house of order $m+1$, and a pair of orthogonal Latin squares of order n , then there exists an $H(2mn+2n-2, 2mn+2n)$ which contains as subarrays an $H(2m, 2m+2)$ and an $H(2n-2, 2n)$, as well as an $H(n, 2n)$.*

Proof. Let $V = \{x_i, y_i | i = 1, 2, \dots, m+1\}$ and let $N = \{1, 2, \dots, n\}$.

Let H_1 be a house of order $m+1$ defined on V where the repeated pairs are $\{x_i, y_i\}$ for $i = 1, 2, \dots, m+1$. Let H_2 be an $H(2m, 2m+2)$ defined on V . The missing pairs for H_2 are $\{x_i, y_i\}$ for $i = 1, 2, \dots, m+1$.

Let A_i be an $H(2n-2, 2n)$ defined on $N \times \{x_i, y_i\}$. Note that A_i is a square array of side $2n-2$.

Let L_1 and L_2 be a pair of orthogonal Latin squares of order n defined on N . L will be the array of pairs formed by the superposition of L_1 and L_2 , $L = L_1 \circ L_2$. We can write L so that the last column contains the pairs (i, i) for $i = 1, 2, \dots, n$. L_{xy} will denote the array of pairs formed by replacing every pair (a, b) in L with the pair $((a, x), (b, y))$. L_{xy} can be written in the following form.

$$L_{xy} = \begin{array}{|c|} \hline B_{xy} \quad C_{xy} \\ \hline \end{array}$$

(Note that C_{xy} contains the pairs $((i, x), (i, y))$ for $i = 1, 2, \dots, m+1$.)

We first use H_1 and H_2 to construct a $(2m+2) \times (4m+2)$ array F .

$$F = \begin{array}{|c|} \hline \begin{array}{ccc} x_1 y_1 & \dots & \mathcal{E} \\ & x_1 y_1 & \dots \\ & & H_1 & H_2 \end{array} \\ \hline \end{array}$$

\mathcal{E} is an empty array of size $2 \times 2m$. Next replace every pair (u, v) which is not a repeated pair in H_1 of F with the $n \times n - 1$ array B_{uv} . Replace every pair (u, v) in H_2 of F with the $n \times 1$ column C_{uv} . Finally, replace the 2×2 array which contains the pair (x_i, y_i) on its diagonal with the $2n-2 \times 2n-2$ array A_i . (Empty cells in H_1 are replaced by $n \times n - 1$ empty arrays and empty cells in H_2 or \mathcal{E} are replaced by $n \times 1$ empty arrays.) Call the resulting array H .

H is a $2mn+2n-2$ square array defined on $N \times V$. Every element of $N \times V$ occurs once in each column of H and once in each row of H . There are $n(m+1)$ pairs which do not occur in H ; these are of the form $\{(j, x_i), (j, y_i)\}$ for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, m+1$. It is straightforward

to verify that every other distinct unordered pair in $N \times V$ occurs precisely once in H . This verifies that H is an $H(2mn + 2n - 2, 2mn + 2n)$. H contains as subarrays an $H(2n - 2, 2n)$ and an $H(2m, 2m + 2)$ (as a copy of H_2). ■

COROLLARY 3.11. *If $n \neq 2, 6$ and $m \geq 2$, then there exists an $H(2mn + 2n - 2, 2mn + 2n)$ which contains as subarrays an $H(2m, 2m + 2)$ and an $H(2n - 2, 2n)$, as well as an $H(n, 2n)$.*

Starters and adders can be used to construct Room squares with sub-Room squares and $H(n, n + \alpha)$ with sub- $H(m, m + \alpha)$. The Room square constructions are described in [19] in terms of balanced tournament designs. Intransitive starters and adders were used in [21] to construct skew $H(n, n + 2)$. We describe the generalization of this construction for $H(n, n + 2k)$, $k \geq 1$.

An intransitive starter over \mathbb{Z}_{2n-2m} for an $H(2n, 2n + 2k)$ written on the symbol set $\mathbb{Z}_{2n-2m} \cup \{\infty_i \mid i = 1, 2, \dots, 2m\} \cup \{\alpha_i \mid i = 1, 2, \dots, 2k\}$ is defined to be a triple (S, R, C) where

$$\begin{aligned}
 S &= \{u_i, v_i \mid i = 1, 2, \dots, n - 3m - k\} \cup \{\infty_i, z_i \mid i = 1, \dots, 2m\} \\
 &\quad \cup \{\alpha_i, w_i \mid i = 1, \dots, 2k\} \\
 C &= \{\{x_i, y_i\} \mid i = 1, 2, \dots, m\} \\
 R &= \{\{x'_i, y'_i\} \mid i = 1, 2, \dots, m\}
 \end{aligned}$$

satisfying the following properties.

- (1) $S \cup C = \mathbb{Z}_{2n-2m} \cup \{\infty_i \mid i = 1, \dots, 2m\} \cup \{\alpha_i \mid i = 1, \dots, 2k\}$
- (2) No element of $\mathbb{Z}_{2n-2m} - \{0, n - m\}$ occurs more than once in $\{\pm(u_i - v_i) \mid i = 1, 2, \dots, n - 3m - k\} \cup \{\pm(x_i - y_i) \mid i = 1, 2, \dots, m\} \cup \{\pm(x'_i - y'_i) \mid i = 1, 2, \dots, m\}$
- (3) $|x_i - y_i| \equiv 1 \pmod{2}$ and $|x'_i - y'_i| \equiv 1 \pmod{2}$ for $i = 1, 2, \dots, m$.

A corresponding adder A is a set $\{c_1, c_2, \dots, c_{n-2m-k}, b_1, b_2, \dots, b_{2m}, a_1, a_2, \dots, a_{2k}\}$ where $a_i, b_i, c_i \in \mathbb{Z}_{2n-2m}$ such that no element of \mathbb{Z}_{2n-2m} occurs more than once in A and $\{u_i + c_i, v_i + c_i \mid i = 1, 2, \dots, n - 3m - k\} \cup \{z_i + b_i \mid i = 1, 2, \dots, 2m\} \cup \{w_i + a_i \mid i = 1, 2, \dots, 2k\} \cup \{x'_i, y'_i \mid i = 1, 2, \dots, m\} = \mathbb{Z}_{2n-2m}$.

Since the proof of the intransitive starter-adder construction is an immediate generalization of the proof for the skew case in [21], we omit it.

THEOREM 3.12 [21, 36]. *If there is an intransitive starter (S, C, R) over \mathbb{Z}_{2n-2m} for an $H(2n, 2n + 2k)$ and a corresponding adder A , and if there is an $H(2m, 2m + 2k)$, then there is an $H(2n, 2n + 2k)$ which contains as a subarray an $H(2m, 2m + 2k)$.*

In order to apply the recursive constructions, we need to construct a number of designs of “small” order. Theorem 3.12 can be used for $H(n, n + \alpha) - H(m, m + \beta)$ when $\alpha = \beta$. Examples of this construction appear in Section 5.

We also need to be able to find some of the more general $H(s, 2n) - H(t, 2m)$ to be used in the recursive constructions. Our main technique for constructing these designs is to use a hill-climbing algorithm on the computer. This algorithm is a modification of the algorithm which has been used successfully to find Room squares, Room squares with subsquares, and frames. We refer the reader to the following papers for discussions of this algorithm: Room squares and Room squares with subsquares [13], frames [10]. In fact, a discussion of the underlying graphs of frames is given in [10] and only minor modifications are needed if the sub-designs are Howell designs of different types. This program is particularly useful in finding Howell designs where the sides are not too large. The existence of many of these designs is stated in this paper; however for reasons of space, we include only three examples. A complete listing of these Howell designs is contained in [7].

4. ROOM SQUARE SUB-DESIGNS

In this section, we investigate the existence of Howell designs which contain Room squares as sub-designs.

We first construct frames which contain as subarrays frames of type 1^n for $n \equiv 1 \pmod{2}$, $n \geq 7$. This construction uses pairwise balanced designs (PBDs) and a more general definition of group divisible designs (GDDs); for definitions and results on these designs, see [35, 34, 15]. Suppose there exists a non-trivial $PBD(v; K)$, D , where $k \equiv 1 \pmod{2}$, $k \geq 7$ for each $k \in K$. If we delete one element from D , the resulting design is a $GDD(v - 1; K; G; 0, 1)$ where $|G_i| \geq 6$, $|G_i| \equiv 0 \pmod{2}$ for each $G_i \in G$. We apply the Fundamental Construction (Theorem 3.6) to construct a frame of type $\{|G_1|, |G_2|, \dots, |G_m|\}$. If there exist $H^*(|G_i|, |G_i| + \alpha)$ for $i = 1, 2, \dots, m$, then there is an $H^*(v - 1, v - 1 + \alpha)$ which contains as sub-designs $RS(k)$ for $k \in K$.

Let P_7 be the set of odd prime powers greater than or equal to 7. Mullin and Stinson have shown that there exists a $PBD(v; P_7)$ for $v \equiv 1 \pmod{2}$, $v \geq 7$ with at most 104 possible exceptions [25]. The existence of these $PBDs$ together with the existence of $H^*(2n, 2n + 2\alpha)$ for $n \geq 3$, $1 \leq \alpha \leq 3$ gives us the following result.

LEMMA 4.1. *If there exists a non-trivial $PBD(v; P_7)$, then there exists an $H^*(v - 1, v - 1 + 2\alpha)$, $1 \leq \alpha \leq 3$, which contains as a sub-design a Room square.*

In general, it is difficult to use this construction if we want to construct an $H(n, n + \alpha)$ with a sub- $RS(m)$ for a particular value of m . However, we can use the direct product to construct infinite classes of $H(n, n + \alpha)$ with $RS(m)$ sub-designs.

LEMMA 4.2. *Let $n \equiv 0 \pmod{2}$, $n \neq 2$ or 6 , and let k be a positive integer, $k \leq n/2$. Let $q \equiv 1 \pmod{2}$, $q \geq 7$. There exists an $H^*(nq, nq + 2k)$ which contains as a sub-design a $RS(q)$.*

Proof. Since there exists a $RS(q)$, there exists a frame of type 1^q . We apply the direct product (Theorem 3.5) to construct a frame of type n^q . If we use a pair of mutually orthogonal Latin squares of side n with the pair $(1, 1)$ in the upper left hand corner and fill in the holes of the resulting design with an $H^*(n, n + 2k)$ in standard form, then the $H^*(nq, nq + 2k)$ contains as a sub-design a $RS(q)$ (in the cells which are in the upper left corner of the $n \times n$ subarrays). ■

These designs can be used to fill in the holes of the frames constructed using Theorem 3.3.

THEOREM 4.3. *Let $q \equiv 1 \pmod{2}$, $q \geq 7$, and let k and l be positive integers which satisfy the inequalities $1 \leq k \leq 2l$ and $3ql \geq 8q + k$. Then there exists an $H^*(n, n + 2k)$ which contains as a sub-design a $RS(q)$ for $n \geq 16ql + \max\{4, 2k\}$.*

Proof. We apply Theorem 3.3 with $m = qj$, $j \geq l$, and $k \leq t \leq 8q + k$. The holes of size $4m = 4qj$ are filled in with the $H^*(4qj, 4qj + 2k)$ constructed in Lemma 4.2. These designs contain as subarrays $RS(q)$. Since $j \geq l$, the bound on k from Lemma 4.2 is $1 \leq k \leq 2l$. The inequality $t \leq 3m$ in Theorem 3.3 gives us the other bound on the parameters, $8q + k \leq 3ql$. The remaining hole is filled in with an $H^*(2t, 2t + 2k)$ (Theorem 1.1). The resulting design is an $H^*(16qj + 2t, 16qj + 2t + 2k)$ where $2k \leq 2t < 16q + 2k$, $j \geq l$ which contains a $RS(q)$. ■

We note that for $1 \leq k \leq 5$, we have $l = 3$, and the construction produces $H^*(n, n + 2k)$ which contain $RS(q)$ s for $n \geq 48q + \max\{4, 2k\}$. We show in Section 6 that if the product kq is large, these bounds can be improved.

The lower bound for n can be improved considerably in special cases. In particular, we will now determine the complete spectrum for $H(n, n + 2)$ with sub- $RS(m)$ for $m = 3, 5$ and 7 . In these cases, we will use Theorem 3.3, but instead of placing the sub-design in the holes of size $4m$, we construct $H^*(2t, 2t + 2) - RS(q)$ for at least eight consecutive values of t . Thus, we only need to construct a small number of designs with the desired subarray in order to establish the existence of the $H^*(n, n + 2)$ for $n \geq n_0$.

LEMMA 4.4. (i) *There exists an $H(n, n + 2)$ which is missing as a subarray a “RS(3)” for n even, $10 \leq n \leq 46$.*

(ii) *There exists an $H(n, n + 2)$ which is missing as a subarray a “RS(5)” for n even, $16 \leq n \leq 54$, and for $n \in \{58, 60, 62, 66\}$.*

Proof. These designs were constructed on the computer using hill climbing and are given in [7]. ■

LEMMA 4.5. *There exists an $H(n, n + 2)$ which is missing as a subarray a RS(3) for $n \in \{48, 50, 52, 54, 56, 58, 60, 62, 64, 66\}$.*

Proof. These cases are described in Table I. ■

THEOREM 4.6. *There exists an $H(n, n + 2)$ which is missing as a subarray a “RS(3)” if and only if $n \equiv 0 \pmod{2}$, $n \geq 10$.*

Proof. Necessity follows from Theorem 2.4.

Lemmas 4.4 and 4.5 provide $H(n, n + 2)$ which are missing RS(3)s for n even, $10 \leq n \leq 66$. If $68 \leq n \leq 86$, we write $n = 16 \cdot 4 + 2t$, $2 \leq t \leq 11$. Since there is an $H(16, 18)$ which is missing a RS(3), we apply Corollary 3.4 with $m = 4$ and $\alpha = 2$. If $88 \leq n \leq 110$, we apply Corollary 3.4 with $m = 5$, using an $H(20, 22)$ which is missing a RS(3) in one of the holes. For $n = 112, 116, 118, 120$, we write $n = 16 \cdot 7 + 2t$ where $t \in \{0, 2, 3, 4\}$ and apply Corollary 3.4 with $m = 7$ using an $H(28, 30) - RS(3)$. The case $n = 114$ is done separately. There exists a $GDD(57; \{5, 6\}; \{2, 11\}; 0, 1)$ (constructed from a $TD(5, 11)$, see [15]) and there exist frames of type 2^5 and 2^6 [12]. Using Theorem 3.6, we construct a frame of type $22^5 4$. If we fill in one of the holes with an $H(22, 24)$ missing a RS(3) and the other

TABLE I

$H(n, n + 2) - RS(3)$ for $n \in \{n \equiv 0 \pmod{2} | 48 \leq n \leq 66\}$

| n | Construction | Comments |
|-----|--------------|------------------------------------------------------------------|
| 48 | 12 · 4 | Theorem 3.5, 4^4 frame, $m = 3$, use $H(12, 14) - RS(3)$ |
| 50 | 10 · 5 | Theorem 3.5, 2^5 frame, $m = 5$, use $H(10, 12) - RS(3)$ |
| 52 | 13 · 4 | Theorem 3.5, 1^{13} frame, $m = 4$, use $RS(13) - RS(3)$ |
| 54 | 12 · 4 + 6 | Theorem 3.5, $4^4 2$ frame, $m = 3$, use $H(12, 14) - RS(3)$ |
| 56 | 8 · 5 + 16 | Theorem 3.5, $2^5 4$ frame, $m = 4$, use $H(16, 18) - RS(3)$ |
| 58 | 8 · 7 + 2 | Theorem 3.7, $n = 7$, $m = 4$, $k = 1$ use $H(10, 12) - RS(3)$ |
| 60 | 12 · 5 | Theorem 3.5, 3^5 frame, $m = 4$, use $H(12, 14) - RS(3)$ |
| 62 | S-A | [21], use $H(12, 14) - RS(3)$ |
| 64 | 16 · 4 | Theorem 3.5, 4^4 frame, $m = 4$, use $H(16, 18) - RS(3)$ |
| 66 | 12 · 5 + 6 | Theorem 3.5, $4^5 2$ frame, $m = 3$, use $H(12, 14) - RS(3)$ |

holes with an $H(22, 24)$ or an $H(4, 6)$, we have an $H(114, 116)$ missing a $RS(3)$.

For $n \geq 122$, we write $n = 16 \cdot m + 2t$. There exist $H(2t, 2t + 2) - RS(3)$ for $5 \leq t \leq 18$. We apply Theorem 3.3 with $m \geq 7$, $m \equiv 1 \pmod{2}$ and $5 \leq t \leq 18$ to construct an $H(n, n + 2)$ which is missing as a subarray a $RS(3)$. ■

THEOREM 4.7. *There exists an $H(n, n + 2)$ which is missing as a subarray a “ $RS(5)$ ” if and only if $n \equiv 0 \pmod{2}$, $n \geq 16$.*

Proof. Necessity follows from Theorem 2.4.

Lemma 4.4 takes care of $H(n, n + 2)$ missing $RS(5)$ s for n even, $16 \leq n \leq 54$, and for $n \in \{58, 60, 62, 66\}$. An $H(56, 58)$ missing a $RS(5)$ can be constructed by using a frame of type $2^{54}1$ [10] in Theorem 3.5 with $m = 4$. We use an $H(16, 18) - RS(5)$ in the hole of size 16 and $H(8, 10)$ in the other holes. Similarly, a frame of type 4^4 can be used for an $H(64, 66) - RS(5)$.

For $68 \leq n \leq 86$, we write $n = 16 \cdot 4 + 2t$, $2 \leq t \leq 12$, and apply Corollary 3.4 using an $H(16, 18) - RS(5)$. For $88 \leq n \leq 110$, we write $n = 16 \cdot 5 + 2t$, $4 \leq t \leq 15$, and apply Corollary 3.4 using an $H(20, 22) - RS(5)$. The $H(114, 116) - RS(5)$ is constructed using the frame of type 22^54 constructed above and an $H(22, 24) - RS(5)$. For $n = 112, 116, \dots, 128$, we write $n = 16 \cdot 7 + 2t$, $t \in \{0, 2, 3, \dots, 8\}$, and apply Corollary 3.4.

For $n \geq 130$, we write $n = 16 \cdot m + 2t$, where $m \geq 7$, $m \neq 10$ and $8 \leq t \leq 15$. To take care of the case $m = 10$, we use $m = 9$ and $16 \leq t \leq 23$. Since there exist $H(2t, 2t + 2) - RS(5)$ for $8 \leq t \leq 23$, we can apply Corollary 3.4 to construct an $H(n, n + 2)$ which is missing a $RS(5)$. ■

LEMMA 4.8. *There exists an $H(n, n + 2)$ which contains as a subarray a $RS(7)$ for $n \in \{22, 24, 26, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 58, 60, 64, 66, 68, 72, 78, 80, 82\}$.*

Proof. These designs were constructed on the computer using hill climbing [7]. ■

We note that several of these large designs can also be easily done using recursive techniques.

LEMMA 4.9. *There exists an $H(n, n + 2)$ which contains as a subarray a $RS(7)$ for $n \in \{28, 56, 62, 70, 74, 76, 84, \dots, 132\}$.*

Proof. The cases for $n \in \{28, 56, 62, 70, 74, 76, 84, 86, 88, 90, 92\}$ are listed in Table II.

TABLE II

$H(n, n + 2) - RS(7)$ for $n \in \{28, 56, 62, 70, 74, 76, 84, 86, 88, 90, 92\}$

| n | Construction | Comments |
|-----|---------------|--------------------------------------------------------------|
| 28 | 7 · 4 | Theorem 4.2, $q = 7, n = 4$ |
| 56 | 7 · 8 | Theorem 4.2, $q = 7, n = 8$ |
| 62 | 7 · 9 - 1 | Theorem 4.1, $63 \in P_7$ with a block of size 7 |
| 70 | 7 · 10 | Theorem 4.2, $q = 7, n = 10$ |
| 74 | 7 · 10 + 4 | Corollary 3.2, Theorem 3.5, using an $H(14, 16) - H(4, 6)$ |
| 76 | 7 · 11 - 1 | Theorem 4.1, $77 \in P_7$ with a block of size 7 |
| 84 | 7 · 12 | Theorem 4.2, $q = 7, n = 12$ |
| 86 | 2 · 8 · 5 + 6 | Theorem 3.7, $n = 5, m = 8, k = 3$, use $H(22, 24) - RS(7)$ |
| 88 | 2 · 8 · 5 + 8 | Theorem 3.7, $n = 5, m = 8, k = 4$, use $H(24, 26) - RS(7)$ |
| 90 | 7 · 13 - 1 | Theorem 4.1, $91 \in P_7$ with a block of size 7 |
| 92 | 23 · 4 | Theorem 3.5, $n = 23, m = 4$, then use a $RS(23) - RS(7)$ |

If $94 \leq n \leq 132$, we use a transversal design $TD(6, 7)$. Give all of the points weights 2 and 4 in such a way that the sum of the weights of the points in group G_1 is 22 and the sum of the other points is $n - 22$. Now use Theorem 3.6 and fill in the blocks with frames of types $2^a 4^{6-a}$ [11]. The resulting frame has holes of size 22 and size s_i for $i = 2, \dots, 7$, where $12 \leq s_i \leq 24$. We apply Theorem 3.1 filling in the groups for $i = 2, \dots, 7$ with $H(s_i, s_i + 2)$ where $s_i = \sum_{x \in G} w(x)$ and filling in the group G_1 with the $H(22, 24) - H(7, 8)$. ■

THEOREM 4.10. *There exists an $H(n, n + 2)$ which contains as a subarray a $RS(7)$ if and only if $n \equiv 0 \pmod{2}, n \geq 22$.*

Proof. Again necessity follows from Theorem 2.4.

If $22 \leq n \leq 132$, we use Lemmas 4.8 and 4.9. Let $n \geq 132$. We write $n = 16 \cdot m + 2t$ where $m \geq 7, m \neq 10$ and $11 \leq t \leq 18$. To take care of the case $m = 10$, we use $m = 9$ and $19 \leq t \leq 26$. Then there exist $H(2t, 2t + 2)$ which contain as subarrays $RS(7)$. We apply Corollary 3.4 to construct $H(n, n + 2)$ which contain as subarrays $RS(7)$. ■

5. LATIN SQUARE SUB-DESIGNS

In Section 3, we noted that several of the constructions can be used to produce frames (and Howell designs) which contain as sub-arrays $H^*(m, 2m)$ or a pair of mutually orthogonal Latin squares of side m . In this section, we concentrate on two particular cases, $H(n, n + 2)$ and $H(n, n + 1)$ (or $RS(n)$). We recall from Theorem 2.5 that a necessary condition for the existence of a $RS(n)$ which contains an $H^*(m, 2m)$ is

| | | | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| | | 7, 9 | | | | | 8, 10 | 5, 6 |
| | | | | 6, 8 | 7, 10 | 5, 9 | | |
| 5, 7 | | 4, 10 | 6, 9 | 1, 2 | | 3, 8 | | |
| | 9, 10 | 3, 6 | 2, 5 | | 1, 8 | | 4, 7 | |
| | | | | 4, 9 | 2, 5 | 1, 10 | 3, 5 | |
| | 5, 8 | | 1, 7 | | 3, 9 | 4, 6 | | 2, 10 |
| 8, 9 | | | | 5, 10 | | 2, 7 | 1, 6 | 3, 4 |
| 6, 10 | | 1, 5 | 4, 8 | 3, 7 | | | 2, 9 | |
| | 6, 7 | 2, 8 | 3, 10 | 4, 5 | | | | 1, 9 |

FIG. 2. An $H(9, 10) - H(2, 4)$.

$n \geq 4m + 1$ while the necessary condition for the existence of an $H(n, n + 2)$ which contains an $H^*(m, 2m)$ is $n \geq 4m$.

We first show that we can determine the spectrum completely for $RS(2n + 1)$ and $H(2n, 2n + 2)$ missing an $H(2, 4)$ and for $RS(2n + 1)$ and $H(2n, 2n + 2)$ containing $H^*(m, 2m)$ sub-designs for $m = 3$ and 4.

THEOREM 5.1. (i) *There exists a $RS(n)$ which is missing as a subarray an $H(2, 4)$ if and only if $n \equiv 1 \pmod{2}$, $n \geq 9$.* (ii) *There exists an $H(n, n + 2)$ which is missing as a subarray an $H(2, 4)$ if and only if $n \equiv 0 \pmod{2}$, $n \geq 8$.*

Proof. (i) Necessity follows from Theorem 2.5.

A $RS(9)$ missing an $H(2, 4)$ (which was found by the hill-climbing algorithm) is displayed in Fig. 2. $RS(n)$ missing $H(2, 4)$ for $n \in \{11, 13, \dots, 27\}$ were found using the hill-climbing algorithm and are given in [7]. Now for all $n \equiv 1 \pmod{2}$, $n \geq 29$, there exists a $RS(n) - RS(9)$ [10, 11]. We can fill in the $RS(9)$ with a $RS(9) - H(2, 4)$ to construct a $RS(n) - H(2, 4)$.

(ii) Necessity follows from Theorem 2.5.

An $H(8, 10)$ which is missing as a subarray an $H(2, 4)$ was found by hill-climbing and is given in [7]. For $10 \leq n \leq 24$ with $n \equiv 0 \pmod{2}$, starter-adders for $H(n, n + 2)$ which contain $H(2, 4)$ are given in Appendix 1. Orders $n = 26, 30, 34$, and 36 were done by hill-climbing and are given in [7]. Constructions for the remaining orders of $n \leq 62$ are listed in Table III.

For $50 \leq n \leq 60$ we first start with a $TD(6, 5)$ and give every point weight 2 except for $5 - i$ ($0 \leq i \leq 5$) points in one group which get weight 0. Since there exist frames of type 2^5 and 2^6 we apply Theorem 3.6 to get frames of type $10^5(2i)^1$. Now insert the appropriate Howell designs in the holes including one $H(10, 12) - H(2, 4)$ (except when $i = 1$).

TABLE III

$H(n, n + 2) - H(2, 4)$ for Some $8 \leq n \leq 62$

| n | Construction | Comments |
|-----|-------------------------|------------------------------------------------------------------------|
| 28 | $2 \cdot 3(4 + 1) - 2$ | Corollary 3.11, $m = 4, n = 3$, use $H(8, 10) - H(2, 4)$ |
| 32 | $2 \cdot 3 \cdot 5 + 2$ | Theorem 3.7, $m = 3, n = 5, k = 1$, use $H(8, 10) - H(2, 4)$ |
| 38 | $(2 \cdot 6)3 + 2$ | Corollary 3.2, Theorem 3.5, $n = 6, m = 3$, use $H(8, 10) - H(2, 4)$ |
| 40 | $2 \cdot 4 \cdot 5$ | Theorem 3.5, $m = 4, n = 5$, use $H(8, 10) - H(2, 4)$ |
| 42 | $2 \cdot 4 \cdot 5 + 2$ | Corollary 3.2, Theorem 3.5, $m = 4, n = 5$, use $H(10, 12) - H(2, 4)$ |
| 44 | $2 \cdot 4 \cdot 5 + 4$ | Theorem 3.7, $m = 4, n = 5, k = 2$, use $H(8, 10) - H(2, 4)$ |
| 46 | $2 \cdot 3(7 + 1) - 2$ | Corollary 3.11, $m = 7, n = 3$, use $H(14, 16) - H(2, 4)$ |
| 48 | $(2 \cdot 6)4$ | Theorem 3.5, $n = 6, m = 4$ |
| 62 | $S-A$ | Theorem 3.12, [21], use $H(12, 14) - H(2, 4)$ |

Let $n \geq 64$. We write $n = 16m + 2t, m \geq 4, m \neq 6, 10$. For $m = 4$, let $0 \leq t \leq 8$. We can apply Theorems 3.3 and 3.1 (using an $H(16, 18) - H(2, 4)$ for $t \neq 1$ and an $H(16, 18)$ for $t = 1$) to construct an $H(n, n + 2)$ which is missing as a subarray an $H(2, 4)$ for $64 \leq n \leq 80$. For $m = 5$, let $0 \leq t \leq 15$. Since there is an $H(20, 22) - H(2, 4)$, we can construct an $H(n, n + 2)$ which is missing an $H(2, 4)$ for $80 \leq n \leq 110$ using Theorem 3.3. (For the case $t = 1$, we use an $H(20, 22)$.) For $m \geq 7, m \neq 10$, let $4 \leq t \leq 20$. We use Corollary 3.4 to construct an $H(16m + 2t, 16m + 2t + 2)$ which contains as a subarray an $H(2t, 2t + 2)$. Since there exist $H(2t, 2t + 2) - H(2, 4)$ for $4 \leq t \leq 20$, there is an $H(n, n + 2)$ which is missing as a subarray an $H(2, 4)$ for $n \geq 112$. ■

THEOREM 5.2. (i) *There exists a $RS(n)$ which is missing as a subarray an $H^*(3, 6)$ if and only if $n \equiv 1 \pmod{2}, n \geq 13$.*

(ii) *There exists an $H(n, n + 2)$ which is missing as a subarray an $H^*(3, 6)$ if and only if $n \equiv 0 \pmod{2}, n \geq 12$.*

Proof. For both cases, necessity follows from Theorem 2.5.

(i) A $RS(13)$ missing an $H^*(3, 6)$ was found by the hill-climbing algorithm and is displayed in Fig. 1. $RS(n)$ missing $H^*(3, 6)$ for $n \in \{15, \dots, 39\}$ were found using the hill-climbing algorithm and are given in [7]. Now for all $n \equiv 1 \pmod{2}, n \geq 41$, there exists a $RS(n) - RS(13)$ [10, 11]; we fill in the $RS(13)$ with a $RS(13)$ which is missing an $H^*(3, 6)$ to construct $RS(n) - H^*(3, 6)$.

(ii) For all $n \equiv 4 \pmod{6}, 16 \leq n \leq 82$, there exists an $H(n, n + 2)$ which contains an $H^*(3, 6)$ by Corollary 3.11. Since there exist frames of type 2^l for $l \geq 5$ [29, 12], we can use the direct product (Theorem 3.5) with $m = 3$ and the basic frame construction (Theorem 3.1) to construct $H(6l, 6l + 2)$ which contain $H^*(3, 6)$ for $l \geq 5$. The three smallest

$H(n, n+2)$ for $n \equiv 0 \pmod{6}$, $n = 12, 18$, and 24 are all constructed using hill-climbing in [7]. For $n \equiv 2 \pmod{6}$, we first construct $H(n, n+2)$ for $14 \leq n \leq 56$ using hill-climbing; these are all listed in [7]. We consider the four remaining cases, $n = 62, 68, 74$, and 80 . There exists an $H(62, 64) - H(12, 14)$ [21]; so we can fill in the $H(12, 14)$ with an $H(12, 14)$ which contains an $H^*(3, 6)$ to construct an $H(62, 64)$ which contains an $H^*(3, 6)$. By (i) there exists a $RS(17)$ which contains an $H^*(3, 6)$. We can use Theorem 3.5 with $m = 4$ and this design to construct an $H(68, 70)$ which contains an $H^*(3, 6)$. To construct an $H(74, 76)$ which contains an $H^*(3, 6)$, we use Theorem 3.7 with $t = 1, n = 5, l = 1$, and $k = 2$ filling in the holes with $H(14, 16)$ and $H(16, 18)$ where at least one of these designs contains an $H^*(3, 6)$. An $H(80, 82)$ which contains an $H^*(3, 6)$ is easily constructed using Theorem 3.5 with a 2^5 frame and $m = 8$; we fill in the holes with an $H(16, 18)$ which contains an $H^*(3, 6)$. We have constructed $H(n, n+2)$ which contain $H^*(3, 6)$ for $n \equiv 0 \pmod{2}$, $12 \leq n \leq 82$. Using a $TD(6, 7)$, we let $w(x) = 2$ for every $x \in G_1$ and let $w(x) = 2$ or 4 for all other points. Then by Theorems 3.6 and 3.1 we can construct an $H(n, n+2) - H(14, 16)$ for every $n \equiv 0 \pmod{2}$, $84 \leq n \leq 154$. Now use the $H(14, 16)$ which contains an $H^*(3, 6)$ to get an $H(n, n+2)$ which contains an $H^*(3, 6)$ for all $n \equiv 0 \pmod{2}$, $84 \leq n \leq 154$.

Let $n \geq 124$, $n \equiv 0 \pmod{2}$. Write $n = 16m + 2t$ where $m \geq 7$, $m \neq 10$ and $12 \leq 2t \leq 26$. To take care of the case $m = 10$, we use $m = 9$ and $28 \leq 2t \leq 42$. By Corollary 3.4, there exist $H(n, n+2)$ which contain $H(2t, 2t+2)$ sub-designs. Now use the $H(2t, 2t+2)$ containing an $H^*(3, 6)$. ■

THEOREM 5.3. (i) *There exists a $RS(n)$ which is missing as a subarray an $H^*(4, 8)$ if and only if $n \equiv 1 \pmod{2}$, $n \geq 17$.*

(ii) *There exists an $H(n, n+2)$ which is missing as a subarray an $H^*(4, 8)$ if and only if $n \equiv 0 \pmod{2}$, $n \geq 16$.*

Proof. Necessity follows from Theorem 2.5.

(i) $RS(n)$ missing $H^*(4, 8)$ for $n \in \{17, \dots, 65\}$ were found using the hill-climbing algorithm and are given in [7]. For all $n \equiv 1 \pmod{2}$, $n \geq 67$, there exists a $RS(n) - RS(17)$ [10]. Plug in the $RS(17)$ missing the $H^*(4, 8)$ into these arrays.

(ii) Since there exist frames of type 1^k for $k \equiv 1 \pmod{2}$, $k \geq 7$ [26] and 2^l for $l \geq 5$ [29, 12], we can use the direct product (Theorem 3.5) with $m = 4$ and the basic frame construction (Theorem 3.1) to construct $H(4n, 4n+2)$ which contain $H^*(4, 8)$ for n a positive integer, $n \geq 9$ and $n = 7$.

Let $n \geq 116$. We write $n = 16 \cdot m + 2t$. Now we apply Corollary 3.4 with $2 \leq t \leq 17$ for $m = 9$ and $m = 7$ and with $2 \leq t \leq 9$ for $m \geq 11$ to construct $H(n, n+2)$ which contain $H^*(4, 8)$.

We consider the remaining cases. If $n \geq 22$ and $n \equiv 6 \pmod{8}$, then we can use Corollary 3.11 to construct $H(n, n+2)$ which contain $H^*(4, 8)$. If $n \equiv 4 \pmod{8}$, $n \geq 28$ or $n \equiv 0 \pmod{8}$, $n \geq 40$, then the frame construction described above provides the $H(n, n+2)$ with $H^*(4, 8)$ sub-designs.

The hill-climbing algorithm was used to construct $H(n, n+2) - H^*(4, 8)$ for $n \in \{16, 18, 20, 24, 26, 32, 34, 50\}$ [7]. We can construct $H(n, n+2)$ which contain $H^*(4, 8)$ for $n = 42$ and $n = 58$ by using Theorem 3.7 with $t = 1$, $n = 5$ or 7 [8] respectively, $m = 4$ and $k = 1$. The holes of the frames are filled in with $H(8, 10)$ and $H(10, 12)$. An $H(66, 68)$ which contains an $H^*(4, 8)$ can be constructed from a frame of type 4^6 [29] using Theorem 3.5 with $m = 3$ to construct a frame of type $12^4 18$ and filling in the holes with $H(14, 16)$ and an $H(18, 20)$ which contains an $H^*(4, 8)$. We construct a frame of type $20^5 14$ by using Theorem 3.6 with a $TD(6, 7)$ where $w(x) = 2$ for every $x \in G_1$ and $w(x) = 2$ or 4 for all other points so that $\sum_{x \in G_i} w(x) = 20$ for $2 \leq i \leq 6$. We fill in the holes of this frame with $H(14, 16)$ and $H(20, 22)$ where at least one of these designs contains an $H^*(4, 8)$ to construct an $H(114, 116)$ which contains an $H^*(4, 8)$. To construct the remaining designs, let $n = 16 \cdot m + 2t$. For $n \in \{74, 82\}$, we apply Corollary 3.4 with $m = 4$ and $t = 5$ and 9 respectively. For $n \in \{90, 98, 106\}$, we apply Corollary 3.4 with $m = 5$ and $t = 5, 9$ and 13 respectively.

Thus, we have constructed $H(n, n+2)$ which contain $H^*(4, 8)$ sub-designs for n a positive integer and $n \geq 16$. ■

In Section 2, we showed that the necessary condition for an $H(n, n+1)$ or an $H(n, n+2)$ to contain an $H(m, 2m)$ sub-design is that $n \geq 4m + 1$ or that $n \geq 4m + 2$, respectively. The most general lower bounds, at the present, are the following:

THEOREM 5.4. (i) *If $n \equiv 1 \pmod{2}$, $n \geq 16m + 1$ or $10m \leq n < 12m$ and $m \geq 7$, then there exists a skew $RS(n)$ which contains as a subarray an $H^*(m, 2m)$. If $n \equiv 1 \pmod{2}$, $n = 12m + 1$, $n = 14m + 1$, $12m + 7 \leq n < 13m$ or $14m + 7 \leq n < 15m$ and $m \geq 7$, then there exists a $RS(n)$ which contains as a subarray an $H^*(m, 2m)$.*

(ii) *If $n \equiv 0 \pmod{2}$, $10m \leq n \leq 12m$, $12m + 4 \leq n < 13m$, $14m + 4 \leq n < 15m$ or $n \geq 16m$ and $m \geq 7$, then there exists an $H(n, n+2)$ which contains as a subarray an $H^*(m, 2m)$.*

Proof. Let $w \geq 3m$ and let m be an integer, $m \geq 3$, $m \neq 6$. There exists a skew frame of type 2^5 with an ordered partitionable transversal [22]. For $w \geq 9$ and $k = 0, 1, 2, 3, 4$, there exist $IA(w+k, k, 4)$, [16]. So we can use Theorem 3.7 to construct a skew frame of type $(2w)^4 (2w+2k)$. Since $w \geq 3m$, there exists an $IA(w, m, 4)$ which contains as a subarray an $H^*(m, 2m)$ (for $m \neq 2, 6$) [16]. If we replace at least one of the

$IA(w, 0, 4)$ s in this construction with an $IA(w, m, 4)$ which contains as a subarray an $H^*(m, 2m)$, the resulting skew frame will contain at least one $H^*(m, 2m)$ as a subarray. We add two new elements, ∞_1 and ∞_2 , to the design.

(i) If we fill in the holes of the frame with skew $RS(2w + 1)$ and $RS(2w + 2k + 1)$ which contain $\{\infty_1, \infty_2\}$ in the lower right hand corner (Corollary 3.2), we construct skew $RS(10w + 2k + 1)$ which contain as subarrays $H^*(m, 2m)$. This provides $RS(n)$ which contain as subarrays $H^*(m, 2m)$ for $n \equiv 1 \pmod{2}$ and $n \geq 30m + 1$.

(ii) If we fill in the holes with $H(2w, 2w + 2)$ and $H(2w + 2k, 2w + 2k + 2)$ which are missing the pair $\{\infty_1, \infty_2\}$, then we construct an $H(10w + 2k, 10w + 2k + 2)$ which contains an $H^*(m, 2m)$ as a subarray. This provides $H(n, n + 2)$ which contain as subarrays $H^*(m, 2m)$ for $n \equiv 0 \pmod{2}$ and $n \geq 30m$.

Let $m \geq 7$. We can improve these bounds by further use of Theorems 3.7 and 3.8. If there exists a (skew) frame of type 2^s with l_1 ordered partitionable transversals and a (skew) frame of type 2^s with l_2 complete ordered partitionable transversals, then there exist (skew) frames of type $(2m)^{s-1}(2m + 2k)$ for $k = 0, 1, \dots, l_1 \lfloor m/2 \rfloor$ and (skew) frames of type $(2m)^s(2k)$ for $k = 0, 1, \dots, l_2 \lfloor m/2 \rfloor$ which contain as subarrays $H^*(m, 2m)$. There exist skew frames of type 2^s with l_1 ordered partitioned transversals and skew frames of type 2^s with l_2 complete ordered transversals for the following (s, l_1, l_2) triples [8]: $(5, 1, 2)$, $(8, 1, 4)$, $(9, 1, 4)$, $(10, 2, 0)$, $(11, 3, 0)$, $(12, 0, 6)$, and $(13, 1, 6)$. Using Theorems 3.7 and 3.8 as described above, we can construct skew frames on n elements where $16m \leq n \leq 32m - 6$ with holes of sizes $2m$ and $2m + 2k$ or $2k$. We fill in the holes of the frames with skew Room squares and Howell designs as described in cases (i) and (ii) above. This provides $RS(n)$ which contain as subarrays $H^*(m, 2m)$ for $n \equiv 1 \pmod{2}$ and $10m \leq n < 12m$ and $16m \leq n < 32m - 6$ and $H(n, n + 2)$ which contain as subarrays $H^*(m, 2m)$ for $n \equiv 0 \pmod{2}$ and $10m \leq n \leq 12m$ and $16m \leq n < 32m - 6$.

Frames of types 2^6 and 2^7 with one complete ordered partitioned transversal can be found in [8]. Using these frames and Theorem 3.8, we can construct $RS(n)$ which contain as subarrays $H^*(m, 2m)$ for $n \equiv 1 \pmod{2}$, $n = 12m + 1$, $14m + 1$, $12m + 7 \leq n < 13m$, and $14m + 7 \leq n < 15m$ and $H(n, n + 2)$ which contain as subarrays $H^*(m, 2m)$ for $n \equiv 0 \pmod{2}$, $n = 12m$, $14m$, $12m + 4 \leq n < 13m$, and $14m + 4 \leq n < 15m$. ■

These results can be improved in certain cases by using the house construction (Theorems 3.9 and 3.10). We get a better result in the case of $H(n, n + 2)$.

THEOREM 5.5. *Let n be a positive integer, $n \geq 3$, $n \neq 6$.*

(i) *There exists a $RS(8n + 1)$ which contains as a subarray an $H^*(n, 2n)$.*

(ii) *There exists an $H(6n - 2, 6n)$ which contains as a subarray an $H^*(n, 2n)$.*

Proof. (i) We apply the house construction for Room squares, Theorem 3.9 [32]; the smallest order house we can use is 4.

(ii) We apply Theorem 3.10 with $m = 2$ (using a house of order 3 and an $H(4, 6)$). ■

The next construction shows that in special cases, we can also produce Room squares of order $6m$ which contain as subarrays $H^*(m, 2m)$.

THEOREM 5.6. *Suppose there exist a pair of mutually orthogonal Latin squares of side n and an $IA(n, m, 4)$.*

(i) *If there exists a $RS(2n + 1)$ which is missing as a subarray an $H^*(m, 2m)$, then there is a $RS(14n + 1)$ which contains as a subarray an $H^*(7m, 14m)$.*

(ii) *If there exists an $H(2n, 2n + 2)$ which is missing as a subarray an $H^*(m, 2m)$, then there is an $H(14n, 14n + 2)$ which contains as a subarray an $H^*(7m, 14m)$.*

Proof. Let F be a frame of type 2^7 which is constructed from a skew Room square of order 7 and a pair of orthogonal partitioned incomplete Latin squares (OPILS) of type 1^7 . F can be written in the following form, where B contains the pair of OPILS of type 1^7 .

$$F = \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array}$$

Suppose F is defined on $V = W \times \bar{W}$ where $W = \{0, 1, \dots, 6\}$ and $\bar{W} = \{\bar{0}, \bar{1}, \dots, \bar{6}\}$. The elements associated with the i th hole of F are $i - 1$ and $\bar{i} - \bar{1}$.

Let $N = \{1, 2, \dots, n\}$. Let L_1 and L_2 be a pair of orthogonal Latin squares of side n defined on N . L will be the array of pairs formed by the superposition of L_1 and L_2 ; $L = L_1 \circ L_2$. L_{xy} is the array of pairs formed by replacing each pair (a, b) in L with the pair $((a, x), (b, y))$.

We use an $IA(n, m, 4)$ to construct a pair of orthogonal Latin squares of side n defined on N which is missing a pair of orthogonal Latin squares of

side m defined on $M = \{1, 2, \dots, m\}$. Let I denote the $n \times n$ array of pairs formed by superimposing the Latin squares. I can be written so that the $m \times m$ empty array is in the lower right hand corner of the array. I_{xy} will be the array of pairs formed by replacing each pair (a, b) with the pair $((a, x), (b, y))$.

We first construct a “frame” of type $(2n)^7$ as follows. Replace each pair (x, y) in A with L_{xy} and replace each pair (x, y) in B with I_{xy} . Denote the resulting array by F' .

(i) Let R_i be a $RS(2n+1)$ which is missing as a subarray an $H^*(m, 2m)$. R_i is defined on $N \times \{i-1, \overline{i-1}\} \cup \{\alpha, \infty\}$ and the subarray is defined on $M \times \{i-1, \overline{i-1}\}$. R_i can be written in the following form where R'_i is an $n \times n$ array and the $H^*(m, 2m)$ is in the lower right hand corner of R'_i .

$$R_i = \begin{array}{ccc} & & C_i^1 \\ & R'_i & \\ D_i^1 & D_i^2 & \{ \alpha, \infty \} \end{array}$$

We fill in the i th hole of F' with R'_i and add a new column $[C_1^1, C_2^1, \dots, C_7^1, C_1^2, \dots, C_7^2]^T$ and a new row $[D_1^1, D_2^1, \dots, D_7^1, D_1^2, \dots, D_7^2, \{ \alpha, \infty \}]$ to F' . The resulting design R is a $RS(14n+1)$ which is missing a $7m \times 7m$ array K defined on $M \times V$.

(ii) Let H_i be an $H(2n, 2n+2)$ defined on $N \times \{i-1, \overline{i-1}\} \cup \{\alpha, \infty\}$ where $\{\alpha, \infty\}$ does not occur in H_i . H_i is missing as a subarray an $H^*(m, 2m)$ which is defined on $M \times \{i-1, \overline{i-1}\}$. H_i can be written so that the empty $m \times m$ array occurs in the lower right hand corner of H_i . We fill in the i th hole of F' with H_i . The resulting design H is an $H(14n, 14n+2)$ which is missing as a subarray a $7m \times 7m$ array K defined on $M \times V$.

Since the pairs of an $H^*(7m, 14m)$ are missing from both H and R , we can fill K in with an $H^*(7m, 14m)$ defined on $M \times V$ (the mutually orthogonal Latin squares are defined on $M \times W$ and $M \times \overline{W}$ respectively). Thus, we have constructed a $RS(14n+1)$ and an $H(14n, 14n+2)$ which contain an $H^*(7m, 14m)$ sub-design. ■

We can apply this construction using the results of Theorems 5.1, 5.2, and 5.3. Since necessary and sufficient conditions for the existence of an $IA(n, m, 4)$ ($n \neq 6, m \neq 1$) is $n \geq 3m$ [16], the best we can expect from this

construction is $RS(6m + 1)$ and $H(6m, 6m + 2)$ which contain as subarrays $H^*(m, 2m)$.

COROLLARY 5.7. *Let n be a positive integer.*

(i) *If $n > 6$, there exists a $RS(14n + 1)$ and an $H(14n, 14n + 2)$ which contain as subarrays an $H^*(14, 28)$.*

(ii) *If $n \geq 9$, there exists a $RS(14n + 1)$ and an $H(14n, 14n + 2)$ which contain as subarrays an $H^*(21, 42)$.*

(iii) *If $n \geq 12$, there exists a $RS(14n + 1)$ and an $H(14n, 14n + 2)$ which contain as subarrays an $H^*(28, 56)$.*

6. $H(n, n + \alpha)$ WITH $H(m, m + \alpha)$ SUB-DESIGNS

In this section we describe constructions for $H^*(n, n + \alpha) - H^*(m, m + \alpha)$. We recall that the bound for n in this case is $3m + 2$. We begin with a bound for the case where n and α are even.

THEOREM 6.1. *If $\alpha \equiv 0 \pmod{2}$ and $m \geq 90$ with $m \equiv 0 \pmod{2}$, then there exists an $H^*(n, n + \alpha)$ containing an $H^*(m, m + \alpha)$ for all even $n \geq 6m$.*

Proof. Since $m \geq 90$, there exists a $TD(6, m/2)$ [1, 4, 18] with groups G_1, \dots, G_6 . Let $6m \leq n \leq 11m$. Write $n = m + 2a + 4b$ where $a + b = 5 \cdot m/2$. Now assign $w(x) = 2$ for every $x \in G_1$; also let $w(x) = 2$ for any a elements and $w(x) = 4$ for the remaining b elements of the TD . Then since there exist frames of type $2^i 4^{6-i}$ for $0 \leq i \leq 6$ [11], by Theorem 3.6 there exists a frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$. Now fill in the hole corresponding to group G_i with $H^*(\sum_{x \in G_i} w(x), \sum_{x \in G_i} w(x) + \alpha)$ to obtain an $H^*(n, n + \alpha)$ which contains an $H^*(m, m + \alpha)$ for every even $6m \leq n \leq 11m$.

Now let $n \geq 11m$. Write $n = 7m + 12a + 2b$ where $a \geq 18$ and $0 \leq b \leq 3m + 6a$. Since $m/2 + a \geq 63$, there exists a $TD(7, m/2 + a)$ [4, 18] with groups G_1, \dots, G_7 . Let $w(x) = 2$ for exactly $m/2$ points of G_1 and $w(x) = 0$ for the remaining a points of G_1 . In the remainder of the TD let $w(x) = 4$ for any b of the points and let $w(x) = 2$ for the remaining $3m + 6a - b$ points. Now since there exist frames of type $2^i 4^j$ for $i + j = 6$ or 7 , we can use Theorem 3.6 to construct a frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$. As above, we fill in the holes with Howell designs to obtain an $H^*(n, n + \alpha)$ which contains an $H^*(m, m + \alpha)$ for every $n \geq 11m$. This completes the proof. ■

We can do better if we restrict the size of α .

THEOREM 6.2. *Let $\alpha \equiv 0 \pmod{2}$.*

(a) *If $m \equiv 0 \pmod{4}$, $m \geq 180$, and $\alpha \leq m/2$, then there exists an $H^*(n, n + \alpha)$ containing an $H^*(m, m + \alpha)$ for all even $n \geq \frac{7}{2}m$.*

(b) *If $m \equiv 2 \pmod{4}$, $m \geq 178$, and $\alpha \leq m/2$, then there exists an $H^*(n, n + \alpha)$ containing an $H^*(m, m + \alpha)$ for all even $n \geq \frac{7}{2}m + 5$.*

Proof. (a) Since $m \geq 180$ and $m \equiv 0 \pmod{4}$, there exists a $TD(6, m/4)$ $[1, 4, 18]$ with groups G_1, \dots, G_6 . Let $\frac{7}{2}m \leq n \leq 6m$. Write $n = m + 2a + 4b$, where $a + b = 5 \cdot m/4$. Now assign $w(x) = 4$ for every $x \in G_1$; also let $w(x) = 2$ for any a elements and $w(x) = 4$ for the remaining b elements of the TD . Then since there exist frames of type $2^4 4^{6-t}$ for $0 \leq t \leq 6$, by Theorem 3.6 there exists a frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$. Now fill in the hole corresponding to group G_i with $H^*(\sum_{x \in G_i} w(x), \sum_{x \in G_i} w(x) + \alpha)$. Note that it is here that we use $\alpha \leq m/2$ since the smallest possible size for a hole in this frame is $m/2$. We obtain an $H^*(n, n + \alpha)$ which contains an $H^*(m, m + \alpha)$ for every even $\frac{7}{2}m \leq n \leq 6m$. Now using Theorem 6.1 finishes this case.

(b) This is very similar to part (a). Since $m \geq 178$ and $m \equiv 2 \pmod{4}$, there exists a $TD(6, (m + 2)/4)$ with groups G_1, \dots, G_6 . Now assign $w(x) = 4$ for every $x \in G_1$ except for one element $x_0 \in G_1$ which has $w(x_0) = 2$ and proceed as in part (a). ■

We are now in a position to improve upon Theorem 4.3 when the product kq is large. We wish to construct $H^*(n, n + 2k)$ which contain $RS(q)$. We do this by embedding the $RS(q)$ in an $H^*(m, m + 2k)$ for a relatively small value of m and then using Theorem 6.2 to embed the $H^*(m, m + 2k)$ into $H^*(n, n + 2k)$ for every even $n \geq \frac{7}{2}m$. The next result gives the details of this construction.

COROLLARY 6.3. *Let $q \equiv 1 \pmod{2}$, $q \geq 7$.*

(a) *If $k \equiv 0 \pmod{2}$, with $2kq \geq 180$, then there exists an $H^*(n, n + 2k)$ which contains a $RS(q)$ for all $n \equiv 0 \pmod{2}$, $n \geq 7kq$.*

(b) *If $k \equiv 1 \pmod{2}$, with $kq \geq 178$, then there exists an $H^*(n, n + 2k)$ which contains a $RS(q)$ for all $n \equiv 0 \pmod{2}$, $n \geq 7kq + 16$.*

Proof. (a) First use Lemma 4.2 with $n = 2k$ to construct an $H^*(2kq, 2kq + 2k)$ which contains a $RS(q)$. Now since $2kq \geq 212$, we can use Theorem 6.2(a) to construct an $H^*(n, n + 2k)$ containing the $H^*(2kq, 2kq + 2k)$ (which contains the $RS(q)$) for all even $n \geq \frac{7}{2}(2kq) = 7kq$.

(b) Proceed as in part (a) but use Theorem 6.2(b). ■

We conclude this section by noting that we can improve these results for a few special cases. In particular, we can determine the spectrum for $H(n, n+2) - H(m, m+2)$ for $m=2$ and $m=4$. We have already given necessary and sufficient conditions for the case of $m=2$ in Theorem 5.1(ii). In the next theorem we will do the same for the case $m=4$.

THEOREM 6.4. *There exists an $H(n, n+2)$ which is missing as a subarray an $H(4, 6)$ if and only if $n \equiv 0 \pmod{2}$, $n \geq 14$.*

Proof. All of these orders are done in [21] except for $n \in \{14, 18, 42, 46, 48, 50, 54, 56, 58, 62, 66, 82\}$. For $n \in \{14, 18, 42, 46, 48, 50, 56\}$ these designs are listed in [7]. We can use Corollary 3.11 to construct $H(n, n+2) - H(4, 6)$ for $n=58$ and $n=82$. To construct an $H(54, 56) - H(4, 6)$, we use a frame of type 2^5 and Theorem 3.5 with $m=5$ to construct a frame of type 10^5 and then use an $H(14, 16) - H(4, 6)$ in the holes (Corollary 3.2). An $H(62, 64) - H(4, 6)$ can be constructed using Theorem 3.7 with $t=2$, $n=5$, $l=1$, $m=3$, and $k=1$ to construct a $12^4 14$ frame; we then use an $H(14, 16) - H(4, 6)$ in the hole of size 14. (A 4^5 frame with 2 ordered partitionable transversals is constructed using [21, Theorem 3.3].) Finally, an $H(66, 68) - H(4, 6)$ is easily constructed using a $4^4 6$ frame, [29], and Theorem 3.5 with $m=3$. This time we fill in the hole of size 18 with an $H(18, 20) - H(4, 6)$. ■

7. CONCLUSIONS

In this paper, we have investigated the existence of incomplete Howell designs, $H(n, n+\alpha) - H(m, m+\beta)$. We have barely scratched the surface of this very large problem and a great deal of work remains to be done to determine the spectrum of incomplete Howell designs. Two of the problems we would like to concentrate on in the future include finding better bounds for the case where α is large (Section 2) and improving our bounds for constructions in the case when $\beta=m$ (Section 5). We would also like to extend our results for the case $\alpha=\beta$ (Section 6). For small values of α , the lower bound in Theorem 6.2 can be improved considerably by using constructions similar to those used for the Room square with subsquare results in [10]. It is also possible to apply constructions similar to those in Theorems 6.1 and 6.2 to produce $H(n, n+\alpha)$ when n is odd.

The majority of the constructions used in this paper are based on frame constructions. Almost all of the constructions for Howell designs in the existence papers [2, 27] and the more specialized papers [21] actually produce Howell designs with sub-designs. Thus, another area of

investigation is to find out how much information we can get about incomplete Howell designs from the available constructions combined with more recent existence results.

APPENDIX 1: STARTER-ADDERS FOR $H(n, n+2) - H(2, 4)$

1. $n = 10$.

| | | | | | | | | | | |
|-----|---|---|------------|---|------------|---|------------|---|------------|---|
| S | 4 | 6 | ∞_1 | 0 | ∞_2 | 3 | ∞_3 | 5 | ∞_4 | 7 |
| A | 0 | | 1 | | 2 | | 3 | | 4 | |
| C | 1 | 2 | | | | | | | | |
| R | 2 | 7 | | | | | | | | |

2. $n = 12$.

| | | | | | | | | | | | | |
|-----|---|---|---|---|------------|---|------------|---|------------|---|------------|---|
| S | 6 | 8 | 0 | 4 | ∞_1 | 3 | ∞_2 | 5 | ∞_3 | 7 | ∞_4 | 9 |
| A | 1 | | 0 | | 3 | | 8 | | 5 | | 2 | |
| C | 1 | 2 | | | | | | | | | | |
| R | 5 | 8 | | | | | | | | | | |

3. $n = 14$.

| | | | | | | | | | | | | | | |
|-----|---|---|---|---|---|---|------------|---|------------|---|------------|----|------------|----|
| S | 6 | 8 | 0 | 5 | 3 | 7 | ∞_1 | 4 | ∞_2 | 9 | ∞_3 | 10 | ∞_4 | 11 |
| A | 1 | | 0 | | 3 | | 11 | | 2 | | 4 | | 9 | |
| C | 1 | 2 | | | | | | | | | | | | |
| R | 1 | 4 | | | | | | | | | | | | |

4. $n = 16$.

| | | | | | | | | | | | | | | | | |
|-----|---|---|---|---|---|---|---|----|------------|---|------------|----|------------|----|------------|----|
| S | 7 | 9 | 0 | 6 | 4 | 8 | 5 | 10 | ∞_1 | 3 | ∞_2 | 11 | ∞_3 | 12 | ∞_4 | 13 |
| A | 1 | | 0 | | 5 | | 2 | | 8 | | 6 | | 7 | | 3 | |
| C | 1 | 2 | | | | | | | | | | | | | | |
| R | 1 | 4 | | | | | | | | | | | | | | |

5. $n = 18$.

| | | | | | | | | | | |
|-----|------------|----|------------|----|------------|----|------------|----|---|----|
| S | 0 | 7 | 6 | 8 | 5 | 9 | 4 | 10 | 3 | 14 |
| A | 0 | | 2 | | 4 | | 1 | | 3 | |
| S | ∞_1 | 11 | ∞_2 | 12 | ∞_3 | 13 | ∞_4 | 15 | | |
| A | | 8 | | 6 | | 7 | | 15 | | |
| C | 1 | 2 | | | | | | | | |
| R | 12 | 15 | | | | | | | | |

6. $n = 20$.

| | | | | | | |
|-----|--------------|---------------|---------------|---------------|-------|------|
| S | 0 8 | 9 16 | 3 5 | 4 10 | 11 15 | 7 12 |
| A | 0 | 1 | 2 | 10 | 4 | 9 |
| S | ∞_1 6 | ∞_2 13 | ∞_3 14 | ∞_4 17 | | |
| A | 7 | 17 | 8 | 12 | | |
| C | 1 2 | | | | | |
| R | 6 9 | | | | | |

7. $n = 22$.

| | | | | | | | |
|-----|--------------|---------------|---------------|---------------|------|-----|-------|
| S | 0 9 | 3 11 | 12 19 | 7 13 | 5 10 | 4 8 | 14 16 |
| A | 0 | 1 | 2 | 3 | 8 | 7 | 12 |
| S | ∞_1 6 | ∞_2 15 | ∞_3 17 | ∞_4 18 | | | |
| A | 17 | 4 | 10 | 19 | | | |
| C | 1 2 | | | | | | |
| R | 2 5 | | | | | | |

8. $n = 24$.

| | | | | | | | | |
|-----|--------------|--------------|--------------|---------------|-------|------|-------|-------|
| S | 0 10 | 3 12 | 5 13 | 7 14 | 15 21 | 6 11 | 16 20 | 17 19 |
| A | 6 | 1 | 2 | 4 | 9 | 16 | 5 | 19 |
| S | ∞_1 4 | ∞_2 8 | ∞_3 9 | ∞_4 18 | | | | |
| A | 8 | 11 | 14 | 13 | | | | |
| C | 1 2 | | | | | | | |
| R | 17 20 | | | | | | | |

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REFERENCES

1. R. J. R. ABEL, Four MOLS of orders 28 and 52, *J. Combin. Theory Ser. A*, **64** (1993), 144–148, to appear.
2. B. A. ANDERSON, P. J. SCHELLENBERG, AND D. R. STINSON, The existence of Howell designs of even side, *J. Combin. Theory Ser. A* **36** (1984), 23–55.

3. R. C. BOSE, E. T. PARKER, AND S. SHRIKHANDE, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Canad. J. Math.* **12** (1960), 189–203.
4. A. E. BROUWER AND G. H. J. VAN REES, More mutually orthogonal Latin squares, *Discrete Math.* **39** (1982), 263–281.
5. R. J. COLLINS AND R. C. MULLIN, Some properties of Room squares—A computer search, in “First Louisiana Conference on Combinatorics, Graph Theory and Computation, Baton Rouge, 1970,” pp. 87–111.
6. J. H. DINITZ, “Lower Bounds for the Number of Pairwise Symmetric Latin Squares,” Ph.D. Thesis, Ohio State University, Columbus, OH, 1980.
7. J. H. DINITZ AND E. R. LAMKEN, “Howell Designs with Sub-designs: The Research Report,” Research Report 90–15, Dept. of Mathematics, Univ. of Vermont.
8. J. H. DINITZ AND E. R. LAMKEN, Frames with partitionable transversals, in preparation.
9. J. H. DINITZ AND D. R. STINSON, The construction and uses of frames, *Ars. Combin.* **10** (1980) 31–54.
10. J. H. DINITZ AND D. R. STINSON, On the existence of Room squares with subsquares, *Contemp. Math.* **111** (1990), 73–92.
11. J. H. DINITZ AND D. R. STINSON, A few more frames, in “Graphs, Matrices, and Designs,” Dekker, New York, to appear.
12. J. H. DINITZ AND D. R. STINSON, Further results on frames, *Ars. Combin.* **11** (1981), 275–288.
13. J. H. DINITZ AND D. R. STINSON, A Hill-climbing algorithm for the construction of one-factorizations and Room squares, *SIAM J. Algebraic Discrete Methods* **8** (1987), 430–438.
14. J. H. DINITZ AND D. R. STINSON, A survey of Room squares and related designs, in “Contemporary, Design Theory: A Collection of Surveys,” pp. 137–204, Wiley, New York, 1992.
15. H. HANANI, Balanced incomplete block designs, *Discrete Math.* **11** (1975), 255–369.
16. K. HEINRICH AND L. ZHU, Existence of orthogonal Latin squares with aligned subsquares, *Discrete Math.* **59** (1986), 69–78.
17. S. H. Y. HUNG AND N. S. MENDELSON, On Howell designs, *J. Combin. Theory Ser. A* **16** (1974), 174–198.
18. D. JUNGnickel, Latin squares, their geometries and their groups, in “Coding Theory and Design Theory, Part II: Design Theory,” pp. 166–225, IMA Volumes in Mathematics and Its Applications, Vol. 21, Springer-Verlag, New York, 1990.
19. E. R. LAMKEN, Constructions for generalized balanced tournament designs, submitted for publication.
20. E. R. LAMKEN, The existence of 3 orthogonal partitioned incomplete Latin squares of type t^n , *Discrete Math.* **89** (1991), 231–251.
21. E. R. LAMKEN AND S. A. VANSTONE, The existence of skew Howell designs of side $2n$ and order $2n + 2$, *J. Combin. Theory Ser. A* **54** (1990), 20–40.
22. E. R. LAMKEN AND S. A. VANSTONE, Skew transversals in frames, *J. Combin. Math. Combin. Comput.* **2** (1987), 37–50.
23. R. C. MULLIN AND E. NEMETH, An existence theory for Room squares, *Canad. Math. Bull.* **12** (1969), 493–497.
24. R. C. MULLIN, P. J. SCHELLENBERG, S. A. VANSTONE, AND W. D. WALLIS, On the existence of frames, *Discrete Math.* **37** (1981), 79–104.
25. R. C. MULLIN AND D. R. STINSON, Pairwise balanced designs with odd block sizes exceeding 5, *Discrete Math.* **84** (1990), 47–62.
26. R. C. MULLIN AND W. D. WALLIS, The existence of Room squares, *Aequationes Math.* **13** (1975), 1–7.

27. D. R. STINSON, The existence of Howell designs of odd side, *J. Combin. Theory Ser. A* **32** (1982), 53–65.
28. D. R. STINSON, Room squares and subsquares, in “Combinatorial Mathematics X,” pp. 86–95, Lecture Notes in Mathematics, Vol. 1036, Springer-Verlag, Berlin/New York, 1983.
29. D. R. STINSON, “Some Classes of Frames, and the Spectra of Skew Room Squares and Howell Designs,” Ph.D. Thesis, University of Waterloo, 1981.
30. D. R. STINSON, Some constructions for frames, Room squares and subsquares, *Ars. Combin.* **12** (1981), 229–267.
31. D. R. STINSON, The spectrum of skew room squares, *J. Austral. Math. Soc. Ser. A* **31** (1981), 475–480.
32. D. R. STINSON AND W. D. WALLIS, An even side analogue of Room squares, *Aequationes Math.* **27** (1984), 201–213.
33. D. R. STINSON AND L. ZHU, Towards the spectrum of Room squares with subsquares, preprint.
34. S. A. VANSTONE, Doubly resolvable designs, *Discrete Math.* **29** (1980), 77–86.
35. R. M. WILSON, The construction and use of pairwise balanced designs, in “Math. Centre Tracts” No. 55, pp. 18–41, 1974.
36. L. ZHU, Constructions of Howell designs with subdesigns, *Ars. Combin.* **11** (1981), 235–238.