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Indecomposable Triple Systems with $\lambda = 6$

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Abstract

A triple system $B[3,\lambda;v]$ is indecomposable if it is not the union of two triple systems $B[3,\lambda_1;v]$ and $B[3,\lambda_2;v]$ with $\lambda = \lambda_1 + \lambda_2$. We prove that indecomposable triple systems with $\lambda = 6$ exist for $v = 8, 14$ and for all $v \geq 17$.

1. Introduction

A *balanced incomplete block design*, denoted $B[k,\lambda;v]$, is a pair (V,B) ; V is a v -set of *elements* and B is a collection of k -element subsets of V called *blocks*. Each 2-subset of V appears in precisely λ blocks. When B contains no repeated blocks we say the block design is *simple*. When $k = 3$ block designs are called *triple systems*.

A $B[k,\lambda;v]$ design (V,B) is *decomposable* if $B = B_1 \cup B_2$ and (V,B_1) is a $B[k,\lambda_1;v]$ design and (V,B_2) is a $B[k,\lambda_2;v]$ design with $\lambda_1 + \lambda_2 = \lambda$. If a balanced incomplete block design is not decomposable then it is termed *indecomposable*.

It would be of interest to find all orders v for which there exist simple indecomposable triple systems. Kramer [5] solved this problem for $\lambda = 2$ and 3. He showed that when $\lambda = 2$, simple indecomposable $B[3,2;v]$ exist for all $v \equiv 0,1 \pmod{3}$, except for $v = 7$ and when $\lambda = 3$, simple indecomposable $B[3,3;v]$ exist for all $v \equiv 1 \pmod{2}$, $v \geq 5$. The case of $\lambda = 4$ was solved by Colbourn and Rosa [2] who showed that simple indecomposable $B[3,4;v]$ exist if and only if $v \equiv 0,1 \pmod{3}$ and $v \geq 10$. When λ is odd and v is sufficiently large, Colbourn [1] proved the existence of indecomposable $B[3,\lambda;v]$. These triple systems are not necessarily simple, however. We see that the case of $\lambda = 6$ is the smallest unknown case. In this paper, we will establish that simple indecomposable $B[3,6;v]$ exists for $v = 8, 14$ and for all $v \geq 17$.

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1. Main Result

In this section we will establish our main Theorem. We begin with the cases of $v = 8$ and $v = 14$.

Lemma 2.1 *There exists simple indecomposable $B[9,6,8]$ and $B[9,6,14]$.*

Proof. Consider the complete design consisting of all 56 triples on a set of size 8. This is a simple $B[3,6;8]$. Furthermore it is indecomposable since by the usual numerical conditions on v , k and λ , it is easy to see that there does not exist any design $B[3,\lambda;8]$ with $\lambda < 6$. By the same numerical considerations, any simple $B[3,6,14]$ must be indecomposable. In [3] Dehon deduces the existence of simple $B[3,\lambda,v]$ for all λ and v which satisfy the necessary conditions. Thus a simple (indecomposable) $B[3,6,14]$ exists. \square

The following lemma is the main tool in constructing indecomposable designs for larger orders. It makes note of the fact that if a subdesign of a design is indecomposable, then the entire design is indecomposable. A $B[k,\lambda;v']$ design (V',B') is a *subdesign* of a $B[k,\lambda;v]$ design (V,B) if $V' \subset V$ and $B' \subset B$. In this case we also say that the design $B[k,\lambda;v']$ is *embedded* in the design $B[k,\lambda;v]$.

Lemma 2.2 *If a $B[k,\lambda;v]$ design D contains an indecomposable subdesign $B[k,\lambda;v']$, then D is indecomposable.*

Proof. A decomposition of the entire system would necessitate a decomposition of the (indecomposable) subsystem. \square

It is our plan to construct triple systems $B[3,6,v]$ for $17 \leq v \leq 36$ which contain the indecomposable subsystem $B[3,6;8]$. In order to do so we use a slightly modified version of the Stinson hill-climbing algorithm for Steiner triple systems (i.e. triple systems with $\lambda = 1$). In [7] Stinson describes a hill-climbing algorithm which is very successful at finding Steiner triple systems even with large fixed subsystems. We modified that algorithm to work for higher λ . For a given v we construct a simple triple system on v points which contains all of the triples from the set $\{1,\dots,8\}$. By the previous lemma we have that these systems are indecomposable.

Lemma 2.3 *There exist simple indecomposable triple systems $B[3,6;v]$ for $v = 18, 19, 21, 22, 24, 25, 26, 27, 28, 30, 31, 33, 34$, and 36 .*

Proof. These designs are all given in [4]. They all contain the indecomposable subsystem $B[3,6;8]$ on the set $\{1,\dots,8\}$. Lemmata 2.1 and 2.2 assure the indecomposability of the triple systems. \square

In order to proceed recursively we cite the following results of Colbourn and Rosa [2].

Theorem 2.4 *A simple $B[9,\lambda;v]$ can be embedded in a simple $B[9,\lambda;2v+1]$, a simple $B[9,\lambda;2v+4]$, and a simple $B[9,\lambda;2v+7]$.*

Lemma 2.5 *There exist simple indecomposable triple systems $B[3,6;v]$ for $v = 17, 20, 28, 29, 32$ and 35 .*

Proof. Embed the simple indecomposable $B[3,6;8]$ into simple (and thus indecomposable) designs $B[3,6;17]$, $B[3,6;20]$, and $B[3,6;23]$. Embed the simple indecomposable $B[3,6;14]$ into simple indecomposable designs $B[3,6;29]$, $B[3,6;32]$, and $B[3,6;35]$. \square

Just to reiterate what was shown in Lemmata 2.3 and 2.5, we now have that there exist simple indecomposable triple systems $B[3,6;v]$ for all $17 \leq v \leq 36$. We are now in a position to prove the main theorem.

Theorem 2.6 *There exist simple indecomposable $B[9,6,v]$ exists for $v = 8, 14$ and for all $v \geq 17$.*

Proof. If $v \leq 36$ then the theorem holds by Lemmata 2.1, 2.3 or 2.5. Now consider an arbitrary $v \geq 37$. By way of induction assume that there exists simple indecomposable triple systems $B[3,6;v']$ for all $v' < v$. If v is even, then $17 \leq (v-4)/2 < v$ and thus there exists a simple indecomposable $B[3,6;(v-4)/2]$. By Theorem 2.4 this design can be embedded into a simple $B[3,6;v]$ which is therefore indecomposable. If v is odd, then $17 \leq (v-1)/2 < v$ and thus again there exists a simple indecomposable $B[3,6;(v-1)/2]$ which embeds into a simple (indecomposable) $B[3,6;v]$. \square

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Addendum. We have learned that this same result has recently been proven independently by S. Milici [6]. In his proof he derives recursive constructions which embed simple $B[3,6;v]$ into simple $B[3,6;w]$ for $w = 2v+1$, $2v+2$ and $2v+4$. He then constructs roughly the same set of "small" orders as were constructed in this paper but without the use of a computer.

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